Overlap distribution in the Spherical Sherrington-Kirkpatrick model

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The model

2-spin Spherical Sherrington-Kirkpatrick model with zero magnetic field

$$Z_N(\beta) = \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} e^{-\beta H_N(\sigma)} \, \mathrm{d}\omega_N(\sigma),$$

Spins:

$$\sigma \in \mathbb{S}^{N-1} := \{\sigma \in \mathbb{R}^N, |\sigma| = \sqrt{N}\}.$$

Hamiltonian:

$$H_N(\sigma) = - \sum_{1 \leq i \neq j \leq N} \frac{1}{\sqrt{2N}} g_{ij} \sigma_i \sigma_j.$$

Disorder: g_{ij}, $1 \leq i \neq j \leq N$ i.i.d. Gaussian.

Interpretation: "soft maximization"

"Low temperature" limit $\beta \to \infty$ of free energy density is the top eigenvalue of GOE minus diagonal:

$$\frac{1}{\beta N} \log Z_N(\beta) \to \max_{|\sigma|=1} \langle \sigma, M\sigma \rangle,$$

$$M_{ij} = g_{ij}, \quad M_{ii} = 0.$$

SSK model partition function is a positive temperature version of the top eigenvalue.

Model features

$$H_N(\sigma) = -\sum_{1\le i\neq j\le N} \frac{1}{\sqrt{2N}} g_{ij}\sigma_i\sigma_j.$$

Mean field model: all spins interact, no geometry, like the Curie-Weiss model (easier than lattice)

Disordered model: interactions are random, with no sign (much harder than ferromagnetic models)

History

Introduced by Kosterlitz, Thouless and Jones (1976) after the original spin glass model with Ising spins $\sigma_i \in \{\pm 1\}$ by Sherrington and Kirkpatrick (1975).

More explicit computations are possible than for Sherrington-Kirkpatrick (SK) model. KTJ compute the limit of the partition function $\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$.

The corresponding computation for SK is much harder. Sherrington-Kirkpatrick (1975) for high-temperature phase (β small). For all β Parisi (2003), Guerra, Talagrand (2008), Panchenko (2011).

p-spin models

Both SK ($\sigma \in \{\pm 1\}$) and SSK ($\sigma \in \mathbb{S}^{N-1}$) can be generalized to p-spin models:

$$H_N(\sigma) = \frac{1}{\sqrt{q} N^{(p-1)/2}} \sum_{1 \le i_1 < \ldots < i_p \le N} g_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}.$$

Crisanti-Sommers (1992) give a variational formula for the free energy in this case. Proved rigorously by Talagrand (2006) and Chen (2013).

Connection to spiked models

Spiked Wigner model:

$$Y = M + \sqrt{\frac{\lambda}{N}} x^*(x^*)^t.$$

M: traceless GOE
x^*: $N \times 1$ vector from a distribution $P(\mathrm{d}x)$.

Can the spike be detected?

Density of Y:

$$P_\lambda(\mathrm{d}Y) = Z^{-1} \int \exp\Big(-\frac{1}{2} \sum_{i<j} (Y_{ij} - \sqrt{\frac{\lambda}{N}} x_i x_j)^2 \Big) P(\mathrm{d}x)$$

Connection to spiked models

Likelihood ratio:

$$L(\lambda) = \frac{P_\lambda}{P_0} = \int \exp\big(\sqrt{\frac{\lambda}{N}} \sum_{i<j} Y_{ij} x_i x_j - \lambda \sum_{i<j} x_i^2 x_j^2\big) P(\mathrm{d}x).$$

If x is uniform on the sphere, this is the partition function of the SSK model.

Likelihood ratio test: compare $L(\lambda)$ to threshold and reject $\lambda = 0$ if $L(\lambda) > c$.

SSK phase transition corresponds to detectability of spike: if signal is too small (high temperature), likelihood ratio test fails. (Onatski, Moreira, Hamlin (2014), Johnstone, Onatski (2015)).

SSK

Kosterlitz-Thouless-Jones:

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) := F(\beta) = \begin{cases} \frac{\beta^2}{2} & \beta < 1 \\ 2(\frac{\beta}{2} - \frac{3}{2} - \log 2) & \beta > 1. \end{cases} \qquad (1)$$

Phase transition at $\beta = 1$.

For $\sigma \in \{\pm 1\}$, limit is given by an infinite dimensional variational formula (Parisi functional).

