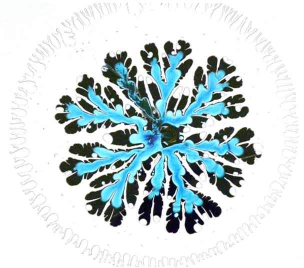
Coulomb Gas and Planar Orthogonal Polynomials



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- Background potential Q. Smooth, strictly subharmonic with growth $Q(z) \geq (1+\epsilon)\log|z| + \mathrm{O}(1)$ at infinity.
- Equal point charges $z_1, \ldots, z_n \in \mathbb{C}$, in external field $2nQ$ has energy

$$\mathcal{H}_m(z_1, \ldots, z_n) = \sum_{1\leq i\neq j\leq n} \log \frac{1}{|z_i - z_j|} + 2m \sum_{1\leq j\leq n} Q(z_j).$$

- The *Coulomb gas* is the random point process determined by the Gibbs measure

$$\mathrm{d}\mathcal{P}_n = \frac{1}{\mathcal{Z}_n}\, \mathrm{e}^{-\frac{\beta}{2}\mathcal{H}_n} \mathrm{dA}^n.$$

  Interested in *large n behaviour*.
- Fekete points ($\beta = \infty$), Ginibre ensemble, RNM ensembles etc ($\beta = 2$), Kähler-Einstein metrics ($\beta \to 0$).

- *Empirical measure* $\mu_n = \frac{1}{n}\sum_{j=1}^n \delta_{z_j}$. Write the energy

$$\mathcal{H}_n(z_1,\ldots,z_n) = n^2\Big\{\int_{z\neq w}\log\frac{1}{|z-w|}\mathrm{d}\mu_n(z)\mathrm{d}\mu_n(w) + 2\int Q(z)\,\mathrm{d}\mu_n.\Big\}$$

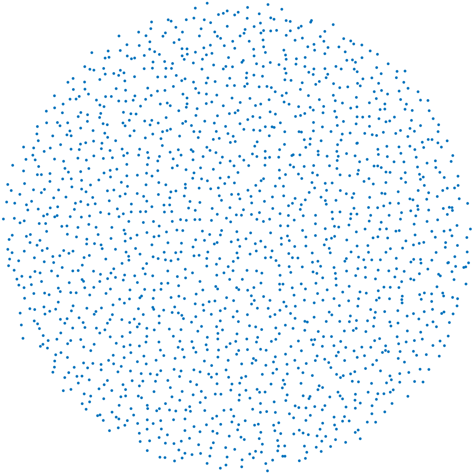
- Logarithmic energy problem with external field: Minimize

$$I(\mu) = \int_{\mathbb{C}^2}\log\frac{1}{|z-w|}\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w) + 2\int_{\mathbb{C}} Q(z)\,\mathrm{d}\mu(z).$$

  Unique *Frostman equilibrium measure* μ_Q.

Theorem (Hedenmalm–Makarov, 2004)

The empirical measures converge to μ_Q *as* $n\to\infty$.



Sample of Ginibre ensemble, $Q(z) = |z|^2/2, \beta = 2$, with $n = 1700$.

- Euler-Lagrange equation:

$$\begin{cases} U^{\mu_Q}(z) = Q(z) + c, & z \in \mathcal{S} = \operatorname{supp} \mu_Q, \\ U^{\mu_Q}(z) \leq Q(z) + c, & z \in \mathbb{C}. \end{cases}$$

- Explicit equilibrium measure difficult to find. If we can find the support $\mathcal{S}$, then

$$\mathrm{d}\mu_Q(z) = 2\Delta Q(z)\, 1_{\mathcal{S}}(z)\, \mathrm{dA}(z).$$

- If Q is real-analytic with $\Delta Q > 0$, then $\partial\mathcal{S}$ is piecewise real-analytic. Only singularities are cusps, double points (Sakai, 1993).

