Limit theorems for determinantal point processes

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[1] Joint work with **Makoto Katori** (Chuo University)

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Content of this talk

1. Brief review on determinantal point processes (DPPs)
2. L^1-limit for generalized accumulated spectrograms
3. Circular Unitary Ensemble (CUE)
4. Two DPPs on the 2-dimensional sphere and limit theorems
5. An extension to the d-dimensional sphere
6. An extension to compact Riemannian manifolds

Reproducing kernel Hilbert space (RKHS)

- Let S be a set and H a Hilbert space of $\mathbb{C}$-valued functions on S.
- H is said to be a reproducing kernel Hilbert space (RKHS) if, for every $y \in S$, the point evaluation map $L_y : H \to \mathbb{C}$

$$L_y(f) = f(y) \quad (f \in H)$$

  is bounded (continuous).
- Since L_y is a bounded linear functional, by Riesz's theorem, we have $K_y \in H$ such that

$$L_y(f) = (f, K_y)_H.$$

- $K(x, y) := K_y(x)$ is called a reproducing kernel in the sense that

$$f(y) = (f, K(\cdot, y))_H \quad \forall f \in H, \ \forall y \in S.$$

Theorem (Moore-Aronszain)

Let K be a Hermitian positive definite kernel $K : S \times S \to \mathbb{C}$. Then, there exists a unique Hilbert space H_K of $\mathbb{C}$-valued functions on S for which K is a reproducing kernel.

Example: Paley-Wiener space

- Band limited functions:

$$PW_a = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-a, a]\},$$

where $\widehat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$.

$$|f(x)| \leq \frac{1}{2\pi} \left| \int_{-a}^{a} \widehat{f}(\omega) e^{i\omega x} d\omega \right| \leq \sqrt{\frac{a}{\pi}} \|f\|.$$

- Reproducing kernel:

$$K_a(x, y) = \frac{\sin a(x-y)}{\pi(x-y)} \to \delta_y(x) \quad (a \to \infty).$$

- RKHS (PW_a, K_a) is called the Paley-Wiener space.

Determinantal point processes (DPPs)

We recall determinantal point processes (DPPs) on S.

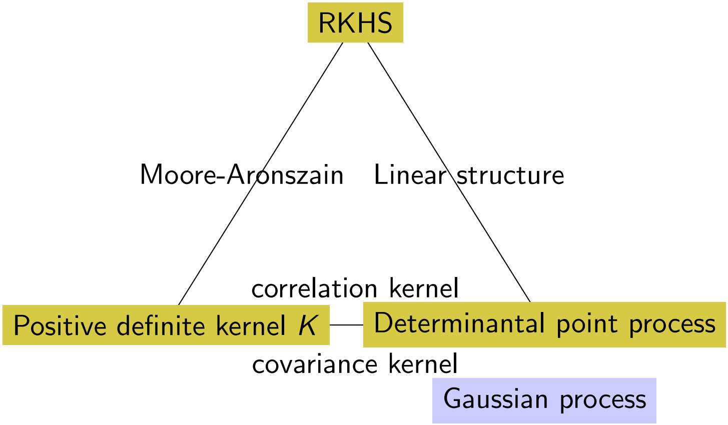
- S: a base space (locally compact Polish space)
- $\lambda(ds)$: Radon measure on S
- $\mathrm{Conf}(S) = \{\xi = \sum_i \delta_{x_i} : x_i \in S, \xi(K) < \infty$ for all bounded set $K\}$: the set of $\mathbb{Z}_{\geq 0}$-valued Radon measures
- $H_K \subset L^2(S, \lambda)$: reproducing kernel Hilbert space (RKHS) with kernel $K(\cdot, \cdot) : S \times S \to \mathbb{C}$.

Theorem (Determinantal point process with (K, λ) or H_K)

There exists a point process $\xi = \xi(\omega)$ on S, i.e., a $\mathrm{Conf}(S)$-valued random variable such that the nth correlation function w.r.t. $\lambda^{\otimes n}$ is given by

$$\rho_n(s_1, \ldots, s_n) = \det(K(s_i, s_j))_{i,j=1}^n.$$

DPP and Gaussian process – RKHS



Determinantal point processes (DPPs) II

- Example. (Paley-Wiener Space): $S = \mathbb{R}$, $\lambda(dx) = dx$ and

$$K(x, y) = \frac{\sin a(x-y)}{x-y}.$$

The RHKS H_K is PW_a, then the corresponding DPP is the limiting CUE (also GUE) eigenvalues process.
Later we will discuss a generalization of this procss.