Spiked model

For the spiked model with i.i.d. prior x^*, Lelarge and Miolane (2016) prove:

$$\frac{1}{N} E_\lambda[\log L(\lambda)] \to \phi(\lambda) = \sup_{q \geq 0} F(\lambda, q),$$

where

$$F(\lambda, q) = \psi(\lambda q) - \lambda \frac{q^2}{4},$$

$$\psi(r) = E[\log \int \exp(\sqrt{r} zx + rxx^* - rxx^* - \frac{r}{2} x^2) \, \mathrm{d}P(\mathrm{d}x)].$$

Hre $z \sim N(0,1)$ and $x^* \sim P(\mathrm{d}x)$.

Fluctuations: high temperature

Aizenman-Lebowitz-Ruelle (1987): central limit theorem for the partition function for Ising spins $\sigma \in \pm 1$ in the low temperature phase:

$$\log Z_N - N(\log 2 + \frac{\beta^2}{2}) \to \mathcal{N}\big((\frac{1}{4}(\log(1-\beta^2)+\beta^2), -\frac{1}{2}(\log(1-\beta^2)+\beta^2)\big).$$

Stein's method approach: Talagrand (2011). May be applied to spherical spin glasses.

For general spherial p-spin model, Chen and Sen prove that the fluctuations of the partition function are Gaussian in the high temperature phase.

Baik-Lee results

Baik and Lee (2015) find the fluctuations of the spherical model for any (off-critical) temperature:

Theorem

For the SSK model, when $\beta < 1$

$$\log Z_N - NF(\beta) \to \mathcal{N}\big((\frac{1}{4}(\log(1-\beta^2)+\beta^2), -\frac{1}{2}(\log(1-\beta^2)+\beta^2)\big)$$

When $\beta > 1$,

$$\frac{N^{2/3}}{\beta-1}(\frac{1}{N}\log Z_N - F(\beta)) \to \mathrm{TW}_{\mathrm{GOE}}.$$

In contrast with this, for $\sigma \in \{\pm 1\}$ (Ising spins), we have almost no information about the fluctuations in the low temperature phase.

Chatterjee (2009) shows

$$\mathrm{Var}(\log Z_N) \leq \frac{C(\beta)N}{\log N}$$

for any temperature for the SK model with Ising spins.

For p-spin models with $p \geq 3$, Subag (2016) shows that the free energy fluctuations are $O(1)$ (tight) in the low temperature phase.

Overlaps

Phase transition occurs also in the geometry of spins.

Consider $\sigma^{(1)}, \sigma^{(2)}$ sampled from Gibbs measure

$$\mu_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_N(\beta)}.$$

$\sigma^{(i)}$ are "near minimizers" of $H(\sigma)$, called replicas.

Overlap: inner product between replicas $\sigma^{(1)}$, $\sigma^{(2)}$.

$$\begin{aligned} R_{1,2} &= \frac{1}{N}\sigma^{(1)} \cdot \sigma^{(2)} \\ &= \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{(1)} \sigma_i^{(2)}. \end{aligned}$$

Overlap in spiked model

The overlap $R_{1,2}$ in the spiked model corresponds to the overlap of a sample x from the posterior distribution

$$P(\mathrm{d}x \mid M),$$

with the spike x^* (Nishimori identity).

$$R_{1,2} \stackrel{d}{=} \frac{1}{N} x \cdot x^*.$$

Overlap asymptotics

Panchenko and Talagrand (2003) proved

$$E[\langle R_{1,2}^2 \rangle] \to \begin{cases} 0, & \beta < 1 \\ q^2, & \beta > 1 \end{cases},$$

for

$$q = \frac{\beta - 1}{\beta}.$$

They showed that overlaps concentrate around two values $\pm q$:

$$E[\langle |R_{12}^2 - q^2| \geq \epsilon \rangle] \to 0.$$

$\langle \cdot \rangle$: Gibbs average.

Spiked model

For the spiked model with i.i.d. prior x^* and x sampled from $P(\mathrm{d}x \mid M)$, Lelarge-Miolane show that

$$\frac{1}{N}(x^* x^t)^2 \to q^*(\lambda),$$

where q^* is the maximizer in the limiting free energy density.

