Random normal matrix models with a soft edge of the spectrum

Joint work with Yacin Ameur and Nam-Gyu Kang

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Random Matrices and Related Topics

Random normal matrix ensembles

2D Coulomb gas model

Consider n point charges on $\mathbb{C}$ influenced by an external potential Q. The energy of a configuration $(\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$ is given by

$$H_n(\zeta_1, \cdots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j).$$

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The Boltzmann-Gibbs distribution at inverse temperature $\beta > 0$:

$$d\mathbf{P}_n^{\beta}(\zeta_1, \cdots, \zeta_n) = \frac{1}{Z_{n,\beta}} e^{-\beta H_n(\zeta_1, \cdots, \zeta_n)} \, dA(\zeta_1) \cdots dA(\zeta_n),$$

where $Z_{n,\beta}$ is the normalizing constant and dA is the area measure in $\mathbb{C}$.

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where $Z_{n,\beta}$ is the normalizing constant and dA is the area measure in $\mathbb{C}$.

If $\beta = 1$, the point process $\{\zeta_j\}_{j=1}^{n}$ is determinantal and it represents the eigenvalues of random normal matrix models.

Correlation functions

The k-point correlation function (for $1 \leq k \leq n$) of the point process:

$$\mathbf{R}_{n,k}(\zeta_1, \cdots, \zeta_k) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2k}} \mathbf{P}_n \left[\mathcal{N}(D(\zeta_j, \epsilon)) \geq 1 \text{ for all } 0 \leq j \leq k \right],$$

where $\mathcal{N}(D)$ is the number of eigenvalues in $D \subset \mathbb{C}$.

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The system of eigenvalues forms a determinantal point process:

$$\mathbf{R}_{n,k}(\zeta_1, \cdots, \zeta_k) = \det \left(\mathbf{K}_n(\zeta_i, \zeta_j) \right)_{i,j=1}^{k},$$

where $\mathbf{K}_n : \mathbb{C}^2 \to \mathbb{C}$ is called a correlation kernel.

Global properties

For a smooth potential Q satisfying $\liminf_{\zeta\to\infty} Q(\zeta)/\log|\zeta| > 2$, the eigenvalues accumulate on a compact set when n goes to ∞.

- Macroscopic asymptotics: [Johansson, Hedenmalm–Makarov]

$$\frac{1}{n}\mathbb{E}\sum_{j=1}^{n}\delta_{\zeta_j} \to \sigma \quad \text{as} \quad n\to\infty \text{(in the weak-star sense of measures)}$$

- $\sigma = \sigma_Q$ is the Frostman's equilibrium measure. It is the unique probability measure which minimizes the weighted logarithmic energy,

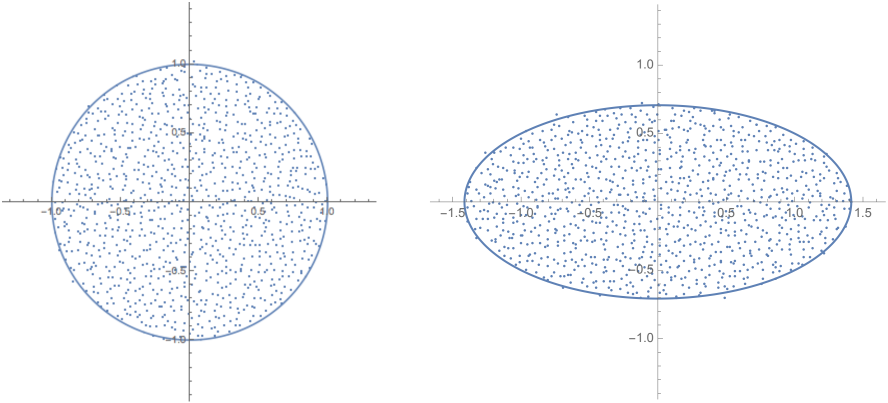
$$I_Q(\mu) = \iint_{\mathbb{C}^2} \log\frac{1}{|\zeta-\eta|}d\mu(\zeta)d\mu(\eta) + \int_{\mathbb{C}} Q\,d\mu,$$

among all compactly supported Borel probability measure μ on $\mathbb{C}$.