- Example (Bargmann-Fock space): $S = \mathbb{C}$ and $\lambda(dz) = \pi^{-1} e^{-|z|^2} dz$ and

$$K(z, w) = e^{z\overline{w}}.$$

The RKHS H_K is the Bargmann-Fock space, i.e.,

$$H_K := \{f \in L^2(\mathbb{C}, \lambda) : f \text{ is entire}\}$$

The DPP in this case is the Ginibre point process.

Determinantal point processes (DPPs) III

- Number of points: If K is of rank N, i.e., $\dim H_K$ is N, then the number of points is N a.s.
- Density of points w.r.t. $\lambda(dx)$ and negative correlation:

$$\rho_1(x) = K(x,x)$$
$$\rho_2(x,y) = K(x,x)K(y,y) - |K(x,y)|^2 \leq \rho_1(x)\rho_1(y)$$

- Gauge invariance: For $u : S \to U(1)$, a gauge transformation

$$K(s,t) \mapsto \tilde{K}(s,t) := u(s)K(s,t)\overline{u(t)}$$

  does not change the law of DPP.
- Scaling property: When $S = \mathbb{R}^d$ and $\lambda(dx) = dx$, for a configuration $\xi = \sum_i \delta_{x_i}$, we define

$$S_c(\xi) = \sum_i \delta_{cx_i}.$$

  If $\xi(\omega)$ is DPP with K, then $S_c(\xi(\omega))$ is also DPP with

$$K_c(x,y) = c^{-d}K(c^{-1}x, c^{-1}y)$$

DPP associated with partial isometry

- We say that $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ is partial isometry if
$$\|\mathcal{W}f\|_{L^2(S_2,\lambda_2)} = \|f\|_{L^2(S_1,\lambda_1)} \quad \text{for all } f \in (\ker \mathcal{W})^\perp$$
- Let $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ and its dual $\mathcal{W}^* : L^2(S_2, \lambda_2) \to L^2(S_1, \lambda_1)$ be partial isometries, or equivalently,
$$\mathcal{K}_1 = \mathcal{W}^*\mathcal{W}, \quad \mathcal{K}_2 := \mathcal{W}\mathcal{W}^* \text{ (orthogonal projections)}$$
- Suppose that both $\mathcal{K}_1$ and $\mathcal{K}_2$ are of locally trace class, i.e.,
$$\mathcal{P}_{\Lambda_1}\mathcal{K}_1\mathcal{P}_{\Lambda_1}, \ \mathcal{P}_{\Lambda_2}\mathcal{K}_2\mathcal{P}_{\Lambda_2} \text{ are of trace class}$$
for any bounded set $\Lambda_i \subset S_i$ $(i = 1, 2)$.
- Then $\mathcal{K}_1$ and $\mathcal{K}_2$ admit kernel $K_1(x, x')$ and $K_2(y, y')$, which are reproducing kernels of $(\ker \mathcal{W})^\perp$ and $(\ker \mathcal{W}^*)^\perp$, respectively.
- Let Ξ_i $(i = 1, 2)$ be the DPPs associated with (K_i, λ_i) $(i = 1, 2)$, respectively.

M.Katori-T.Shirai, Partial Isometry, Duality, and Determinantal Point Processes, available at `https://arxiv.org/abs/1903.04945`

(1) Orthogonal polynomial ensemble.

$$\mathcal{W} : L^2(\mathbb{R}, \lambda) \to \ell^2(\mathbb{Z}_{\geq 0})$$

defined by the kernel

$$(\mathcal{W}f)(n) = \int_R \overline{\varphi_n(x)} f(x) \lambda(dx)$$

where $\{\varphi_n(x)\}_{n \in \mathbb{Z}_{\geq 0}}$ are orthonormal polynomials for $L^2(\mathbb{R}, \lambda)$.

$$K_1^{\{0,1,\dots,N-1\}}(x, y) = \sum_{j=0}^{N-1} \varphi_j(x)\overline{\varphi_j(y)} \Longrightarrow \text{DPP } \Xi_1 \text{ on } \mathbb{R}.$$

$$K_2^{[r,\infty)}(n, m) = \int_r^\infty \overline{\varphi_n(x)} \varphi_m(x) \lambda(dx) \Longrightarrow \text{DPP } \Xi_2 \text{ on } \mathbb{Z}_{\geq 0}.$$

Duality relation: for any $m = 0, 1, \dots,$

$$\mathbb{P}\Big(\Xi_1([r, \infty)) = m\Big) = \mathbb{P}\Big(\Xi_2(\{0, 1, \dots, N-1\}) = m\Big)$$

(2) Weyl-Heisenberg ensemble (Abreu-Pereira-Romero-Torquato('17)):

- $\mathcal{W} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is the short-time Fourier transform defined by

$$\mathcal{W}f(z) = \int_{\mathbb{R}^d} f(t)g(t-x)e^{2\pi i\xi t}dt, \quad z := (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

  where g is a window function such that $\|g\|_{L^2(\mathbb{R}^d)} = 1$.
- It is easy to see that

$$\mathcal{W}^*\mathcal{W} = I_{L^2(\mathbb{R}^d)}, \quad \mathcal{K} = \mathcal{W}\mathcal{W}^*(\text{orthogonal proj. on } L^2(\mathbb{R}^d \times \mathbb{R}^d)).$$

- DPP on $\mathbb{R}^d \times \mathbb{R}^d$ associated with $\mathcal{K}$ is called Weyl-Heisenberg ensemble.

