Properties of Free Multiplicative Convolution

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Random Matrices and Related Topics
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Wishart ensemble

- Suppose that we have sample N independent random vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$ from N-dimensional complex standard Gaussian distribution.
- Then their *sample covariance matrix* is defined by

$$XX^*, \quad \text{where} \quad X = (x_{ij})_{1\le i,j\le N} = (\mathbf{x}_1, \cdots, \mathbf{x}_N).$$

- X and XX^* are known as (complex) *Ginibre* and *Wishart ensemble*.

Marčenko-Pastur law

The empirical distribution of eigenvalues of $N^{-1}XX^*$ converges to the Marčenko-Pastur distribution μ_{MP}:

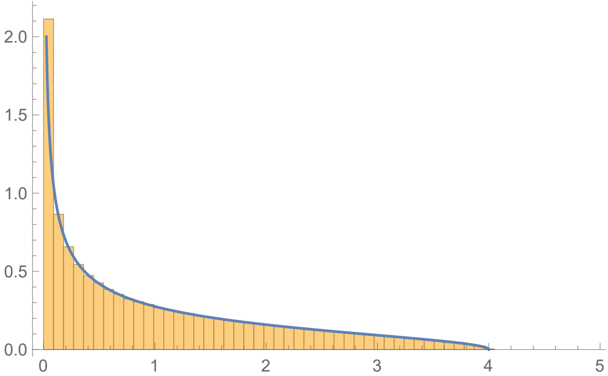


Figure: Histogram of eigenvalues of $N^{-1}XX^*$ with $N = 5000$ and density $\frac{1}{2\pi}\sqrt{\frac{(4-x)}{x}}$ of Marčenko-Pastur distribution.

Sample covariance matrix with general population

- In some occasions, we wish to consider the case in which the law of $\mathbf{x}$ is *non-standard* Gaussian, so that the variables are dependent.
- Thus we take $\mathbf{y}_i := D\mathbf{x}_i$ where D is another $(N \times N)$ matrix, called *population matrix*. In this case, the sample covariance matrix becomes

$$YY^* = (\mathbf{y}_1, \cdots, \mathbf{y}_N)(\mathbf{y}_1, \cdots, \mathbf{y}_N)^T = DXX^*D^*,$$

referred as *non-white Wishart ensemble*.

deformed Marčenko-Pastur law

- If the e.s.d. of D^*D converges to a probability measure ν, then that of DXX^*D^* also converges(Marčenko and Pastur, 1967 [8]).
- The limiting measure was characterized by an integral equation satisfied by its Stieltjes transform.

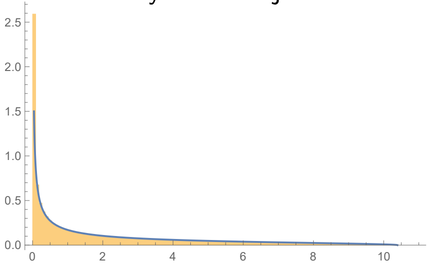


Figure: Eigenvalues of DXX^*D^* where e.s.d. of DD^* converges to the arcsine distribution $\mu_{\mathrm{AS}}(\mathrm{d}x) := \frac{1}{\pi}\frac{1}{\sqrt{x(4-x)}}\mathrm{d}x$

The limit is *"free multiplicative convolution"* of ν and μ_{MP}.

Stieltjes transform and M-function

Definition 1

For a probability measure μ on $\mathbb{R}_+ := [0, \infty)$, we define its *Stieltjes transform* and *M-function* by, for $z \in \mathbb{C} \setminus \mathbb{R}_+$,

$$m_\mu(z) := \int \frac{1}{x - z} \mathrm{d}\mu(x), \quad \text{and} \quad M_\mu(z) = 1 - \left(\int \frac{x}{x - z} \mathrm{d}\mu(x) \right)^{-1}.$$

Remark

- $M_\mu(z) = 1 - (zm_\mu(z) + 1)^{-1}$
- $m_\mu, M_\mu : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C} \setminus \mathbb{R}_+$ are analytic.
- $M_\mu : (-\infty, 0) \xrightarrow{\sim} (-\infty, -\mu(0)/(1 - \mu(0)))$ is increasing.
- M_μ^{-1} is analytic in a neighborhood of $(-\infty, -\mu(0)/(1 - \mu(0)))$.