- σ has compact support $S = S_Q$ (droplet).
- $d\sigma(\zeta) = 1_S \Delta Q(\zeta)\,dA(\zeta)$ where $\Delta = \partial\bar{\partial}$ and $dA(x+iy) = dx\,dy/\pi$.

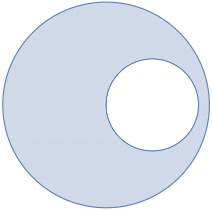
Global properties: examples

$\{\zeta_j\}_1^n$: the eigenvalues of random normal matrices associated with Q.



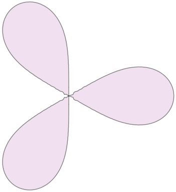
$Q(\zeta) = |\zeta|^2 \quad (n = 1024)$ $\qquad$ $Q(\zeta) = |\zeta|^2 - \frac{1}{3}\,\mathrm{Re}\,\zeta^2 \ (n = 1024)$

Droplets



$$Q(z) = |z|^2 - \frac{1}{2}\log|z - \frac{1}{2}|$$

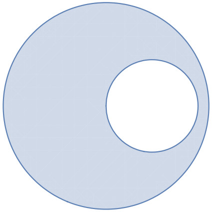
[Balogh–Bertola–Lee–McLaughlin]



$$Q(z) = |z|^6 - \frac{2}{\sqrt{3}}\,\mathrm{Re}\, z^3$$

[Balogh–Merzi]

Droplets



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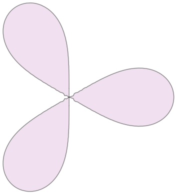


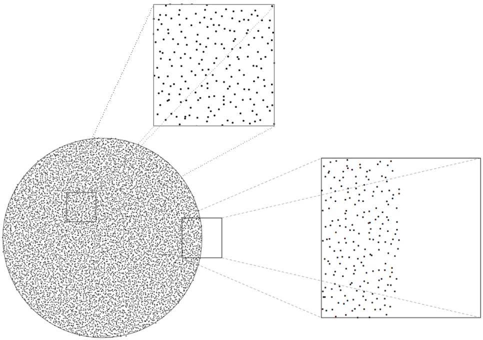
$$Q(z) = |z|^6 - \frac{2}{\sqrt{3}}\,\mathrm{Re}\, z^3$$

[Balogh–Merzi]

If Q is real analytic (except at finitely many points), then the boundary ∂S is a finite union of analytic arcs with at most a finite number of singularities.

Local properties

Define a rescaled eigenvalue system $\{z_j\}_1^n$ at a point in the droplet S.



If p is a "regular" point,

⋆ Bulk scaling ($p \in \operatorname{Int} S$)

$$z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p).$$

⋆ Edge scaling ($p \in \partial S$)

$$z_j = \sqrt{n\Delta Q(p)}\ e^{-i\theta}(\zeta_j - p),$$

$e^{i\theta}$: outer normal to ∂S at p.

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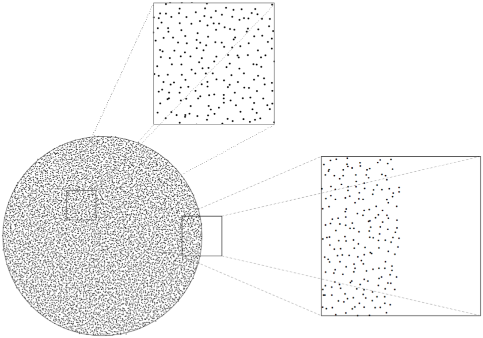
Define a rescaled eigenvalue system $\{z_j\}_1^n$ at a point in the droplet S.



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$e^{i\theta}$: outer normal to ∂S at p.

Universality phenomenon: in the large n limit the microscopic behavior of eigenvalues does not depend on the specific potential.

Universality results

Bulk scaling limits ($p \in \operatorname{Int} S$)

- $\Delta Q(p) > 0$: [Berman '08, Ameur–Hedenmalm–Makarov '11].
  The rescaled process $z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p)$ converges to the determinantal point process with correlation kernel

$$G(z,w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2} \quad \text{(Ginibre kernel).}$$

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$$G(z,w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2} \quad \text{(Ginibre kernel).}$$

- $Q(\zeta) = Q_r(\zeta - p) + h(\zeta) - (2c/n)\log|\zeta - p|$ for $d > 0$, $c > -1$:
  - ∘ Q_r is a radially symmetric function with $Q_r(\zeta) = |\zeta|^{2d} + O(|\zeta|^{2d+\epsilon})$
  - ∘ h is a harmonic polynomial [Ameur–Kang–S. '18].