Example: When $d = 1$, $g(t) = 2^{1/4}e^{-\pi t^2}$, by identifying $\mathbb{R} \times \mathbb{R}$ with $\mathbb{C}$, we have

$$K_2(z,w) = \frac{e^{i\pi \operatorname{Re} z \operatorname{Im} z}}{e^{i\pi \operatorname{Re} w \operatorname{Im} w}} e^{\pi\{z\overline{w} - \frac{1}{2}(|z|^2+|w|^2)\}} \quad (z,w \in \mathbb{C}).$$

The corresponding Weyl-Heisenberg ensemble is the Ginibre point process.

DPP associated with partial isometry

We focus on a generalized framework of Weyl-Heisenberg ensembles.

- Let $\mathcal{W} : L^2(S_1, \lambda_1) \to L^2(S_2, \lambda_2)$ be an isometry and its dual $\mathcal{W}^* : L^2(S_2, \lambda_2) \to L^2(S_1, \lambda_1)$ be a partial isometry, i.e.,

$$\mathcal{W}^*\mathcal{W} = I_{L^2(S_1,\lambda_1)},$$
$$\mathcal{W}\mathcal{W}^* =: \mathcal{K}_2(\text{orthogonal projection on } (\ker \mathcal{W}^*)^\perp$$

- Suppose that $\mathcal{K}_2$ is of locally trace class, i.e., $\mathcal{K}_2$ admits a kernel $K_2(y, y')$.
- Let Ξ_2 the DPP on S_2 associated with (K_2, λ_2).

Generalized accumulated spectrogram

- Ξ_2 is the DPP on S_2 associated with (K_2, λ_2).
- For $\Lambda \subset S_2$ such that $\mathbb{E}[\Xi_2(\Lambda)] < \infty$, we define the restriction

$$(\mathcal{K}_2)_\Lambda := \mathcal{P}_\Lambda \mathcal{K}_2 \mathcal{P}_\Lambda \quad \text{(trace class)}$$

and consider the eigenvalue problem

$$(\mathcal{K}_2)_\Lambda \Phi_j^{(\Lambda)} = \mu_j^{(\Lambda)} \Phi_j^{(\Lambda)} \quad (j = 1, 2, \dots)$$

such that

$$1 \geq \mu_1^{(\Lambda)} \geq \mu_2^{(\Lambda)} \geq \cdots \geq 0$$

and $\Phi_j^{(\Lambda)}$ is the normalized eigenfunction for $\mu_j^{(\Lambda)}$.
- Set $N_\Lambda = \lceil \mathbb{E}[\Xi_2(\Lambda)] \rceil$ and define a generalized accumulated spectrogram

$$\rho_\Lambda(y) := \sum_{j=1}^{N_\Lambda} |\Phi_j^{(\Lambda)}(y)|^2 \quad (y \in S_2).$$

Example 1

Weyl-Heisenberg case (Ginibre case):

- For $\Lambda \subset \mathbb{R} \times \mathbb{R} \simeq \mathbb{C}$, we set $N_\Lambda = \lceil \mathbb{E}[\Xi(\Lambda)] \rceil$ and define

$$\rho_\Lambda(z) := \sum_{j=1}^{N_\Lambda} \frac{(\pi z)^j}{j!} |z|^{2j} e^{-\pi |z|^2} \quad \text{(accumulated spectrogram)},$$

  where $N_\Lambda = \lceil \mathbb{E}[\Xi(\Lambda)] \rceil$.

- (Corresponding to Circular law for Ginibre)
  Let $\mathbb{D}_1 = \{(x, \xi) \in \mathbb{R}^2 : x^2 + \xi^2 \le 1\} \subset \mathbb{C}$. As $R \to \infty$,

$$\rho_{R\mathbb{D}_1}(R\cdot) \to \mathbf{1}_{\mathbb{D}_1} \quad \text{in } L^1(\mathbb{C}),$$

  where $N_{R\mathbb{D}_1} \approx \pi R^2$.

Example 2

Weyl-Heisenberg case (Ginibre case):

- For $\Lambda =$ star, we have the following figure.