"Reconstruction threshold" has the equivalent definitions:

$$\lambda_c = \sup\{\lambda > 0 : q^*(\lambda) = 0\}$$
$$= \sup\{\lambda > 0 : \phi(\lambda) = 0\}.$$

Observation (El Alaoui, Krzkala, Jordan):

$$\lambda_c (E[x^2])^2 \le 1.$$

$\lambda = 1/(E[x^2])^2$ is the spectral threshold where $\lambda_1(Y)$ leaves the bulk and the first eigenvector is uniformative about x^* (Péché, Capitaine, Feral, 200x, Benaych-Georges and Nadakuditi, 2011).

For some distributions like sparse Rademacher, the inequality $\lambda_c < 1/E[x^2]^2$ is strict. There is a conjectural "hardness" phase transition between the spectral threshold and λ_c.

Results: high temperature

We describe the fluctuations of the overlaps.

Theorem (Nguyen-S., 2018)

For $\beta = \beta_N = 1 - cN^{-1/3+\tau}$, $\tau > 0$,

$$\langle e^{\sqrt{N(1-\beta_N^2}tR_{12}} \rangle_{\beta,N} = e^{t^2} + o(1)$$

with very high probability.

The overlap remains Gaussian close to the critical temperature (but no closer than $O(N^{-1/3+})$.

Talagrand: moments $E[\langle(\sqrt{N(1-\beta_N^2}R_{12})^p\rangle]$ are Gaussian up to $\beta_N \ll 1 - N^{-1/3}$, not Gaussian for $\beta_N = 1 - cN^{-1/3}$.

Results: low temperature

With Benjamin Landon: describe the annealed fluctuations of $R_{1,2}$ for $\beta > 1$ fixed.

We give expansions up to $o(N^{-2/3})$, and identify the limiting distribution of $|R_{1,2}|$. ($\langle R_{12} \rangle = 0$ by symmetry of the Gibbs measure.)

Results: low temperature

Theorem (Landon, S. 2019)

Asymptotically with probability 1:

$$\begin{aligned}\langle R_{12}^2 \rangle = &\left(\frac{1-\beta}{\beta}\right)^2 + 2\frac{\beta-1}{\beta^2}\cdot\left(\frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j(M)-\lambda_1(M)}+1\right)\\ &-\frac{1}{N\beta^2}\left(\frac{1}{N}\sum_{j=2}^{N}\frac{1}{(\lambda_j(M)-\lambda_1(M))^2}\right)\\ &+\frac{1}{\beta^2}\left(\frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j(M)-\lambda_1(M)}+1\right)^2\\ &+O(N^{-1+}).\end{aligned}$$

Results: low temperature

Theorem (Landon, S. 2019)

Asymptotically with probability 1:

$$\langle |R_{12}| \rangle = q + \frac{1}{\beta} \cdot \left(\frac{1}{N} \sum_{j=2}^{N} \frac{1}{\lambda_j(M) - \lambda_1(M)} + 1 \right) + O(N^{-1+}).$$

Similar formulas were found using physics methods by Baik, Le Doussal and Wu. They also have precise predictions for the case with non-zero magnetic field.

Orders of magnitude

By level repulsion

$$\lambda_1 - \lambda_2 \geq N^{-2/3+\epsilon},$$

and rigidity, we have

$$\frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j(M)-\lambda_1(M)}+1=O(N^{-1/3+}),$$

and

$$\frac{1}{N^2}\sum_{j=2}^{N}\frac{1}{(\lambda_j-\lambda_1)^2}=O(N^{-2/3+}).$$

From the first term, we might expect tightness of $N^{1/3}(\langle R_{12}^2\rangle - q^2)$, $N^{1/3}(\langle |R_{12}|\rangle - q)$.

"Renormalized" Airy series

The term

$$\frac{1}{N}\tilde{m}_N(\lambda_1) = \frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j(M)-\lambda_1(M)} - 1 \sim N^{-1/3}$$

will be approximated by the Airy process. By rigidity:

$$\lambda_j - \lambda_1 \sim (j/N)^{2/3}.$$

Let $\chi_1 \geq \chi_2 \geq \dots$ be the points of an Airy GOE process. Define:

$$\Xi := \lim_{n\to\infty}\left(\sum_{j=2}^{n}\frac{1}{\chi_j-\chi_1} - \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}}\frac{1}{\pi\sqrt{x}}dx\right).$$

Landon-S. (2019): The limit exists almost surely.