  The rescaled process $z_j = n^{1/2d}(\zeta_j - p)$ converges to the determinantal point process with correlation kernel

$$K(z,w) = \sum_{j=0}^{\infty} \frac{d(z\bar{w})^j}{\Gamma((j+c+1)/d)} e^{-|z|^{2d}/2 - |w|^{2d}/2 + c\log|z| + c\log|w|}.$$

Universality results

Edge scaling limits ($p \in \partial S$)

- $Q(\zeta) = |\zeta|^2 - a\,\mathrm{Re}\,\zeta^2$: [Lee–Riser, '15] fine asymptotics of eigenvalues.
- $\Delta Q(p) > 0$, regular points:

  [Ameur–Kang–Makarov '14]
  Translation invariant limiting kernels were characterized.

  [Hedenmalm-Wennman '17]
  The rescaled process $z_j = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta_j - p)$ converges to the determinantal point process with correlation kernel

$$e^{z\bar{w}-|z|^2/2-|w|^2/2} \cdot \frac{1}{2}\,\mathrm{erfc}\left(\frac{z+\bar{w}}{\sqrt{2}}\right).$$

- $\Delta Q(p) > 0$, singular points [Ameur–Kang–Makarov–Wennman '15].
- $\Delta Q(p) = 0$?
- Q with a logarithmic singularity on ∂S ?

Free boundary / Hard edge

Random normal matrix models with two different boundary conditions.

⋆ Free boundary RNM model

- External potential Q
- Equilibrium measure

$$\sigma_Q = 1_S \Delta Q dA$$

- Edge scaling

$$z_j = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta - p)$$

- Rescaled correlation kernel

$K(z,w) = G(z,w)\varphi(z+\bar{w})$

where

$G(z,w) = e^{z\bar{w}-|z|^2/2-|w|^2/2}$

$\varphi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-(\xi-z)^2/2}d\xi$

⋆ Hard edge RNM model

- $Q^{\mathrm{H}} = \begin{cases} Q & \text{if} \quad \mathrm{z} \in \mathrm{S} \\ +\infty & \text{otherwise} \end{cases}$
- Equilibrium measure σ_Q
- Edge scaling

$$z_j = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta - p)$$

- Rescaled correlation kernel

$G(z,w)H(z+\bar{w})1_{\mathbb{L}}(z)1_{\mathbb{L}}(w)$

where

$G(z,w) = e^{z\bar{w}-|z|^2/2-|w|^2/2}$

$H(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \frac{e^{-(\xi-z)^2/2}}{\varphi(\xi)}d\xi$

Free boundary / Hard edge

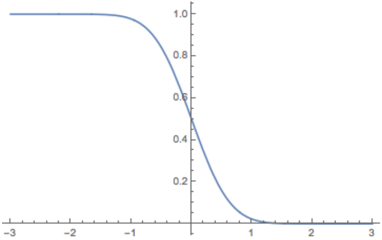
Random normal matrix models with two different boundary conditions.

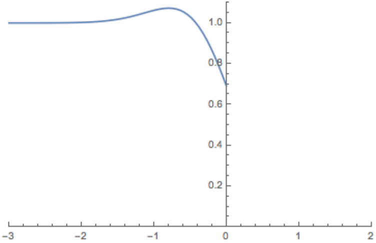
⋆ Free boundary RNM model

- External potential Q
- Rescaled one point function
  $R(z) = K(z, z) = \varphi(z + \bar{z})$

⋆ Hard edge RNM model

- $Q^{\mathrm{H}} = Q + \infty \cdot 1_{\mathbb{C}\setminus S}$
- Rescaled one point function
  $R^{\mathrm{H}}(z) = H(z + \bar{z})1_{\mathbb{L}}(z)$





A scale of boundary conditions

Construct the random normal matrix model which interpolates between the free boundary case and the hard edge case.