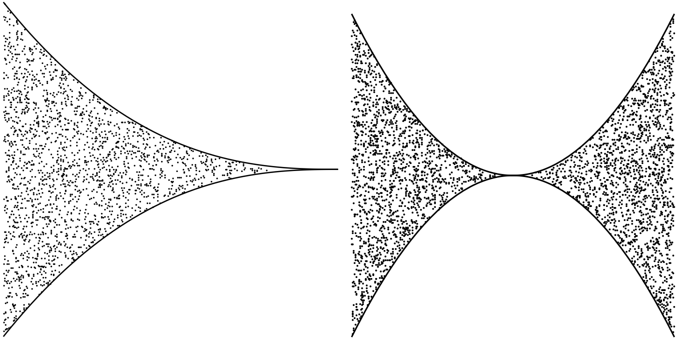
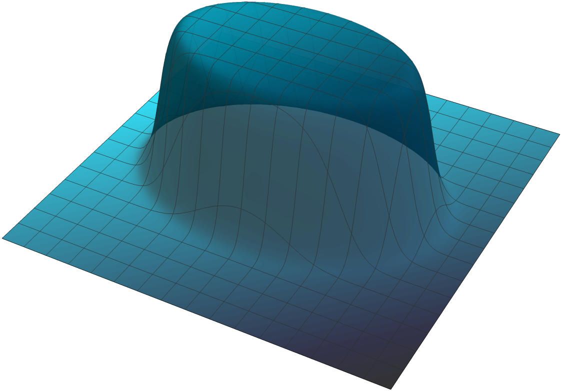


Figure. Illustration of the possible types of singular points.



The density of the Coulomb gas with $\beta = 2$ and $Q(z) = \frac{1}{2}|z|^2 + \frac{1}{5}\mathrm{Re}(z^2)$.

The determinantal case $\beta = 2$.

- *Correlation kernel* given by ONP's P_j

$$\mathcal{K}_n(z,w) = \sum_{j=0}^{n-1} P_j(z)\overline{P_j(w)}\mathrm{e}^{-n(Q(z)+Q(w))}.$$

- The law is given by

$$\mathrm{d}\mathcal{P}_n = \frac{1}{n!}\det(\mathcal{K}_n(z_i,z_j))_{1\le i,j\le n}\mathrm{dA}^n.$$

- The k-point functions $R_{n,k}(z_1,\dots,z_k)$ obtained by integrating out variables $z_n, z_{n-1},\dots,z_{k+1}$

$$R_{n,k}(z_1,\dots,z_k) = \det(\mathcal{K}_n(z_i,z_j))_{1\le i,j\le k}.$$

What do we see in blowup at microscopic scale around z_0 in the bulk?

- Ginibre: at scale $\frac{1}{\sqrt{n}}$, a determinantal random point process $G_\infty(z,w) = \mathrm{e}^{z\bar{w}-\frac{1}{2}(|z|^2+|w|^2)}$.
- For z_0 a bulk point we have universal blow-up $G_\infty(z,w)$ (Ameur–Hedenmalm–Makarov, 2008).
- Asymptotic expansion of *full* Bergman kernel $\mathcal{K}_{n,\infty}$ of (Tian, 1990. Refined by Zelditch, Catlin):

$$\mathcal{K}_{n,\infty}(z,z) = 2n\Delta Q(z) + \mathfrak{b}_0(z) + \frac{1}{n}\mathfrak{b}_1(z) + \ldots .$$

… but what about boundary points?

Rescaling around boundary point z_0 with normal n:

$$k(\xi,\eta)=\frac{1}{2n\Delta Q(z_0)}\mathcal{K}_n\Big(z_0+\mathrm{n}\frac{\xi}{\sqrt{2n\Delta Q(z_0)}},z_0+\mathrm{n}\frac{\eta}{\sqrt{2n\Delta Q(z_0)}}\Big).$$

- Ginibre ensemble: explicit computations give

$$k_n(\xi,\eta)\to\operatorname{erfc}(\xi+\bar\eta)\mathrm{e}^{\xi\bar\eta-\frac{1}{2}(|\xi|^2+|\eta|^2)}.$$

- Similar for $Q(z)=|z|^2+t\,\mathrm{Re}(z^2)$ (Lee and Riser, 2015) and $Q(z)=|z|^2+c\log|z-a|$ (Balogh–Bertola–Lee–McLaughlin, 2012).
- Partial Bergman kernels with S^1-invariance (Ross–Singer, 2015). *Spectral* Bergman kernels (Zelditch–Zhou, 2018)
- A priori assumption of translation invariance (Ameur–Kang–Makarov, 2014, also AKMW, 2015).