Free multiplicative convolution

Definition 2

For two probability measures μ and ν on $[0, \infty)$, both not δ_0, $\mu \boxtimes \nu$ is the unique probability measure satisfying

$$M_{\mu \boxtimes \nu}^{-1}(z) = \frac{1}{z} M_{\mu}^{-1}(z) M_{\nu}^{-1}(z)$$

in a neighborhood of $(-\infty, -C)$.

Remark

If X and Y are *free* random variables with distributions μ and ν, then $\mu \boxtimes \nu$ is the distribution of $\sqrt{X} Y \sqrt{X}$ (or $\sqrt{Y} X \sqrt{Y}$).

Free additive convolution

Definition 3

For two probability measures μ and ν on $\mathbb{R}$, $\mu \boxplus \nu$ is the unique probability measure satisfying

$$F^{-1}_{\mu\boxplus\nu}(z) = F^{-1}_{\mu}(z) + F^{-1}_{\nu}(z) - z$$

in a neighborhood of $(\mathrm{i}M, \mathrm{i}\infty)$, where $F_\mu(z) := -1/m_\mu(z)$.

Remark

- If X and Y are *free* random variables with distributions μ and ν, then $\mu \boxplus \nu$ is the distribution of $X + Y$.
- As X and Y are *noncommutative*, $\log(XY) = \log X + \log Y$ and $\mathrm{e}^{X+Y} = \mathrm{e}^X \mathrm{e}^Y$ are no longer true.

Connection to random matrices

- U_N: $(N \times N)$- Haar distributed random unitary matrix.
- $C_N = \mathrm{diag}(c_1^{(N)}, \cdots, c_N^{(N)}), D_N = \mathrm{diag}(d_1^{(N)}, \cdots, d_N^{(N)})$ such that

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{c_i^{(N)}} \to \mu \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N} \delta_{d_i^{(N)}} \to \nu, \quad \text{as } N \to \infty.$$

- $(\lambda_1^{(N)}, \cdots, \lambda_N^{(N)})$: eigenvalues of $C_N + U_N D_N U_N^*$,
  $(\gamma_1^{(N)}, \cdots, \gamma_N^{(N)})$: those of $\sqrt{C_N} U_N D_N U_N^* \sqrt{C_N}$ (for $C_N, D_N \geq 0$).

Theorem (Voiculescu, 1998 [9])

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^{(N)}} \to \mu \boxplus \nu \quad \textit{and} \quad \frac{1}{N}\sum_{i=1}^{N} \delta_{\gamma_i^{(N)}} \to \mu \boxtimes \nu.$$

We may replace $U_N D_N U_N^*$ with Wishart ensemble and ν with μ_{MP}.

Marčenko-Pastur distribution revisited

- $\mu_{\mathrm{MP}} = \lim_{n\to\infty} ((n-1)\delta_0/n + \delta_1/n)^{\boxplus n}$ is also known as *free Poisson law*.
- In general for any $a \geq 1$, $((n-a)\,\delta_0/n + a\delta_1/n)^{\boxplus n}$ converges to

$$\mu_{\mathrm{MP}}^{(a)}(\mathrm{d}x) := \frac{1}{2\pi x}\sqrt{4a-(x-(a+1))^2}\mathrm{d}x.$$

- The measure $\mu_{\mathrm{MP}}^{(a)}$'s are also the limiting e.s.d. of the general sample covariance $N^{-1}XX^*$, where X is $(N \times M)$ random matrix whose entries are i.i.d. and $M/N \to a$.
- In fact, $\mu_{\mathrm{MP}}^{(a)}$ are also $\boxtimes$-infinitely divisible, so that the common properties of $\mu_{\mathrm{MP}}^{(a)}$ are "desirable" in terms of the operation $\boxtimes$.

Marčenko-Pastur distribution revisited

Examples of “desirable” properties are..

- Having density, which is analytic in the bulk of spectrum.
- The density being bounded by $1/x$.
- The density decaying as square root at the edges.