- External potential $Q_t := \check{Q} + t(Q - \check{Q})$ for $t > 0$.
- $\check{Q}$ is the solution of the obstacle problem:

  $\check{Q}$ is the maximal subharmonic function $\leq Q$ which grows like $\log|z|^2 + O(1)$ when $|z| \to \infty$.
- $\check{Q} = -2U^\sigma + \gamma$ where U^σ is the logarithmic potential of the equilibrium measure σ and γ is the modified Robin constant which makes $Q = \check{Q}$ on S.

A scale of boundary conditions ($Q(z) = |z|^2$)

Construct the random normal matrix model which interpolates between the free boundary case and the hard edge case.

We first consider the Ginibre potential $Q(z) = |z|^2$.

- $\{\zeta_j\}$: eigenvalues of random normal matrices associated with the potential $Q_t = \begin{cases} |z|^2 & \text{if z} \in \mathbb{D} \\ t|z|^2 + (1-t)(\log|z|^2 + 1) & \text{if z} \in \mathbb{D}^{\text{c}}. \end{cases}$
- $\check{Q}(z) = \log|z|^2 + 1$: the obstacle function associated with $Q(z) = |z|^2$. ($\check{Q}$ is the maximal subharmonic function $\leq Q$ which grows like $\log|z|^2 + O(1)$ when $|z| \to \infty$.)

- $t = 1$: $Q_t = Q$ (free boundary)
- $t = \infty$: $Q_t = Q^{\text{H}}$ (hard edge)
- $t = 0$: $Q_t = \log|z|^2 + 1$ (not confining)

Soft edge scaling limits ($Q(z) = |z|^2$)

- $\{\zeta_j\}$: eigenvalues of random normal matrices associated to the potential $Q_t = \begin{cases} |z|^2 & \text{if } z \in \mathbb{D} \\ t|z|^2 + (1-t)(\log|z|^2 + 1) & \text{if } z \in \mathbb{D}^c. \end{cases}$
- Edge scaling ($p \in \partial\mathbb{D}$): $z_j = \sqrt{n}e^{-i\theta}(\zeta_j - p)$.
- The rescaled eigenvalue system $\{z_j\}$ converges to the determinantal point process with correlation kernel

$$K^t(z,w) = G(z,w)S_t(z+\bar{w})\, e^{(1-t)((\operatorname{Re} z)^2 1_{\{\operatorname{Re} z>0\}} + (\operatorname{Re} w)^2 1_{\{\operatorname{Re} w>0\}})}.$$

- $S_t(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 \frac{e^{-(\xi-z)^2/2}}{\Phi_t(\xi)}d\xi$, where

$\Phi_t(\xi) = \varphi(\xi) + \frac{1}{\sqrt{t}}e^{\frac{1-t}{2t}\xi^2}\left(1 - \varphi(\frac{\xi}{\sqrt{t}})\right)$ and $\varphi(\xi) = \frac{1}{\sqrt{2\pi}}\int_\xi^\infty e^{-u^2/2}du.$

Soft edge scaling limits ($Q(z) = |z|^2$)

- Soft edge limiting correlation kernel:

$$K^t(z,w) = G(z,w) S_t(z+\bar{w})\, e^{(1-t)((\mathrm{Re}\, z)^2 1_{\{\mathrm{Re}\, z>0\}} + (\mathrm{Re}\, w)^2 1_{\{\mathrm{Re}\, w>0\}})}.$$

- $S_t(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-(\xi-z)^2/2}}{\Phi_t(\xi)} d\xi$, where

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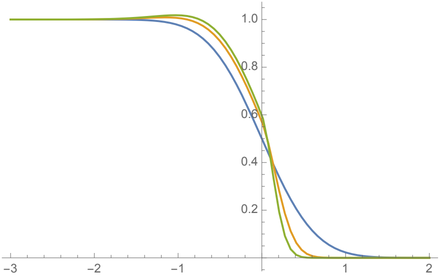
- $t = 1$: $K^t(z,w) = G(z,w)\varphi(z+\bar{w})$ free boundary case.

- $t = \infty$: $K^t(z,w) = G(z,w) H(z+\bar{w}) 1_{\{\mathrm{Re}\, z<0\}} 1_{\{\mathrm{Re}\, w<0\}}$ hard edge case.