Distributional limits

Theorem (Landon, S., 2019)

The following limits hold in distribution:

$$N^{1/3}\big(\langle R_{12}^2\rangle - q^2\big) \to 2\left(\frac{\beta-1}{\beta^2}\right)\Xi,$$

and

$$N^{1/3}\big(\langle|R_{12}|\rangle - \frac{1-\beta}{\beta}\big) \to \frac{1}{\beta}\Xi.$$

Once the existence of Ξ is established, this means we need to show

$$N^{1/3}\left(\frac{1}{N}\sum_{j=2}^{N}\frac{1}{\lambda_j(M)-\lambda_1(M)} - 1\right)$$

has the same limit.

Connecting spin glass to RMT: contour integral

1. Rotational invariance:

$$Z_N = \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} e^{\beta \sum_{i=1}^{N} \lambda_i \sigma_i^2} \, \mathrm{d}\omega_N(\sigma),$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$$

are the eigenvalues of (traceless) GOE. Unlike for other spin distributions, the partition function depends only on the spectrum.

2. Formally replacing $\sigma_i \in \mathbb{S}^{N-1}$ by Gaussian distributed random variables, we can compute Z_N explicitly. The radial part of the Gaussian vector has spherical distribution, and in large dimensions, the radius is concentrated around $\sqrt{N}$.

Using this idea made precise using Laplace inversion, KTJ, and independently Baik and Lee find the representation:

$$Z_N(\beta) = \frac{\Gamma(N/2)2^{N/2-1}}{2\pi i(N\beta)^{N/2-1}} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}G(z)}\,\mathrm{d}z,$$

$$G(z) = \beta z - \frac{1}{N}\sum_{j=1}^{N} \log(z-\lambda_i).$$

This is ready for asymptotic analysis, provided we can control the random quantity $G(z)$.

Contour integral: history

The idea to replace is classic, rediscovered many times. In statistical mechanics, it appeared in work of Berlin-Kac who proposed the Gaussian spin version of the $O(n)$ model and used in to solve the "spherical Curie-Weiss model".

More recently, Duminil-Copin, Goswami, Severo, Yadin (2018) find a new proof of the percolation phase transition using the Gaussian free field, with a similar idea: write the spins of Ising model as the signs of a Gaussian field, then integrate out the modulus.

Random matrix theory

Key tool to analyze the contour (Baik Lee, Baik-Lee-Wu, Nguyen-S.): eigenvalue rigidity.

Erdös-Yau-Yin (2010), with many later versions. If γ_i are the quantiles of the semicircle law:

$$|\lambda_i - \gamma_i| \le N^{-2/3+\delta} \min(i, (N+1-i))^{-1/3}, \quad i = 1, \dots, N,$$

with very high probability.

For our work, we will also need level repulsion, and distributional properties of the edge eigenvalues (Airy process).

Saddle point heuristics

Saddle point equation:

$$G'(\gamma) = \beta - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} = 0.$$

For Wigner matrix

$$x \mapsto \frac{1}{N}\sum_{i=1}^{N}\frac{1}{x-\lambda_i}$$

is decreasing on (λ_1, ∞).

$$E[\sum_{j=1}\frac{1}{\lambda_1-\lambda_j}] = \frac{1}{2}E[\lambda_1] = 1 + O(N^{-2/3+}).$$

The saddle point approaches λ_1 as β approaches 1.

When $\beta < 1$ fixed, the effect of the other eigenvalues on λ_1 is averaged, so we can replace

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} = \beta \rightsquigarrow \int \frac{\rho_{sc}(x)}{z-x}\mathrm{d}x = \beta.$$

The saddle point γ is close to the solution of a deterministic equation.

When $\beta > 1$, main contribution comes from a small neighborhood of λ_1. The saddle point γ is close to λ_1.

Baik-Lee:

$$N(\gamma - \lambda_1) = O(N^{\epsilon})$$

using rigidity.

Contours for replica quantities

Double contour integral representation for the overlap expectation:

$$\langle R_{12}^2 \rangle = \left(\frac{1}{4}\right) \frac{\int\int e^{N(G(z)+G(w))/2}\left(\sum_{i=1}^{N} \frac{1}{\beta^2 N^2 (z-\lambda_i(M))(w-\lambda_i(M))}\right) \mathrm{d}z\mathrm{d}w}{\left(\int e^{NG(z)/2}\mathrm{d}z\right)^2}.$$

Similar contour integrals integrals are available for $\langle R_{12}^4 \rangle$, $\langle e^{tR_{12}} \rangle$, etc.