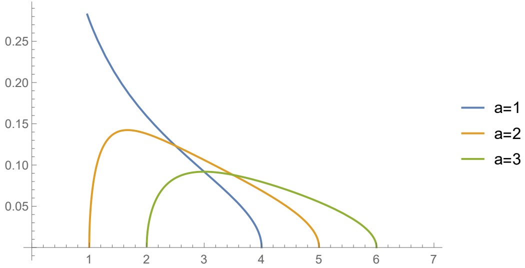


Figure: Densities of $\mu_{\mathrm{MP}}^{(a)}$

These properties of $\mu_{\mathrm{MP}}^{(a)}$ hold even for convolution of *two* measures, under proper assumptions.

Regularity(Lebesgue decomposition) of free convolution

Let $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$. Then $\mu \boxtimes \mu$ can be explicitly calculated as

$$\frac{1}{2}\delta_0 + \frac{1}{2\pi\sqrt{x(4-x)}}\mathbb{1}_{(0,4)}(x)\mathrm{d}x.$$

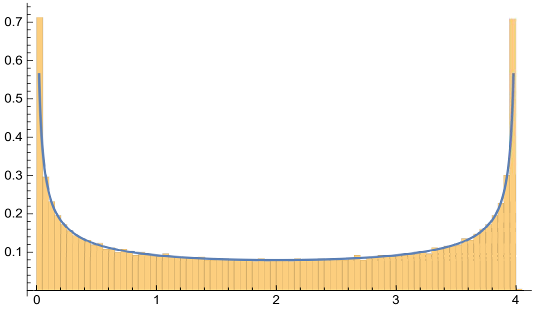


Figure: Nonzero eigenvalues of $(5 \cdot 10^3)$ matrix $\sqrt{C}UDU^*\sqrt{C}$, where $\{c_i, d_i\}$ are i.i.d. with law μ and U is independent of C and D.

Known results for free additive convolution: Lebesgue decomposition

Theorem (Belinschi, 2008 [4])

Let μ and ν be Borel probability measures on $\mathbb{R}$, both not a point mass.

(i) $(\mu \boxplus \nu)(\{a\}) > 0$ *if and only if there exist* $b, c \in \mathbb{R}$ *with* $a = b + c$ *and* $\mu(\{b\}) + \nu(\{c\}) > 1$.
In this case, $(\mu \boxplus \nu)(\{a\}) = \mu(\{b\}) + \nu(\{c\}) - 1$.

(ii) $(\mu \boxplus \nu)^{\mathrm{sc}} \equiv 0$.

(iii) $\frac{\mathrm{d}}{\mathrm{d}x}(\mu \boxplus \nu)^{\mathrm{ac}}(x)$ *is analytic whenever positive and finite.*

Lebesgue decomposition

Theorem 1 (J., 2019 [7])

Let μ and ν be Borel probability measures on $\mathbb{R}_+$, both not a point mass.

(i*) *For $c > 0$, $(\mu \boxtimes \nu)(\{c\}) > 0$ if and only if there exist $u, v \in (0, \infty)$ with $uv = c$ and $\mu(\{u\}) + \nu(\{v\}) > 1$.*
In this case, $(\mu \boxtimes \nu)(\{c\}) = \mu(\{u\}) + \nu(\{v\}) - 1$.

(ii*) $(\mu \boxtimes \nu)(\{0\}) = \max(\mu(\{0\}), \nu(\{0\}))$.

(iii) $(\mu \boxtimes \nu)^{\mathrm{sc}} \equiv 0$.

(iv) $\frac{\mathrm{d}(\mu \boxtimes \nu)^{\mathrm{ac}}(x)}{\mathrm{d}x}$ *is analytic whenever positive and finite.*

*First two statements were proved in Belinschi, 2003 [3].

Boundedness of the density

Letting $\mu = (1-p)\delta_0 + p\delta_{1/p}$, we find that $\mu \boxtimes \mu$ *almost* have an atom at p^{-2} if $p = 1/2$. Figures below show what happens if $p < 1/2$.