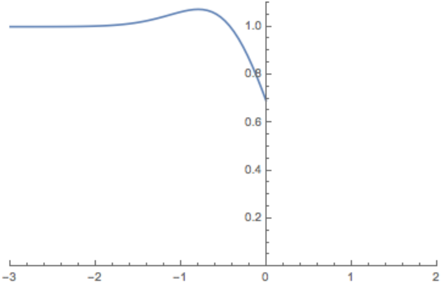
1-point density of the rescaled system

R^t: the limiting 1-point function of the rescaled eigenvalue system $\{z_j\}$.

Graphs of $R^t(x) = K^t(x,x) = S_t(2x)\, e^{2(1-t)x^2 1_{\{x>0\}}}$ for $x \in \mathbb{R}$. $(t \geq 1)$



$t = 1$(blue), 5(orange), and 10(green)

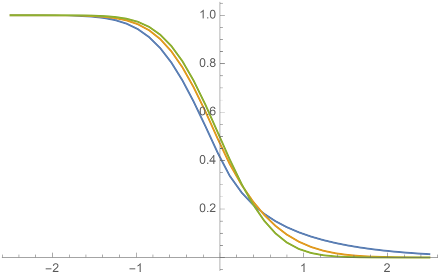


$t = \infty$ (hard edge)

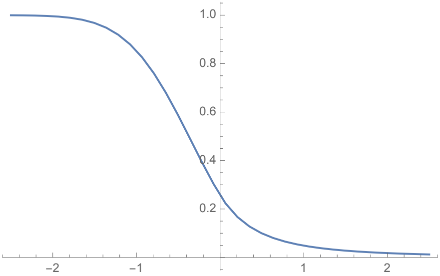
1-point function of the rescaled system

R^t: the limiting 1-point function of the rescaled eigenvalue system.

Graphs of $R^t(x) = K^t(x, x) = S_t(2x)\, e^{2(1-t)x^2 1_{\{x>0\}}}$ for $x \in \mathbb{R}$. $(t < 1)$



$t = 0.1$(blue), 0.5(orange), 0.9(green)



t=0

1-point function of the rescaled system

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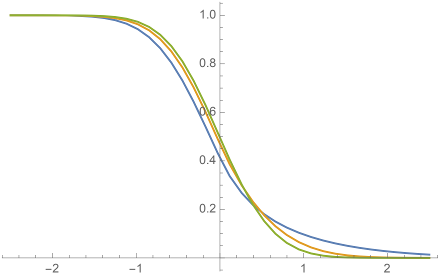

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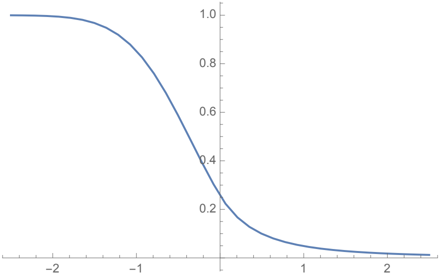


t=0

Exterior estimates:

$R^t(x) \le Ce^{-tx^2}, \quad x > 0$

$R^0(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{4x^2} + O(x^{-3})\right), x \to \infty.$

Correlation kernels

Correlation kernel $\mathbf{K}_n$ is given by

$$\mathbf{K}_n(\zeta,\eta) = \sum_{j=0}^{n-1} p_{n,j}(\zeta)\overline{p_{n,j}(\eta)}e^{-nQ(\zeta)/2-nQ(\eta)/2},$$

where $p_{n,j}$ is an orthonormal polynomial of deg j with respect to $e^{-nQ}dA$:

$$\int_{\mathbb{C}} p_{n,j}(z)\,\overline{p_{n,k}(z)}\,e^{-nQ(z)}dA(z) = \delta_{jk}.$$

We use the recent method of [Hedenmalm-Wennman '17] to obtain the asymptotics of weighted orthogonal polynomials near the boundary of the droplet.