In the talk, I used here the figure 3 in the following paper.

L. D. Abreu, K. Gröchenig, and J. L. Romero, On accumulated spectrograms, Trans. Amer. Math. Soc **368** (2016), 3629-3649.

L^1-limit law of generalized spectrograms

Weyl-Heisenberg case:

- For $\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d (\simeq \mathbb{C}^d)$, we set $N_\Lambda = \lceil \mathbb{E}[\Xi_2(\Lambda)] \rceil$ and define

$$\rho_\Lambda(z) := \sum_{j=1}^{N_\Lambda} |\Phi_j^{(\Lambda)}(z)|^2, \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Theorem (Abreu-Gröchenig-Romero ('16))

Under a mild condition for $\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$, for Weyl-Heisenberg ensemble on $\mathbb{R}^d \times \mathbb{R}^d$, as $R \to \infty$,

$$\rho_{R\Lambda}(R\cdot) \to \mathbf{1}_\Lambda \quad \text{in } L^1(\mathbb{R}^d \times \mathbb{R}^d).$$

CUE eigenvalues and Poisson point process

- CUE (circular unitary ensemble) is the group $\mathscr{U}(N)$ of $N \times N$ unitary matrices with Haar measure.

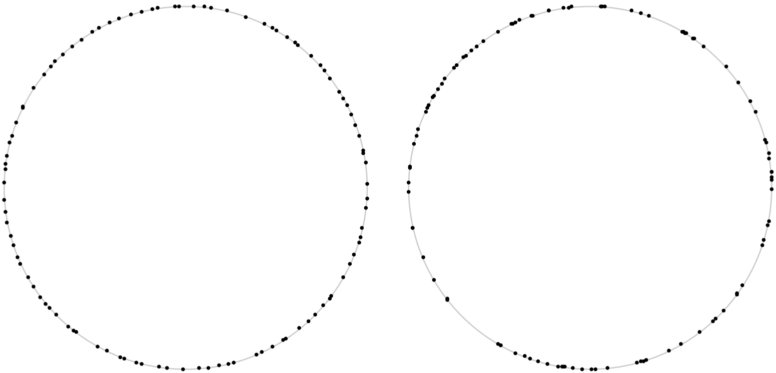


Figure: CUE eigenvalues (left) and Poisson (right) ($N = 100$)

Cirucular Unitary Ensemble (CUE)

- Let $\mathscr{U}(N)$ be the group of $N \times N$ unitary matrices with Haar measure.
- The probability distribution of eigenvalues $\{e^{\sqrt{-1}\theta_j}\}_{j=1}^N$ is

$$\frac{1}{n!(2\pi)^N} \prod_{1 \le j < k \le N} |e^{\sqrt{-1}\theta_j} - e^{\sqrt{-1}\theta_k}|^2 d\theta_1 \dots d\theta_N$$

- They form a DPP on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with $\lambda(d\theta) = d\theta/(2\pi)$ on $\mathbb{T}$ and

$$\begin{aligned} K_N(\theta, \varphi) &= \sum_{k=0}^{N-1} e^{\sqrt{-1}k\theta} \overline{e^{\sqrt{-1}k\varphi}} \\ &= u(\theta) \underbrace{\frac{\sin N(\theta - \varphi)/2}{\sin(\theta - \varphi)/2}}_{:=\tilde{K}_N(\theta,\varphi)} \overline{u(\varphi)}, \end{aligned}$$

  where $u(\theta) = e^{\sqrt{-1}(N-1)\theta/2}$.
- RKHS: $H_K = \mathrm{span}\{e^{\sqrt{-1}k\theta}, k = 0, 1, \dots, N-1\} \subset L^2(\mathbb{T}, d\theta)$.

Limiting DPP for CUE eigenvalues

- CUE eigenvalues form an N-points DPP on $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ with

$$\tilde{K}_N(\theta, \theta') = \frac{\sin N(\theta - \theta')/2}{\sin(\theta - \theta')/2}$$

- $\rho_1(\theta) = \tilde{K}_N(\theta, \theta) = N$ and the empirical dist. of points converges to the uniform dist. on $\mathbb{T}^1$.
- Scaling $\xi = \sum_i \delta_{\theta_i} \mapsto S_N(\xi) = \sum_i \delta_{x_i}$ where $x_i = N\theta_i$,

$$\frac{1}{N}\tilde{K}_N\left(\frac{\theta}{N}, \frac{\theta'}{N}\right) = \frac{1}{N}\frac{\sin(x-y)/2}{\sin(\frac{x}{N} - \frac{y}{N})/2} \to \frac{\sin(x-y)/2}{(x-y)/2} =: K_{\mathrm{sinc}}(x, y).$$

- From this observation, we can see that

Fact:

N-point DPP on $\mathbb{T}^1 \overset{d}{\Rightarrow}$ the DPP on $\mathbb{R}^1$ with K_{sinc} (PW-space)

Two ways of generalizations of CUE

We have two generalizations of CUE on $\mathbb{T} \simeq \mathbb{S}^1$ to the sphere $\mathbb{S}^2$.