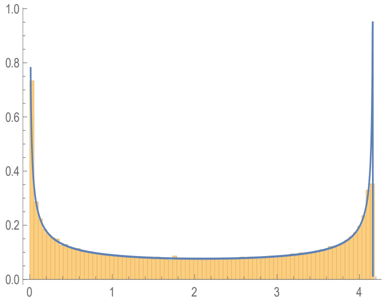


Figure: Density of $(\mu \boxtimes \mu)^{\mathrm{ac}}$ where $\mu = 0.51\delta_0 + 0.49\delta_{1/0.49}$.

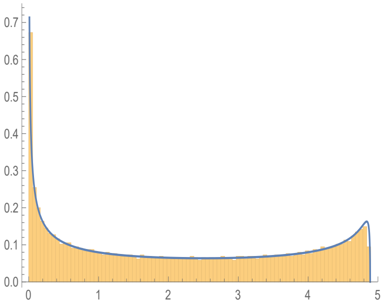


Figure: Density of $(\mu \boxtimes \mu)^{\mathrm{ac}}$ where $\mu = 0.55\delta_0 + 0.45\delta_{1/0.45}$.

Known results for free additive convolution: Boundedness of the density

Theorem (Belinschi, 2013 [5])

Let μ and ν be Borel probability measures on $\mathbb{R}$, both not a point mass. If F_μ and F_ν are continuous at infinity and $\mu(\{b\}) + \nu(\{c\}) < 1$ for all $b, c \in \mathbb{R}$, then $\mu \boxplus \nu = (\mu \boxplus \nu)^{\mathrm{ac}}$ and the density is bounded and continuous.

Remark

The density of μ_{MP} diverges as $x^{-1/2}$ around x, thus we need different statement to cover μ_{MP}.

Boundedness of the density

Theorem 2 (J., 2019 [7])

Let μ and ν be probability measures on $\mathbb{R}_+$ such that M_μ and M_ν are continuous at 0 and ∞. Further assume that $\mu(\{a\}) + \nu(\{b\}) < 1$ for all $a, b \in (0, \infty)$. Then the density of $(\mu \boxtimes \nu)^{\mathrm{ac}}$ is continuous and uniformly $O(x^{-1})$ on $(0, \infty)$.

Remark

By Theorem 1 (i), $\mu \boxtimes \nu$ can have point mass at 0 under the assumptions of Theorem 2.

Square root behavior at the edges

U: Haar unitary matrix, X_1, X_2: Ginibre ensembles

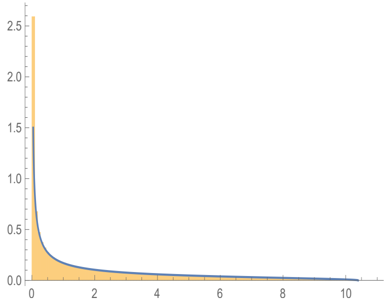
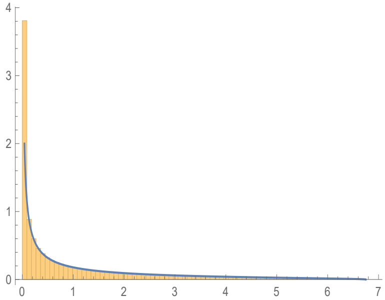


Figure: Limiting e.s.d. $\mu_{\mathrm{MP}} \boxtimes \mu_{\mathrm{AS}}$ of $(I+U)X_1X_1^*(I+U^*)$.



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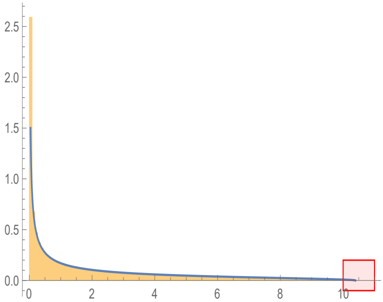

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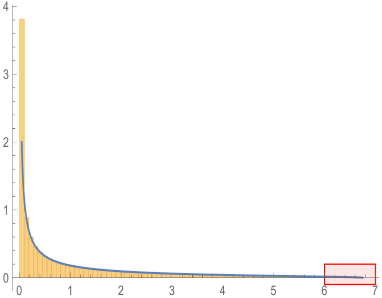