Order-of-magnitude heuristics

When $\beta = 1 - N^{-1/3+}$, one can show that $\gamma - \lambda_1 = O(N^{-2/3+})$: local semicircle law is still useful at this distance from the spectrum.

When $\beta > 1$, we replace the Baik-Lee bound $\gamma - \lambda_1 = O(N^{-1+\epsilon})$ by the use of an approximate saddle point contour taking into account only the effect of λ_1:

$$G(z) = \beta z - \frac{1}{N}\sum_{j=1}^{N} \log(z - \lambda_j) \rightsquigarrow (\beta - 1)z - \frac{1}{N}\log(z - \lambda_1).$$

This leads to an approximate saddle point

$$\gamma' = \lambda_1 + \frac{1}{\beta - 1}\frac{1}{N}.$$

Convergence of "Airy" term

$$\frac{1}{N^{2/3}}\sum_{j=1}^{N}\frac{1}{\lambda_1-\lambda_j}-N^{1/3}$$

$$=\frac{1}{N^{2/3}}\sum_{j=1}^{k}\frac{1}{\lambda_1-\lambda_j}-\int_0^{\left(\frac{3\pi k}{2}\right)^{2/3}}\frac{1}{\pi\sqrt{x}}\,\mathrm{d}x \qquad \text{(convergence to Airy)}$$

$$+\frac{1}{N^{2/3}}\sum_{j=k+1}^{N^\delta}\frac{1}{\lambda_1-\lambda_j}-\int_{\left(\frac{3\pi k}{2}\right)^{2/3}}^{\left(\frac{3\pi N^\delta}{2}\right)^{2/3}}\frac{1}{\pi\sqrt{x}}\,\mathrm{d}x \qquad (*)$$

$$+\frac{1}{N^{2/3}}\sum_{N^\delta}^{N}\frac{1}{\lambda_1-\lambda_j}-N^{1/3}\int_{-2}^{\gamma_{N^\delta}}\frac{\rho_{sc}(x)}{2-x}\,\mathrm{d}x \qquad \text{(rigidity)}$$

$$+\int_0^{\left(\frac{3\pi N^\delta}{2}\right)^{2/3}}\frac{1}{\pi\sqrt{x}}\,\mathrm{d}x-N^{1/3}\int_{-2}^{\gamma_{N^\delta}}\frac{\rho_{sc}(x)}{2-x}\,\mathrm{d}x, \qquad \text{(explicit)}$$

Eigenvalue locations at the edge

Need precise estimates for the number of GOE, Airy_1 eienvalues close to the edge.

We show

$$E[\big|N^{2/3}(\lambda_k - 2) + \big(\frac{3\pi k}{2}\big)^{2/3}\big| \leq C\frac{(\log k)^2}{k^{1/3}}]$$

for $k \leq N^{2/5}$.

The proof uses precise estimates for the counting function of GOE at the egde and its variance, which we could not locate in the literature.

Eigenvalue locations at the edge

For GUE, Gustavsson (2005):

$$|E[\#\{i : \mu_i \geq 2 - sN^{-2/3}\}] - \frac{2}{3\pi}s^{3/2}| = O(1)$$
$$|\mathrm{Var}(\#\{i : \mu_i \geq 2 - sN^{-2/3}\}) - \frac{3}{4\pi^2}\log s| = O(\log\log s).$$

For Airy$_2$, Soshnikov shows

$$|E[\#\{i : \chi_i \leq T\}] - \frac{2}{3\pi}T^{3/2}| = O(1)$$
$$|\mathrm{Var}(\#\{i : \chi_i \leq T\}) - \frac{3}{4\pi^2}\log s| = O(1).$$

From GUE to GOE

Theorem (Forrester-Rains)

Let GOE_n *and* GUE_n *denote the set formed by the union of the eigenvalues of the GOE and GUE, respectively. Then,*

$$\mathrm{GUE}_n \stackrel{d}{=} \mathrm{Even}\left(\mathrm{GOE}_n \cup \mathrm{GOE}_{n+1}\right)$$

where the RHS is the set formed by the second largest, fourth largest, sixth largest, etc. elements of $\mathrm{GOE}_n \cup \mathrm{GOE}_{n+1}$.

Outlook

1. Non-zero magnetic field case is work in progress. See conjectures in J. Baik's talk.
2. Any information closer to $\beta = 1$.
3. *beta* ≥ 1 fluctuations for SK with Ising spins.

Thank you for listening!