1. Vandermonde determinant of distances:

$$\prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 = \prod_{1 \le j < k \le n} \|z_j - z_k\|_{\mathbb{R}^2}^2 \quad (z_j \in \mathbb{S}^1)$$

2. DPP with the projection kernel onto an eigenspace:

$$K_N(\theta, \varphi) = \sum_{k=0}^{N-1} e^{ik\theta} \overline{e^{ik\varphi}}$$

with $\lambda(d\theta) = d\theta/(2\pi)$ on $\mathbb{S}^1$. Here $e^{ik\theta}$ is an eigenfunction of the Laplacian $\Delta_{\mathbb{S}^1} = \frac{d^2}{d\theta^2}$:

$$-\Delta_{\mathbb{S}^1} e^{ik\theta} = k^2 e^{ik\theta}.$$

$L^2(\mathbb{S}^1)$ is spanned by $\{e^{ik\theta}\}_k$.

(1) Spherical ensemble

- Ginibre random matrix:

$$G \sim Ginibre(N) \iff \{G_{ij}\}_{i,j=1}^{N} \text{ are i.i.d. and } G_{ij} \sim N_{\mathbb{C}}(0,1).$$

- Let $A, B \sim Ginibre(N)$ be independent.
- (Krishnapur '09) The eigenvalues of $A^{-1}B$ forms a DPP on $\mathbb{C}$ with

$$K_N(z,w) = (1 + z\overline{w})^{N-1}$$
$$\lambda(dz) = \frac{N}{\pi(1+|z|^2)^{N+1}} dm(z)$$

- Density of points: $K_N(z,z)\lambda(dz) = \frac{N}{\pi(1+|z|^2)^2} dm(z)$.
- The reproducing kernel Hilbert space (RKHS) is the space of polynomials:

$$H_{K_N} = \mathrm{span}\{z^n : n = 0, 1, \ldots, N-1\}$$

Spherical ensemble

- Through the stereographic projection, it is considered as a point process on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
- The distribution w.r.t. the surface measure is given by

$$(\text{const.}) \prod_{1 \le j < k \le N} \|P_j - P_k\|_{\mathbb{R}^3}^2 \quad \text{on } \widehat{\mathbb{C}},$$

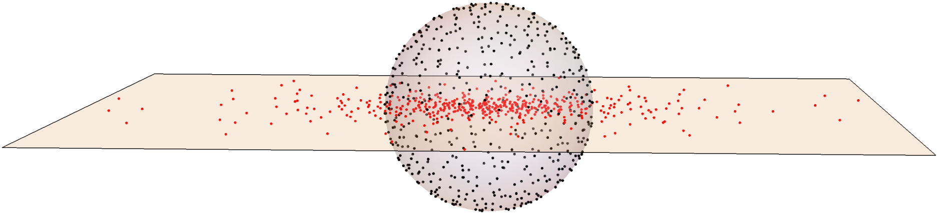


Figure: Pullback of eigenvalues of $A^{-1}B$ by the stereographic projection ($N = 500$)

Spherical ensemble

$$(\text{const.}) \prod_{1 \le j < k \le N} \|P_j - P_k\|_{\mathbb{R}^3}^2 \quad \text{on } \widehat{\mathbb{C}} \simeq \mathbb{S}^2,$$

- This DPP is $O(3)$-invariant, and uniformly distributed with density $N/4\pi$.
- This may be considered as a spherical version of CUE eigenvalues.
- It has been studied as u $2D$ one-component plasma/$2D$ Coulomb gas on $\mathbb{S}^2$.
- The correlation kernel is given by

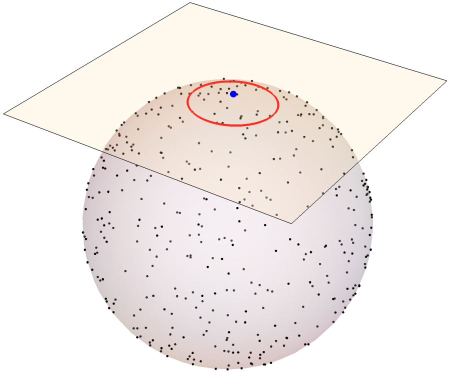
$$\begin{aligned} K(p, p') &= K((\theta, \varphi), (\theta', \varphi')) \\ &= \frac{N}{4\pi} \Big(e^{\sqrt{-1}(\varphi - \varphi')} \sin(\theta/2) \sin(\theta'/2) + \cos(\theta/2) \cos(\theta'/2) \Big)^{N-1} \end{aligned}$$

where $p = (\theta, \varphi)$ is the polar coordinates of $\mathbb{S}^2$.