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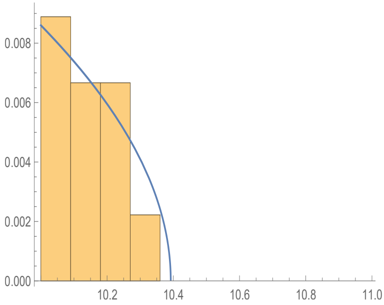

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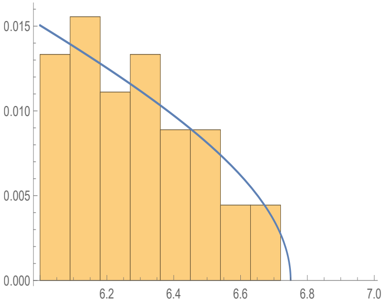


Figure: Limiting e.s.d. $\mu_{\mathrm{MP}} \boxtimes \mu_{\mathrm{MP}}$ of $X_2X_1X_1^*X_2^*$.

Known results for free additive convolution: Square root behavior

Assumption 1

Let μ and ν be Borel probability measures on $\mathbb{R}$ satisfying the following:

(i) They have densities; $\mathrm{d}\mu(x) = \rho_\mu(x)\mathrm{d}x$, $\mathrm{d}\nu(x) = \rho_\nu(x)\mathrm{d}x$.

(ii) $\operatorname{supp}\rho_\mu = [E_-^\mu, E_+^\mu]$, $\operatorname{supp}\rho_\nu = [E_-^\nu, E_+^\nu]$.

(iii) The measures are *Jacobi*; there exist $-1 < t_\pm^\mu, t_\pm^\nu < 1$ and a constant $C > 1$ such that

$$C^{-1} \leq \frac{\rho_\mu(x)}{(x - E_-^\mu)^{t_-^\mu}(E_+^\mu - x)^{t_+^\mu}} \leq C, \quad \text{for a.e. } x \in [E_-^\mu, E_+^\mu],$$

and the same bound holds for ν.

Known results for free additive convolution: Square root behavior

Theorem (Bao, Erdős and Schnelli, 2018 [2])

Under Assumption 1, there exist $E_- < E_+$ *and* $\gamma_+, \gamma_- > 0$ *such that*

- $\{E \in \mathbb{R} : \rho(x) > 0\} = (E_-, E_+)$ *so that* $\mathrm{supp}(\mu \boxplus \nu) = [E_-, E_+]$,
- $$\lim_{x \searrow E_-} \frac{\rho(x)}{\sqrt{x - E_-}} = \gamma_-, \qquad \lim_{x \nearrow E_+} \frac{\rho(x)}{\sqrt{E_+ - x}} = \gamma_+,$$

where $\rho(x)$ *is the continuous density of* $\mu \boxplus \nu$.

Square root behavior at the edges

Theorem 3 (J. 2019 [7])

Let μ and ν be probability measures on $\mathbb{R}_+$ satisfying Assumption 1 and $E_-^{\mu}, E_-^{\nu} > 0$. Then there exist $0 < E_- < E_+$ and $\gamma_+, \gamma_- > 0$ such that

- $\{E \in \mathbb{R} : \rho(x) > 0\} = (E_-, E_+)$ *so that* $\operatorname{supp}(\mu \boxtimes \nu) = [E_-, E_+]$,
- $$\lim_{x \searrow E_-} \frac{\rho(x)}{\sqrt{x - E_-}} = \gamma_-, \qquad \lim_{x \nearrow E_+} \frac{\rho(x)}{\sqrt{E_+ - x}} = \gamma_+,$$

where $\rho(x)$ is the continuous density of $\mu \boxtimes \nu$.

Analytic subordination functions

Proposition 1

Let μ and ν be probability measures on $\mathbb{R}_+$, both not δ_0. There exist unique analytic self-maps Ω_μ and Ω_ν of $\mathbb{C} \setminus \mathbb{R}_+$ satisfying the following:

(i) $\lim_{z\to-\infty} \Omega_\mu(z) = \lim_{z\to-\infty} \Omega_\nu(z) = -\infty$;

