Harmonic Means of Wishart Matrices

Hi! I'm Asad Lodhia, I'm a postdoc at the University of Michigan.

Feel free to email me at alodhia@umich.edu

Based on joint work with Keith Levin and Liza Levina.

Thanks to the organizers for the invitation! I hope you enjoy it.

What's a Harmonic Mean?

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Take two positive definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$

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So the operator norm is smaller...

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Let $\mathbf{X}$ be $P \times N$, i.i.d complex standard normals $P < N$:

$$\left|\frac{P}{N} - \gamma\right| \leq \frac{K}{P^2}, \quad \text{for some } K > 0 \text{ and } \gamma \in (0, 1).$$

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The matrix $\mathbf{W} = \frac{\mathbf{XX}^*}{N}$ has limiting spectral measure

$$\rho_\gamma(x) := \frac{\sqrt{((1+\sqrt{\gamma})^2 - x)(x - (1-\sqrt{\gamma})^2)}}{2\gamma\pi x}$$

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This is invertible with probability 1.

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Optimizing the Frobenius Norm has been done. (Ledoit, Peché, Wolf)

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Closer now.

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Result: the limiting ESD is

$$\frac{n}{2\pi\gamma x}\sqrt{(e_+ - x)(x - e_-)}$$

where

$$e_\pm = 1 - \gamma + \frac{2\gamma}{n} \pm 2\sqrt{\frac{\gamma}{n}}\sqrt{1 - \gamma + \frac{\gamma}{n}}$$

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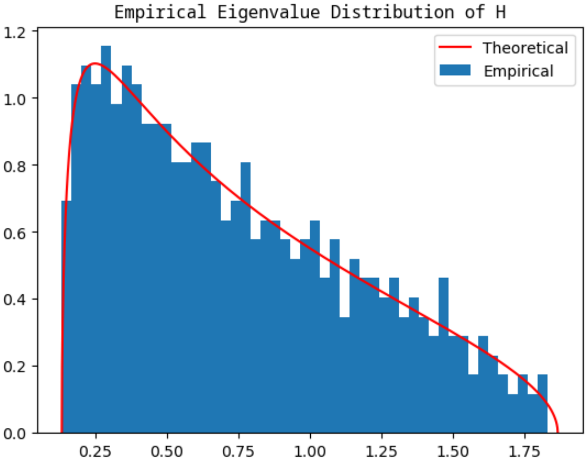
Also:

$$\lim_{P,N\to\infty} \|\mathbf{H} - \mathbf{I}\| = 1 - e_- = \gamma - \frac{2\gamma}{n} + 2\sqrt{\frac{\gamma}{n}}\sqrt{1 - \gamma + \frac{\gamma}{n}}$$

Figure of the ESD vs LSD $P = 500$, $N = 1000$ and $n = 2$



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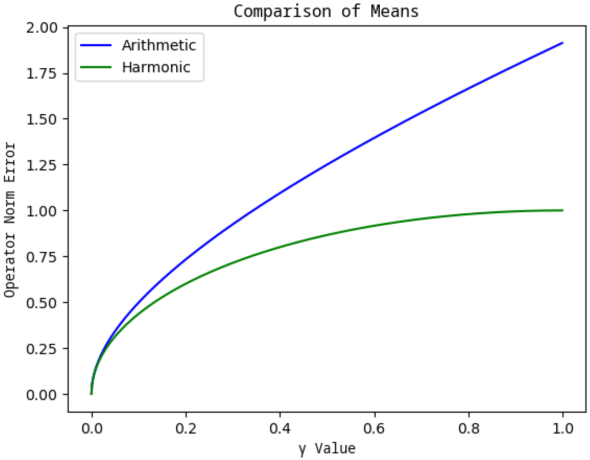
For $n = 2$ it's **always** true for $\gamma \in (0,1)$

$$\sqrt{2\gamma}\sqrt{1-\frac{\gamma}{2}} < \frac{\gamma}{2} + \sqrt{2\gamma}$$

Error Comparison for $n = 2$ as a function of γ



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Submultiplicative bound

$$\left\| \sqrt{\Sigma}\mathbf{H}\sqrt{\Sigma} - \Sigma \right\| \leq \left\| \sqrt{\Sigma}\mathbf{A}\sqrt{\Sigma} - \Sigma \right\| \frac{\|\Sigma\| \|\Sigma^{-1}\| \|\mathbf{H} - \mathbf{I}\|}{\|\mathbf{A} - \mathbf{I}\|}$$