Point process on the tangent space at the north-pole

- As $N \to \infty$, the empirical measure $\frac{1}{N}\sum_{i=1}^{N} \delta_{P_i}$ converges weakly to the uniform measure on $\mathbb{S}^2$ almost surely.
- We consider the pullback of points on the sphere by the exponential map $\exp : T_{e_3}(\mathbb{S}^2) \to \mathbb{S}^2$, i.e., using the polar coordinates (θ, φ),

$$T_{e_3}(\mathbb{S}^2) \ni (\theta \cos\varphi, \theta \sin\varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{S}^2.$$



Limiting point process for spherical ensembles

- $\tilde{\xi}_N$: spherical ensemble, which is the eigenvalues process of $A^{-1}B$ for $A, B \sim Ginibre(N)$. $\quad$ (N-point process on $\mathbb{S}^2$)
- Let $e_3 = (0, 0, 1)$ be the north pole and $T_{e_3}(\mathbb{S}^2)$ be the tangent space at e_3.
- For fixed $\epsilon > 0$, we consider the pull-back of points on $\mathbb{S}^2 \cap B_\epsilon(e_3)$ by the exponetial map $\exp : T_{e_3}(\mathbb{S}^2) \to \mathbb{S}^2$ and denote it by $\xi_N^{(\epsilon)}$.
- Scaling map: For a configuration $\xi = \sum_i \delta_{x_i}$, we define

$$S_c(\xi) = \sum_i \delta_{cx_i}.$$

Theorem (Katori-S.)

The scaled p.p. $S_{\sqrt{N}}(\xi_N^{(\epsilon)})$ converges weakly to the Ginibre DPP.

Recall that the Ginibre DPP is the DPP on $\mathbb{C}$ associated with the kernel

$$K(z, w) = e^{z\overline{w}}, \quad \lambda(dz) = \pi^{-1} e^{-|z|^2} dz.$$

(2) Harmonic ensemble for $\mathbb{S}^2$

- $L^2(\mathbb{S}^2)$: There is a spectral decomposition of $L^2(\mathbb{S}^2)$ as

$$L^2(\mathbb{S}^2) \simeq \bigoplus_{\ell=0}^{\infty} E_\ell,$$

  where E_ℓ is the eigenspace of $-\Delta_{\mathbb{S}^2}$ corresponding to the eigenvalue $\ell(\ell+1)$ and $\dim E_\ell = 2\ell+1$.

- Spherical harmonics:

$$Y_m^\ell(\theta, \varphi) := \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_m^\ell(\cos\theta) e^{im\varphi} \quad (-\ell \le m \le \ell),$$

  where $P_m^\ell(x)$ is the associated Legendre polynomial of degree m.

- Eigenspace E_ℓ: E_ℓ is spanned by the spherical harminoics

$$E_\ell = \mathrm{span}\{Y_m^\ell : m = -\ell, -\ell+1, \dots, \ell\}.$$

Projection kernel

- $L^2(\mathbb{S}^2) = \oplus_{\ell=0}^{\infty} E_\ell$, where E_ℓ is the eigenspace of $-\Delta_{\mathbb{S}^2}$ for $\ell(\ell+1)$.
- Reproducing kernel for $\oplus_{\ell=0}^{N-1} E_\ell$: $\dim E_\ell = 2\ell + 1$.

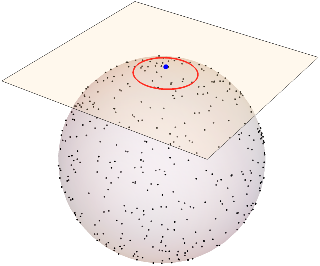
$$K_N(x,y) = \sum_{\ell=0}^{N-1} \underbrace{\sum_{m=-\ell}^{\ell} Y_m^\ell(x)\overline{Y_m^\ell(y)}}_{\text{projection onto } E_\ell} = \sum_{\ell=0}^{N-1} \Psi^\ell(x,y),$$

  where $\Psi^\ell(x,y)$ is the reproducing kernel for E_ℓ.
- DPP on $\mathbb{S}^2$ associated with K_N: The number of points is N^2.
- As $N \to \infty$, the empirical measure converges weakly to the uniform measure on $\mathbb{S}^2$ in law.