Theorem (Hedenmalm–W., 2018)

Assume that $\mathcal{S}$ is simply connected, with no singular points, and that $U^{\mu} < Q + c$ outside $\mathcal{S}$. Then, the rescaled boundary kernel converges

$$k_n(\xi, \eta) \to \operatorname{erfc}(\xi + \bar{\eta})\mathrm{e}^{\xi\bar{\eta} - \frac{1}{2}(|\xi|^2 + |\eta|^2)}.$$

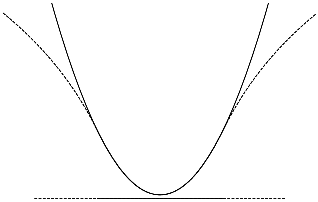
Define the subharmonic envelopes

$$\widehat{Q_\tau}(z) = \sup\left\{u(z) \,:\, u\,\text{subharm.},\ u(z) \le Q(z),\ \limsup_{|z|\to\infty} \frac{u(z)}{\log|z|} \le \tau\right\}.$$

Contact set $\mathcal{S}_\tau$.

- Compare u with $\frac{1}{n}\log|P|$. We have the fundamental bound

$$|P(z)|^2 \mathrm{e}^{-2nQ(z)} \le Cn\,\mathrm{e}^{-2n(Q-\widehat{Q_\tau})(z)}\|P\|_n^2, \qquad P \in \mathcal{P}_{k+1},\ \tau = \frac{k}{n}.$$



- $C^{1,1}$-regularity of $\widehat{Q}$. $\mathcal{S}_1$ is the support of the equilibrium measure (Obst. problem $\leftrightarrow$ EL-eqn.).

- ϕ_τ – conformal mapping $\mathcal{S}_\tau^c \to \mathbb{D}_{\mathrm{e}}$.
- $\mathcal{V}_\tau$ – bounded holomorphic function in exterior domain with real part Q on $\partial\mathcal{S}_\tau$.
- $\breve{Q}_\tau$ – harmonic continuation of $\widehat{Q}_\tau\big|_{\mathcal{S}^c}$ inwards. We have $\operatorname{Re}\mathcal{V}_\tau + \log|\phi_\tau| = \breve{Q}_\tau$.

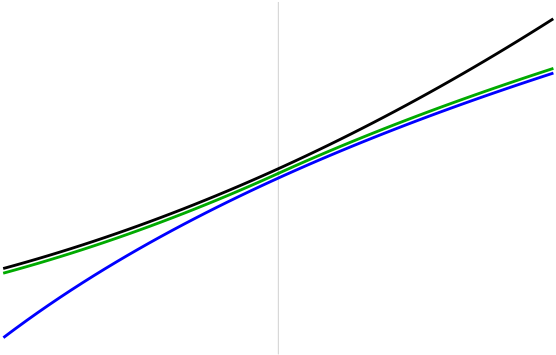


Figure: Q (black), $\widehat{Q}_\tau$ (green) and $\breve{Q}_\tau$ (blue) near $\partial\mathcal{S}_\tau$

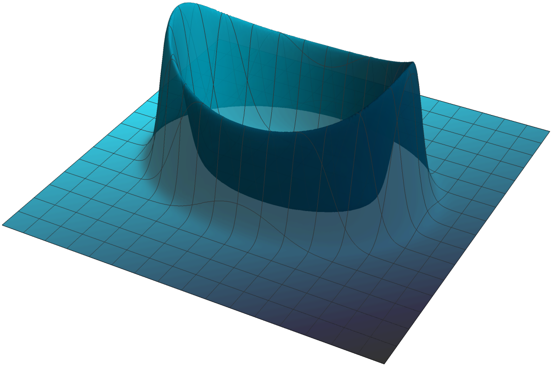
Theorem (Hedenmalm–W., 2018)

There exist bounded holomorphic $\mathcal{B}_{j,\tau}$ such that for any $\kappa \in \mathbb{N}$ we have that