(ii) *For all* $z \in \mathbb{C}_+$, $\Omega_\mu(\bar{z}) = \overline{\Omega_\mu(z)}$, $\Omega_\nu(\bar{z}) = \overline{\Omega_\mu(z)}$, *and*

$$\arg \Omega_\mu(z) \geq \arg z, \quad \arg \Omega_\nu(z) \geq \arg z;$$

(iii) *(Subordination) For all* $z \in \mathbb{C} \setminus \mathbb{R}_+$,

$$M_\mu(\Omega_\nu(z)) = M_\nu(\Omega_\mu(z)) = M_{\mu\boxtimes\nu}(z);$$

(iv) *(Free multiplicative convolution) For all* $z \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\Omega_\mu(z)\Omega_\nu(z) = zM_{\mu\boxtimes\nu}(z).$$

Characterization of the edge

Let μ and ν satisfy the assumptions of Theorem 3.

- By Theorem 1 (iv) and Theorem 2,
  (*Edges* of $\operatorname{supp} \mu \boxtimes \nu$) $\equiv$ (points at which *analyticity* of $M_{\mu\boxtimes\nu}$ breaks).
- Using the subordination functions,

$$M_{\mu\boxtimes\nu}(z) = M_\nu(\Omega_\mu(z)), \quad \forall z \in \mathbb{C} \setminus \mathbb{R}_+.$$

- The subordination functions extend continuously to $\mathbb{R}$.

$E \in \mathbb{R}$ being an edge of $\operatorname{supp} \mu \boxtimes \nu$ implies either

$$(\Omega_\mu \text{ is not analytic at } E) \quad \text{or} \quad (M_\nu \text{ is not analytic at } \Omega_\mu(E)).$$

Stability bounds

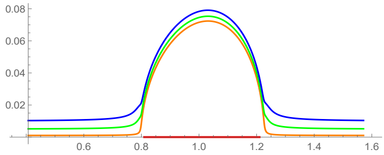


Figure: Graph(orange) of $\Omega_\mu(\cdot + 10^{-3}\mathrm{i})$ for $\mu_{\mathrm{MP}}^{(1.1)} \boxtimes \mu_{\mathrm{MP}}^{(1.1)}$ and the support(red).

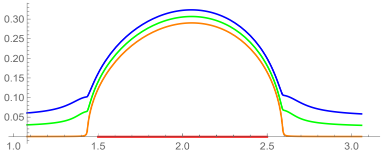


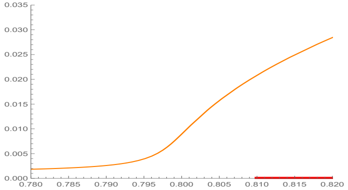
Figure: Graph(orange) of $\Omega_\mu(\cdot + 10^{-3}\mathrm{i})$ for $(\mu_{\mathrm{AS}}^{(1.5,3,5)})^{\boxtimes 2}$ and the support(red).

Proposition 2

Let μ and ν satisfy assumptions of Theorem 3. Then there exists a constant $c > 0$ such that

$$\inf_{z\in\mathbb{C}_+} \operatorname{dist}(\Omega_\mu(z), \operatorname{supp}\nu) \geq c, \qquad \inf_{z\in\mathbb{C}_+} \operatorname{dist}(\Omega_\nu(z), \operatorname{supp}\mu) \geq c.$$

Stability bounds



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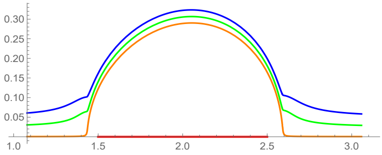

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Singularity of subordination functions

From Proposition 1, we find the following heuristic equality:

$$
\begin{aligned}
&zM_\nu(\Omega_\mu(z)) = zM_{\mu\boxtimes\nu}(z) = \Omega_\mu(z)\Omega_\nu(z) \\
=&\Omega_\mu(z)M_\mu^{-1}(M_{\mu\boxtimes\nu}(z)) = \Omega_\mu(z)(M_\mu^{-1}\circ M_\nu \circ \Omega_\mu)(z),
\end{aligned}
$$

Thus Ω_μ has an inverse $\widetilde{z}$ given by

$$
\widetilde{z}(\Omega) = \frac{\Omega M_\mu^{-1}\circ M_\nu(\Omega)}{M_\nu(\Omega)}.
$$

By inverse function theorem, we can guess that the analyticity of Ω_μ breaks at z if $\widetilde{z}'(\Omega_\mu(z)) = 0$.