Limiting DPP associated with K_N

- For fixed $\epsilon > 0$, $\xi_N^{(\epsilon)}$ is the pull-back of points on $\mathbb{S}^2 \cap B_\epsilon(e_3)$ by the exponential map $\exp : T_{e_3}(\mathbb{S}^2) \to \mathbb{S}^2$.
- For $\xi = \sum_i \delta_{x_i}$,

$$S_N(\xi) = \sum_i \delta_{Nx_i}.$$



Theorem (Katori-S.)

The scaled p.p. $S_N(\xi_N^{(\epsilon)})$ converges weakly to the DPP on $T_{e_3}(\mathbb{S}^2) \simeq \mathbb{R}^2$ associated with the kernel

$$K(x, y) = \frac{1}{2\pi|x - y|} J_1(|x - y|),$$

where $J_1(r)$ is the Bessel function of the first kind.

(3) Harmonic ensemble on $\mathbb{S}^n$

- Eigenspace E_ℓ: E_ℓ is the eigenspace of $-\Delta_{\mathbb{S}^n}$ corresponding to $\ell(\ell+n-1)$.
- Spherical harmonics on $\mathbb{S}^n$: E_ℓ is spanned by the spherical harmonics $\{Y_m^\ell\}_{m=1}^{d_\ell}$, where $d_\ell = \frac{(2\ell+n-1)(\ell+n-2)!}{(n-1)!\ell!}$.
- Spectral decomposition:

$$L^2(\mathbb{S}^n) = \bigoplus_{\ell=0}^{\infty} E_\ell$$

- Projection onto $H_N := \oplus_{\ell=0}^{N-1} E_\ell$:

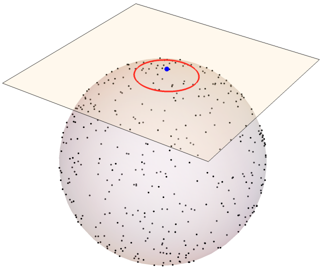
$$K_N(x,y) = \sum_{\ell=0}^{N-1} \underbrace{\sum_{m=1}^{d_\ell} Y_m^\ell(x)\overline{Y_m^\ell(y)}}_{\text{projection onto } E_\ell},$$

- (H_N, K_N) is a RKHS, and then $\exists$ the rotation invariant DPP on $\mathbb{S}^n$.

Limiting DPP for harmonic ensemble on $\mathbb{S}^n$

- $\xi_N^{(\epsilon)}$ is the pull-back of points on $\mathbb{S}^n \cap B_\epsilon(e_{n+1})$ by the exponetial map $\exp : T_{e_{n+1}}(\mathbb{S}^n) \to \mathbb{S}^n$.
- For $\xi = \sum_i \delta_{x_i}$,

$$S_N(\xi) = \sum_i \delta_{Nx_i}.$$



Theorem (Katori-S.)

The scaled p.p. $S_N(\xi_N^{(\epsilon)})$ converges weakly to the DPP on $T_{e_{n+1}}(\mathbb{S}^n) \simeq \mathbb{R}^n$ associated with the kernel

$$K^{(n)}(x, y) = \frac{1}{(2\pi|x-y|)^{\frac{n}{2}}} J_{\frac{n}{2}}(|x-y|),$$

where $J_\nu(r)$ is the Bessel function of the first kind with index ν.

Example: generalized Paley-Wiener space

- Frequency bounded functions: For a bounded Borel set $B \subset \mathbb{R}^n$,

$$PW_B(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \widehat{f} \subset \overline{B}\},$$

where $\widehat{f}(\omega) = \int_{\mathbb{R}^n} f(x) e^{-i\omega\cdot x} dx$.

$$|f(x)| \leq \frac{1}{(2\pi)^n} \left| \int_B \widehat{f}(\omega) e^{i\omega\cdot x} d\omega \right| \leq \sqrt{\frac{|B|}{(2\pi)^n}} \|f\|_{L^2(\mathbb{R}^n)}.$$

- Reproducing kernel:

$$K_B(x, y) = \frac{1}{(2\pi)^n} \int_B e^{i\omega\cdot(x-y)} d\omega$$

- RKHS $(PW_B(\mathbb{R}^n), K_B)$ is a generalization of the Paley-Wiener space.