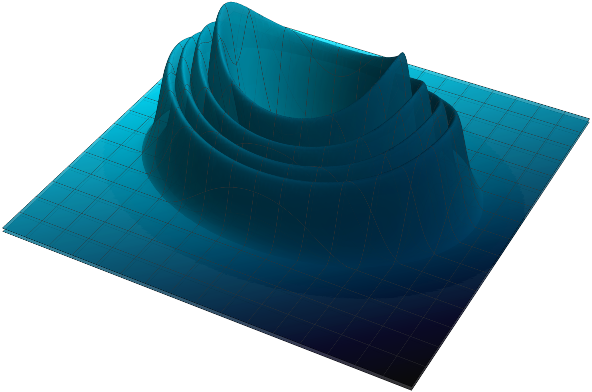
$$P_k(z) = n^{\frac{1}{4}} \sqrt{\phi_\tau'(z)}[\phi_\tau(z)]^k \mathrm{e}^{n\mathcal{V}_\tau(z)} \big(\mathcal{B}_{0,\tau}(z) + \ldots + n^{-\kappa}\mathcal{B}_{\kappa,\tau}(z) + \mathrm{O}(n^{-\kappa-1})\big)$$

as $n \to \infty$. The expansion holds uniformly for $\mathrm{d}(z, \mathcal{S}^c) \le C\sqrt{n^{-1}\log n}$.

Dominant behaviour $|P_k(z)|^2 \mathrm{e}^{-2nQ(z)} \sim \mathrm{e}^{-2n(Q-\breve{Q}_\tau)(z)}$ – a Gaussian ridge around the boundary $\partial\mathcal{S}_\tau$.



Plot of $|P_k(z)|^2 e^{-2nQ(z)}$, with $(k, n) = (25, 20)$, $Q(z) = \frac{1}{2}|z|^2 + \frac{1}{5}\mathrm{Re}(z^2)$.



Plot of $|P_k(z)|^2 \mathrm{e}^{-2nQ}$ for several k.

- Ansatz for $P_n = P_{n,n}$:

$$F_n(z) = n^{\frac{1}{4}}\phi'(z)[\phi(z)]^n \mathrm{e}^{n\mathcal{V}(z)} f_n(\phi(z)),$$

where f_n is an asymptotic expansion of bounded holomorphic functions

$$f_n(z) = B_0(z) + n^{-1}B_1(z) + n^{-2}B_2(z) + \dots, \qquad z \in \mathbb{D}_\mathrm{e}(0,\rho).$$

- Use conformal mapping to pull back to more radial setting:

$$\int_{\mathbb{C}\setminus\mathcal{K}} F_n\overline{Q}\mathrm{e}^{-2nQ}\mathrm{dA} = \int_{\mathbb{D}_\mathrm{e}(0,\rho)} f_n\overline{g_Q}\,\mathrm{e}^{-2nR}\mathrm{dA}$$

where

$$R(z) = (Q - \log|\phi| - \mathrm{Re}\,\mathcal{V})(\varphi(z))$$

is of quadratic growth around $\mathbb{T}$. Here, $g_Q = \frac{Q}{\phi'\,\phi^n \mathrm{e}^{n\mathcal{V}}} \circ \varphi$.

- Foliation $\gamma_{t,n}$, normal velocity ν (scalar field). Then we have

$$\int f\bar{g}\,\mathrm{e}^{-2nR}\mathrm{dA} = \int_{|t|\le\delta_n}\Big(\int_{\gamma_{t,n}} f\bar{g}\,\mathrm{e}^{-2nR}\nu\mathrm{d}\sigma\Big)\mathrm{d}t + \mathrm{O}(n^{-\infty}\|g\|)$$

We aim to construct f_n and $(\gamma_{t,n})$ simultaneously such that

$$\int_{\gamma_t} f\bar{g}\,\mathrm{e}^{-2nR}\nu\mathrm{d}\sigma \sim 0.$$

- In terms of conformal mappings $\varphi_{t,n}:\mathbb{D}\to\mathrm{ext}(\gamma_{t,n})$ we have

$$\begin{aligned}&\int_{\gamma_t} f\bar{g}\,\mathrm{e}^{-2nR}\nu\mathrm{d}\sigma\\ &= \int_{\mathbb{T}}\frac{\bar{g}}{\bar{f}_n}(\varphi_{t,n})\,|f_n(\varphi_{t,n})|^2\,\mathrm{e}^{-2nR(\varphi_{t,n})}\mathrm{Re}\left\{\bar{\zeta}\,\partial_t\varphi_{t,n}(\zeta)\overline{\varphi_{t,n}'(\zeta)}\right\}\mathrm{d}\sigma(\zeta)\,\mathrm{d}t.\end{aligned}$$