Characterization of the edges

Proposition 3

Define $\mathcal{V} := \partial\{x \in \mathbb{R} : \rho(x) > 0\}$. *For* $z \in \mathbb{C}_+ \cup \mathbb{R}$, *the following holds:*

$$\left| \left(\frac{\Omega_\nu(z)}{M_\mu(\Omega_\nu(z))} M_\mu'(\Omega_\nu(z)) - 1 \right) \left(\frac{\Omega_\mu(z)}{M_\nu(\Omega_\mu(z))} M_\nu'(\Omega_\mu(z)) - 1 \right) \right| \leq 1.$$

Furthermore, the equality holds if and only if $z \in \mathcal{V}$. *In this case, the equality remains true without taking the absolute value of LHS.*

Remark

(i) In fact, $\widetilde{z}'(\Omega_\mu(z)) = 0$ is equivalent to the equality without modulus.

(ii) We can prove that $\mathcal{V}$ consists of exactly two points $\{E_-, E_+\}$.

Square root behavior

We can prove that $\widetilde{z}''(\Omega_\mu(E_\pm)) \neq 0$, so that in a neighborhood of E_+,

$$\begin{aligned} z &= \widetilde{z}(\Omega_\mu(z)) \\ &= E_+ + \widetilde{z}''(\Omega_\mu(E_+))(\Omega_\mu(z) - \Omega_\mu(E_+))^2 + o(|\Omega_\mu(z) - \Omega_\mu(E_+)|^3). \end{aligned}$$

Inverting the expansion, we have

$$\Omega_\mu(z) = c\sqrt{z - E_+} + o(|z - E_+|^{3/2}).$$

Recalling that

$$E\rho(E) = \frac{1}{\pi}\operatorname{Im}\Omega_\mu(E + \mathrm{i}0)\int \frac{x}{|x - \Omega_\mu(E_+ + \mathrm{i}0)|^2}\mathrm{d}\nu(x),$$

we have the square root behavior around E_+.

Behavior at the hard edge

U: Haar unitary matrix, X_1, X_2: Ginibre ensembles

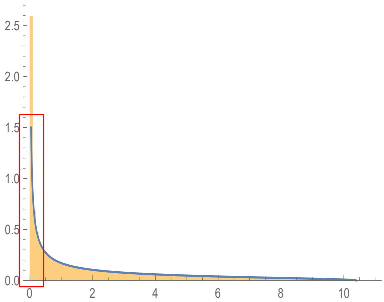

Figure: Limiting e.s.d. $\mu_{\mathrm{MP}} \boxtimes \mu_{\mathrm{AS}}$ of $(I+U)XX^*(I+U^*)$.

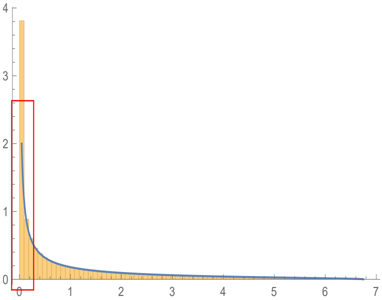

Figure: Limiting e.s.d. $\mu_{\mathrm{MP}} \boxtimes \mu_{\mathrm{MP}}$ of $X_2X_1X_1^*X_2^*$.

Both densities diverge as $x^{-2/3}$ as $x \to 0$.

Behavior at the hard edge

So far, when both of the measures μ and ν are separated from the *hard edge*, $\mu \boxtimes \nu$ shared the same property with $\mu \boxplus \nu$.

Theorem (Banica, Belinschi, Capitaine and Collins, 2011 [1])

The density ρ_s of the fractional power $\mu_{\mathrm{MP}}^{\boxtimes s}$ of $\mu_{\mathrm{MP}} \equiv \mu_{\mathrm{MP}}^{(1)}$ satisfies

$$\rho_s(x) \sim \frac{1}{\pi} x^{-\frac{s}{s+1}} \quad \text{as } x \to 0.$$

Remark

- It implies that the bound $O(1/x)$ in Theorem 2 is optimal.
- If supports of μ and ν touches 0, i.e. $E_-^{\mu} = E_-^{\nu} = 0$, Theorem 3 fails.