Assumptions

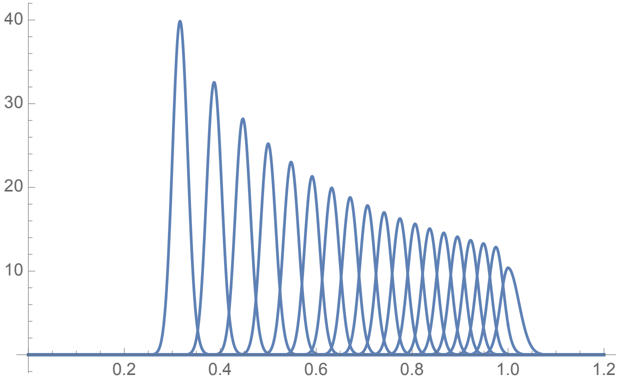
- Assumptions: (i) Growth at infinity: $\liminf_{z\to\infty} Q(z)/\log|z| > 2$.
  (ii) Q is real analytic and strictly subharmonic in S^*.
  (iii) The boundary ∂S is a smooth, simple, and closed curve.
- Fix $p \in \partial S$. Rescale

$$z_j = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta_j - p), \quad e^{i\theta} : \text{ outer normal to } \partial S \text{ at p.}$$

Weighted orthogonal polynomials

For any ζ with $\mathrm{dist}(\zeta, \partial S) \le M/\sqrt{n}$ for large M, the higher degree terms $|p_{n,j}|^2 e^{-nQ_t}$ $(n - \sqrt{n}\log n \le j \le n-1)$ contribute to the kernel.

Graphs of $|p_{n,j}|^2 e^{-nQ_t}$ restricted on $\mathbb{R}$ when $Q(z) = |z|^2$ and $t = 0.5$.
$(n = 1000,\ j = 100, 150, \cdots, 1000)$



Approximate quasi-polynomials

Fix j with $n - \sqrt{n}\log n \le j \le n-1$ and write $\tau = j/n$.

- ∂S_τ is smooth, simple, and closed.
- Let $\phi_\tau \colon S_\tau^c \to \mathbb{D}^c$ be the conformal map s.t. $\phi_\tau(\infty) = \infty$, $\phi_\tau'(\infty) > 0$.
- $\check{Q}_\tau$: the obstacle function s.t. $\check{Q}_\tau = Q$ on S_τ and $\check{Q}_\tau \sim \tau \log |z|^2$ at ∞.
- $\mathcal{Q}_\tau$: the bounded holomorphic function on S_τ^c s.t. $\operatorname{Re}\mathcal{Q}_\tau = Q$ on ∂S_τ.
- $\mathcal{H}_\tau$: the bounded holomorphic function on S_τ^c s.t.

  $\operatorname{Re}\mathcal{H}_\tau = \frac{1}{2}\log \Delta Q - \log \Phi_{j,n}$ on ∂S_τ where $\Phi_{j,n}(\zeta) = \Phi_t\big(\frac{j-n}{\sqrt{n}} \cdot \frac{\phi_\tau'(\zeta)}{\sqrt{\Delta Q(\zeta)}}\big)$

The approximate quasi-polynomial of degree j is defined in a neighborhood of S_τ^c by

$$F_{n,j} = \left(\frac{n}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi_\tau'}\, \phi_\tau^j\, e^{n\mathcal{Q}_\tau/2}\, e^{\mathcal{H}_\tau/2}.$$

Error function approximation

Let $\{\zeta_j\}$ be the eigenvalues of random normal matrices associated with Q_t. The 1-point function of the system $\{\zeta_j\}$ can be approximated by

$$\mathbf{R}_n^t(\zeta) = \sum_{j=0}^{n-1} |p_{n,j}(\zeta)|^2 e^{-nQ_t(\zeta)} = \sum_{j=n-\sqrt{n}\log n}^{n-1} |F_{j,n}(\zeta)|^2 e^{-nQ_t(\zeta)} (1 + O(n^{-\frac{1}{2}+\delta}))$$

for all ζ with $\mathrm{dist}(\zeta, \partial S) = O(n^{-1/2})$.

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for all ζ with $\mathrm{dist}(\zeta, \partial S) = O(n^{-1/2})$.

Fix $p \in \partial S$. Let $\{z_j\}$ be the rescaled system $z_j = \sqrt{n\Delta Q(p)}\, e^{-i\theta}(\zeta_j - p)$. The 1-point function of the system $\{z_j\}$ is approximated by

$$R_n^t(z) = \frac{1}{n\Delta Q(p)} \mathbf{R}_n^t(\zeta) = \frac{1}{n\Delta Q(p)} \sum_{j=n-\sqrt{n}\log n}^{n-1} |F_{j,n}(\zeta)|^2 e^{-nQ_t(\zeta)} + o(1)$$

where $\zeta = p + e^{i\theta} \frac{z}{\sqrt{n\Delta Q(p)}}$.