Multi-dimensional version of Paley-Wiener space

- Correlation kernel:

$$K^{(n)}(x,y) = \frac{1}{(2\pi|x-y|)^{\frac{n}{2}}} J_{\frac{n}{2}}(|x-y|)$$
$$= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \mathbf{1}_{B_1}(u) e^{iu\cdot(x-y)} du.$$

- RKHS: $H_{K^{(n)}}$ is the generalized Paley-Wiener space corresponding to the unit ball B_1.
- Invariance: $K^{(n)}(x,y)$ is motion invariant and hence

$$K^{(n)}(x,y) = k^{(n)}(|x-y|)$$

where

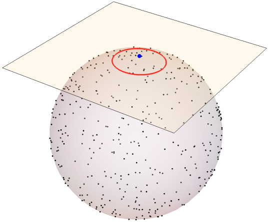
$$k^{(n)}(r) = \frac{1}{(2\pi r)^{\frac{n}{2}}} J_{\frac{n}{2}}(r)$$

- For odd $n = 1, 3, \ldots$, it is simplified as

$$k^{(1)}(r) = \frac{\sin r}{\pi r}, \quad k^{(3)}(r) = \frac{1}{2\pi^2 r^2}\left(\frac{\sin r}{r} - \cos r\right), \ldots$$

Question

- M: compact, smooth Riemannian manifold of dimension n.
- Suppose that on a neighborhood on $B_\epsilon(p)$ of a point $p \in M$, the empirical measure converges to a measure with positive density on $B_\epsilon(p)$.
- For sufficiently small $\epsilon > 0$ (smaller than the injective radius at p), $\xi_N^{(\epsilon)}$ is the pull-back of points on $M \cap B_\epsilon(p)$ by the exponential map $\exp : T_p(M) \to M$.



- For $\xi_N^{(\epsilon)} = \sum_i \delta_{x_i}$,

$$S_{a_N}(\xi_N^{(\epsilon)}) = \sum_i \delta_{a_N x_i} \overset{d}{\Rightarrow} ??$$

The Weyl law and quantum ergodicity

- Let (M, g) be a compact, smooth Riemannian manifold of dimension n and consider the eigenvalue problem

$$-\Delta_M \varphi_j = \lambda_j^2 \varphi_j,$$

where $0 = \lambda_1 \leq \lambda_2 \leq \cdots$ and $\{\varphi_j\}_{j\geq 1}$ is an ONB of $L^2(M)$.

- Weyl law: As $\lambda \to \infty$,

$$N(\lambda) = \#\{j \geq 1 : \lambda_j \leq \lambda\} \sim \frac{|B_1|}{(2\pi)^n} \mathrm{Vol}(M) \lambda^n,$$

where $|B_1|$ is the volume of the unit ball of in $\mathbb{R}^n$.

- Quantum ergodicity: Does the following hold?

$$|\varphi_j(x)|^2 dx \xrightarrow{w} dx \quad \text{as } j \to \infty?$$

Thm. (Shnirelman-Zelditch-Colin de Verdiére) This is true along a subsequence with density 1 if the geodesic flow on M is ergodic.

- E_{λ_i}: the eigenspace of $-\Delta_M$ corresponding to λ_i.
- Reproducing kernel (projection kernel) for $\oplus_{\lambda_i \leq \lambda} E_{\lambda_i}$:

$$K_\lambda(x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)}.$$

- Consider DPP $\xi_\lambda(\omega)$ on M associated with K_λ.
- The counting function is equal to the number of DPP points:

$$N(\lambda) = \int_M K_\lambda(x, x)dx \sim \frac{|B_1|}{(2\pi)^n} \mathrm{Vol}(M)\lambda^n,$$

where B_1 is the unit ball in $\mathbb{R}^n$.

Universality of DPP on Riemannian manifold

- For $\xi_\lambda^{(\epsilon)} = \sum_i \delta_{x_i}$ on the cotangent space $T_p^*(M)$ at p, which is the pullback of the DPP ξ_λ on $M \cap B_p(\epsilon)$.

$$S_\lambda(\xi_\lambda^{(\epsilon)}) = \sum_i \delta_{\lambda x_i} \stackrel{d}{\Rightarrow} ??$$

Theorem (Katori-S.)

The scaled DPP $S_\lambda(\xi_\lambda^{(\epsilon)}(\omega))$ converges weakly to the DPP associated with

$$K^{(n)}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbf{1}_{B_1}(u) e^{iu\cdot(x-y)} du,$$

where B_1 is the unit ball in $\mathbb{R}^n$.

Summary

- We discussed L^1-limit for the accumulated spectrogram for DPPs from the view point of hyperunifomity.

- Two types of DPPs on $\mathbb{S}^2$ are discussed.
  1. through the eigenvalues of $A^{-1}B$ (harmonic ensemble)
  2. through the RKHS spanned by spherical harmonics (spherical ensemble).

  The former converges to DPP associated with the Bessel function J_1, the latter converges to Ginibre DPP.
- The DPP on $\mathbb{S}^n$ associated with RKHS spanned by spherical harmonics is introduced, and show the convergence toawards the DPP associated with the generalized Paley-Wiener space
- Furthermore, we considered the similar problem on compact Riemannian manifolds, and we showed the universality.