Converting the singularity

If a measure $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$ supported on $(0,\infty)$ satisfies $f(x) \sim x^{\alpha-1}$ so that

$$\mu((0,x]) \sim x^{\alpha} \quad \text{as } x \to 0, \tag{1}$$

where $\alpha \in (0,1)$, then for any realization X of μ, the distribution $\mu^{(-1)}$ of X^{-1} satisfies

$$\mu^{(-1)}((x,\infty)) \sim x^{-\alpha} \quad \text{as } x \to \infty.$$

Since $(\mu \boxtimes \nu)^{(-1)} = \mu^{(-1)} \boxtimes \nu^{(-1)}$, the case $E_{-}^{\mu} = E_{-}^{\nu} = 0$ can be converted to the case in which μ and ν are regularly varying around ∞.

Regular variation and M-function

Suppose that $\mu((x,\infty)) \sim x^{-\alpha}$ and $\nu((x,\infty)) \sim x^{-\beta}$, with $\alpha,\beta \in (0,1)$.

- The measures $x\mathrm{d}\mu(x)$ and $x\mathrm{d}\nu(x)$ satisfy, as $y \to +\infty$,

$$\int_0^y x\mathrm{d}\mu(x) \sim \frac{\alpha}{1-\alpha} y\mu((y,\infty)) \sim \frac{\alpha}{1-\alpha} y^{1-\alpha}, \quad \text{and}$$
$$\int_0^y x\mathrm{d}\nu(x) \sim \frac{\beta}{1-\beta} y\mu((y,\infty)) \sim \frac{\beta}{1-\beta} y^{1-\beta}.$$

- By Karamata's Abelian-Tauberian theorem, as $y \to +\infty$,

$$\int \frac{x\mathrm{d}\mu(x)}{x+y} \sim \frac{\alpha\pi}{\sin(\alpha\pi)} y^{-\alpha} \quad \text{and} \quad \int \frac{x\mathrm{d}\nu(x)}{x+y} \sim \frac{\beta\pi}{\sin(\beta\pi)} y^{-\beta}.$$

Regular variation and M-function

Recalling

$$M_\mu(y) = 1 - \left(\int \frac{x \mathrm{d}\mu(x)}{x-y} \right)^{-1}$$

and combining above results, as $w \searrow -\infty$ we get

$$M_{\mu \boxtimes \nu}^{-1}(w) \sim \frac{1}{w} \left(-\frac{\alpha \pi w}{\sin(\alpha\pi)} \right)^{\frac{1}{\alpha}} \left(-\frac{\beta \pi w}{\sin(\beta\pi)} \right)^{\frac{1}{\beta}} = -C_{\alpha,\beta}(-w)^{\frac{1}{\alpha}+\frac{1}{\beta}-1}.$$

Going backwards, we conclude

$$(\mu \boxtimes \nu)(x,\infty) \sim D_{\alpha,\beta} x^{-(\alpha^{-1}+\beta^{-1}-1)^{-1}}. \tag{2}$$

Remark

Hazra and Maulik [6] showed that for all $\alpha \geq 0$, any regualy varying μ with tail index α is *free subexponential*, i.e. $\mu^{\boxplus n}(x,\infty) \sim n\mu(x,\infty)$.

Behavior at the hard edge

Now substituting μ and ν by $\mu^{(-1)}$ and $\nu^{(-1)}$,

If μ and ν satisfy $\mu(0,x) \sim x^{\alpha}$ and $\nu(0,x) \sim x^{\beta}$ for some $\alpha, \beta \in (0,1)$, then

$$(\mu \boxtimes \nu)(0,x) \sim D_{\alpha,\beta} x^{(\alpha^{-1}+\beta^{-1}-1)^{-1}}.$$

- If we plug in $\alpha = \beta = 1/2$, then

$$(\alpha^{-1} + \beta^{-1} - 1)^{-1} = \frac{1}{3}, \tag{3}$$

which coincides with the case of $\mu_{\mathrm{MP}} \boxtimes \mu_{\mathrm{MP}}$ and $\mu_{\mathrm{MP}} \boxtimes \mu_{\mathrm{AS}}$.

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Thank you for listening!