Lemma (Existence of orthogonal foliation)

There exist bounded holomorphic functions B_j and $\hat{\varphi}_{j,k}$, such that

$$\varphi_{t,n}(z) \sim \sum_{j,k\geq 0} n^{-j} t^k \hat{\varphi}_{j,k}(z), \qquad |t| \leq n^{-\frac{1}{2}} \log n$$

defines a family of conformal mappings which foliates the $n^{-\frac{1}{2}} \log m$-band around $\mathbb{T}$, and such that with $f = \sum_{j\leq\kappa} n^{-j} B_j$ we have

$$|f(\varphi_{t,n}(\zeta))|^2 \mathrm{e}^{-2nR(\varphi_{t,n}(\zeta))} \Re\{\bar{\zeta}\, \dot{\varphi}_{t,n}(\zeta) \overline{\varphi'_{t,n}(\zeta)}\} = \mathrm{e}^{-nt^2} \big(1 + \mathrm{O}(n^{-\kappa-1})\big).$$

- In other words, there is a substitute for the polar coordinates, up to $\mathrm{O}(n^{-\kappa})$-error! The choice of f_n determined by the foliation.

A droplet Ω_0 of incompressible fluid between two plates. in a medium with permeability $\kappa = \omega^{-1}$. Fluid is injected at $z_0 \in \Omega_0$, producing a *Hele-Shaw* flow $(\Omega_t)_t$.

- Normalized conformal mappings $\varphi_t : \mathbb{D}_\mathrm{e} \to \Omega_t$, with $\infty \mapsto z_0$ satisfies

$$\omega(\varphi_t(\zeta))\Re\{\bar{\zeta}\,\partial_t\varphi_t(\zeta)\overline{\varphi_t'(\zeta)}\} = 1, \qquad \zeta \in \mathbb{T}$$

- The flow $(\varphi_{t,n})_t$ is a Hele-Shaw flow (approx.), with weight

$$w_n(z) = |\,f_n(z)\,|^2 \mathrm{e}^{-2nR(z)},$$

  and injection at infinity.

- *Reformulation of Theorem*
  Once can choose f_n, such that the flow exists to prescribed accuracy, long enough to cover δ_n-neighbourhood of $\mathbb{T}$.

How close to a polynomial is F_n?

- Analytic in $\mathbb{C} \setminus \mathcal{K}$ for some compact. Cut-off function χ vanishes on $\mathcal{K}$.
- Correction $\chi F - v$, where $\bar\partial v = \bar\partial(\chi F) = F \bar\partial \chi$.
- Hörmander's estimate: smooth strictly s.h. weight V on $\mathbb{C}$: The equation $\bar\partial u = f$ has a solution u which satisfies

$$\int |u|^2 \mathrm{e}^{-V} \mathrm{dA} \leq \int |f|^2 \frac{\mathrm{e}^{-V}}{\Delta V} \mathrm{dA}.$$

- Now $\chi F - v$ is holomorphic, and v is small. Growth might be more than polynomial!

- Apply estimate to $V = 2n\widehat{Q}$. Then

$$\int |v|^2 \mathrm{e}^{-2nQ} \leq \int |v|^2 \mathrm{e}^{-2n\widehat{Q}} \leq \frac{1}{2n}\int_{\mathcal{S}} |\bar\partial\chi|^2 \, |F|^2 \frac{\mathrm{e}^{-2nQ}}{\Delta Q}$$

  and with our ansatz, $|P|^2\mathrm{e}^{-2nQ}$ is very small away from $\partial\mathcal{S}$.

- *The key:* the support of $\nabla\chi$ cannot intersect $\mathcal{S}^c$, if we want to localize in polynomial space.

- Bergman kernel asymptotics – boundary localization not possible.

Thank you for listening!