Error function approximation

Applying Taylor expansion to the quasi-polynomial,

$$R_n^t(z) = \frac{1}{\sqrt{2\pi}} \frac{|\phi'(p)|}{\sqrt{n\Delta Q(p)}} \sum_{k=1}^{\sqrt{n}\log n} \frac{e^{-\frac{1}{2}(\xi_k - 2\,\mathrm{Re}\,z)^2}}{\Phi_t(\xi_k)} e^{2(1-t)(\mathrm{Re}\,z)^2 1_{\{\mathrm{Re}\,z<0\}}} + o(1),$$

where $\xi_k = -\frac{k}{\sqrt{n}} \cdot \frac{|\phi'(p)|}{\sqrt{\Delta Q(p)}}$.

By the Riemann sum approximation,

$$\begin{aligned}\lim_{n\to\infty} R_n^t(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-\frac{1}{2}(\xi - 2\,\mathrm{Re}\,z)^2}}{\Phi_t(\xi)}\, d\xi \;\cdot e^{2(1-t)(\mathrm{Re}\,z)^2 1_{\{\mathrm{Re}\,z<0\}}} \\ &= S_t(2\,\mathrm{Re}\,z)\; e^{2(1-t)(\mathrm{Re}\,z)^2 1_{\{\mathrm{Re}\,z<0\}}}.\end{aligned}$$

Universality results for radially symmetric potentials

Theorem [Ameur–Kang–S., 19]

Suppose that Q is radially symmetric. For $0 < t < \infty$, the rescaled process $\{z_j\}_1^n$ converges to a determinantal point process with correlation kernel

$$K^t(z,w) = G(z,w)S_t(z+\bar{w})e^{(1-t)((\operatorname{Re} z)^2 1_{\{\operatorname{Re} z<0\}} + (\operatorname{Re} w)^2 1_{\{\operatorname{Re} w<0\}})}$$

with locally uniform convergence of correlation functions.

- In the free boundary case ($t = 1$), the above theorem is proved for general potentials. [Hedenmalm–Wennman, 17]

Spectral radius

Assume that the potential Q is radially symmetric.

Let $|\zeta|_n = \max_{1\leq j\leq n}|\zeta_j|$ be the maximal modulus of eigenvalues associated with Q_t and r_0 be the radius of the outer boundary of the droplet.

Theorem [Ameur–Kang–S., 19]

Write $\gamma_n = \log(n/2\pi) - 2\log\log n + 2\log(r_0\sqrt{t\Delta Q(r_0)}/(\sqrt{t}+1))$. The random variable $\omega_n = \sqrt{4nt\gamma_n\Delta Q(r_0)}\left(|\zeta|_n - r_0 - \sqrt{\frac{\gamma_n}{4nt\Delta Q(r_0)}}\right)$ converges in distribution to the Gumbel distribution: for $x\in\mathbb{R}$

$$\lim_{n\to\infty}\mathbf{P}_n(\omega_n\leq x) = e^{-e^{-x}}.$$

- The fluctuation of the maximal modulus of Ginibre ensemble converges in distribution to the Gumbel distribution. [Rider, 03]
- General radially symmetric potentials and $t=1$, [Chafai–Peche, 14]

Related results

- [S., 15] For $t = \infty$, put $\omega_n = \log 4 \cdot r_0 n \Delta Q(r_0)(|\zeta|_n - r_0)$. Then
$$\lim_{n\to\infty} \mathbf{P}_n(\omega_n \le x) = \min\{e^x, 1\}.$$
- [Butez, Garcia-Zeleda, 18] Consider $Q_n = (1 + \frac{1}{n})\check{Q}$ and suppose that the outer boundary of the droplet is unit circle. Then
$$\lim_{n\to\infty} \mathbf{P}_n(|\zeta|_n < x) = \prod_{k=1}^{\infty}(1 - x^{-2k}), \quad x > 1.$$

Thank you for your attention!