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So if we have

$$\limsup_{P,N\to\infty} \frac{\|\Sigma\|\|\Sigma^{-1}\|\|\mathbf{H}-\mathbf{I}\|}{\|\mathbf{A}-\mathbf{I}\|} < 1$$

then

$$\limsup_{P,N\to\infty} \frac{\left\|\sqrt{\Sigma}\mathbf{H}\sqrt{\Sigma}-\Sigma\right\|}{\left\|\sqrt{\Sigma}\mathbf{A}\sqrt{\Sigma}-\Sigma\right\|} < 1$$

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$$c < \frac{5}{4}\sqrt{\frac{4}{3}} \approx 1.44337567\ldots$$

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More on general Σ later.

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Then

$$\lim_{P,T\to\infty} \|\mathbf{A} - \mathbf{I}\| = \Gamma + 2\sqrt{\Gamma}$$

and

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The argmin is 2! If T is at least twice P split your data in two and take the harmonic mean.

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Then $\mathbf{W}_i$ are asymptotically free and Q is a non-commutative polynomial (result due to Donati-Martin and Capitaine)

$$\lim_{P,N\to\infty} \|Q(\mathbf{W}_1, \dots, \mathbf{W}_n)\| = \|Q(\mathbf{p}_1, \dots, \mathbf{p}_n)\|_{\mathcal{F}}$$

the variables $\mathbf{p}_j$ are freely independent non-commutative Free Poisson Random Variables.

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There is a unit vector e_0 in the hilbert space such that

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Free independence means

$$\nu\left(\prod_{l=1}^{n} \left\{ Q_l(\mathbf{p}_l) - \nu[Q_l(\mathbf{p}_l)] \right\} \right) = 0$$

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Let μ be a measure compactly supported on the positive reals,

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Define $\mu_1 \boxplus \mu_2$ as the measure such that

$$R_\mu(z) := K_\mu(z) - \frac{1}{z}$$
$$R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z)$$

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If we compute

$$nR_{\mathbf{p}^{-1}}(z) = \sum_{i=1}^{n} R_{\mathbf{p}_i^{-1}}(z)$$

we can obtain the Stieltjest transform of

$$n\mathbf{h}^{-1}$$

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Plug in $z = K_{\mathbf{p}_j^{-1}}(w)$ and then plug in the R-transform.

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Calculation is quick but how to justify and why does the operator norm converge?

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We prove that there exists a $\kappa > 0$:

$$\mathbf{P}(\{\max(\|\mathbf{W}_i\|, \|\mathbf{W}_i^{-1}\|, \|\mathbf{H}\|, \|\mathbf{H}^{-1}\|) > \kappa\})$$

is summable in P.

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Pair of fixed point equation for $\mathbf{e} = \lim \sqrt{\Sigma}\mathbf{H}\sqrt{\Sigma} - \Sigma$

$$m_{\mathbf{e}}(z) = \int \frac{F(dx)}{z - \gamma x\{\frac{1}{n}(zm_{\mathbf{e}}(z)-1)S_{\mathbf{h}-1}(zm_{\mathbf{e}}(z)-1) + \frac{1}{n}zm_{\mathbf{e}}(z)-1\}},$$

and

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Acknowledgments and Advertisement

Anna Maltsev is looking for a PhD Student at Queen Mary University of London. Ask me more questions

Thanks to Alice Guionnet, Alan Edelman and Jinho Baik for helpful comments and suggestions.