Off-spectral analysis of Bergman kernels

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Subharmonic potentials

We let $\mathbb{C}$ be the complex plane. We let $Q : \mathbb{C} \to \mathbb{R}$ be a C^2-smooth potential, with sufficient growth at infinity:

$$\tau_Q := \liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 0.$$

CONES OF SUBHARMONIC FUNCTIONS

Let $\mathrm{SH}(\mathbb{C})$ denote the cone of subharmonic functions. Moreover, for $w_0 \in \mathbb{C}$ and $0 \le \tau < +\infty$, let

$$\mathrm{SH}_{\tau,w_0}(\mathbb{C}) := \big\{ q \in \mathrm{SH}(\mathbb{C}) : \ q(z) \le \tau \log|z - w_0| + \mathrm{O}(1) \ \text{ as } \ z \to w_0 \big\},$$

which we may refer to as the (τ, w_0)-pinched subharmonic functions. Informally, these are the subharmonic functions whose Laplacian has a point mass of magnitude at least τ at the point w_0.

Exponentially varying weights and associated Bergman spaces

Exponentially varying weights and Bergman spaces

For a positive real parameter m, let A^2_{mQ} denote the space of entire functions with finite norm

$$\|f\|^2_{mQ} := \int_{\mathbb{C}} |f(z)|^2 \mathrm{e}^{-2mQ(z)} \mathrm{d}A(z) < +\infty.$$

Here, $\mathrm{d}A$ denotes the area element, normalized so that the unit disk $\mathbb{D}$ gets unit area.

Nested subspaces of functions vanishing at a point

Let A^2_{mQ,n,w_0} denote the subspace

$$A^2_{mQ,n,w_0} := \left\{ f \in A^2_{mQ} : \ f(z) = \mathrm{O}(z - w_0)^n \text{ as } z \to w_0 \right\}.$$

This is a closed subspace of A^2_{mQ}.

Bergman kernels

Associated with the Hilbert space A^2_{mQ} is the inner product

$$\langle f, g\rangle_{mQ} := \int_{\mathbb{C}} f(z)\bar{g}(z)\,\mathrm{e}^{-2mQ(z)}\mathrm{d}A(z).$$

The Bergman kernel

The Bergman kernel for A^2_{mQ} is the function $K_m(\cdot,\zeta) \in A^2_{mQ}$ with

$$f(\zeta) = \langle f, K_m(\cdot,\zeta)\rangle_{mQ}, \qquad \zeta \in \mathbb{C}.$$

Correspondingly, the Bergman kernel for the subspace A^2_{mQ,n,w_0} is denoted by $K_{m,n,w_0}(\cdot,\zeta)$.

Root functions

The root function of of order n at w_0, denoted k_{m,n,w_0} is the solution to the optimization problem

$$\max\left\{ \operatorname{Re} f^{(n)}(w_0) :\ f \in A^2_{mQ,n,w_0},\ \|f\|_{mQ} \le 1 \right\}$$

provided the maximum exists as a positive number. Otherwise, we declare $\mathrm{k}_{m,n,w_0} = 0$.

The Bergman kernel in $\mathbb{C}^2$ for a tubular domain

Consider the tubular domain in $\mathbb{C}^2$ given by

$$\Omega_{mQ} := \big\{(z,w) \in \mathbb{C}^2 : |w| < \mathrm{e}^{-mQ(z)}\big\}.$$

Then the Bergman kernel $K_{\Omega_{mQ}}$ for the space $A^2(\Omega_{mQ})$ is such that

$$K_{\Omega_{mQ}}((z,0),(\zeta,0)) = \mathrm{const}\, K_{mQ}(z,\zeta),$$

where the constant depends on normalizations.

Pseudoconvexity

We use the defining function $\rho = \log|w| + mQ(z)$, in which case (when $w \neq 0$) the associated quadratic form of $(a,b) \in \mathbb{C}^2$ on the complex tangent plane $\frac{a}{2w} + m\frac{\partial Q}{\partial z} b = 0$ is

$$L(\rho)(a,b) = m\frac{\partial^2 Q}{\partial z \partial \bar{z}}|a|^2.$$

This quadratic form is positive semidefinite if and only if $\Delta Q \geq 0$, and consequently Ω_{mQ} is pseudoconvex if and only if $\Delta Q \geq 0$. Moreover, Ω_{mQ} is locally pseudoconvex wherever $\Delta Q \geq 0$.

Root functions and the Berman kernel

LEMMA

We have that

$$\frac{K_{m,n,w_0}(z,\zeta)}{K_{m,n,w_0}(\zeta,\zeta)^{1/2}} \to \mathrm{k}_{m,n,w_0}(z)$$

if ζ approaches w_0 along an appropriate direction.

Expansion of Bergman kernel in root functions

We have that

$$K_{m,n,w_0}(z,\zeta) = \sum_{l=n}^{+\infty} \mathrm{k}_{m,n,w_0}(z)\overline{\mathrm{k}_{m,n,w_0}(\zeta)}.$$

An obstacle problem

Let

$$\hat{Q}(z) := \sup\left\{q(z)\ q \in \mathrm{SH}(\mathbb{C}),\ q \le Q \text{ on } \mathbb{C}\right\},$$

and, analogously, for the (τ, w_0)-pinched problem,

$$\hat{Q}_{\tau,w_0}(z) := \sup\left\{q(z)\ q \in \mathrm{SH}_{\tau,w_0}(\mathbb{C}),\ q \le Q \text{ on } \mathbb{C}\right\}.$$

Spectral droplets

We put

$$\mathcal{S} := \{z \in \mathbb{C} :\ Q(z) = \hat{Q}(z)\}, \quad \mathcal{S}_{\tau,w_0} := \{z \in \mathbb{C} :\ Q(z) = \hat{Q}_{\tau,w_0}(z)\},$$

and call these spectral droplets (or spectra). The bulk of the spectral droplet $\mathcal{S}$ is the set

$$\mathrm{bulk}(\mathcal{S}) := \{z \in \mathrm{int}(\mathcal{S}) :\ \Delta Q(z) > 0\},$$

with the obvious modifications in the case of $\mathcal{S}_{\tau,w_0}$.

Note that $\mathcal{S}_{\tau,w_0}$ gets smaller as τ increases, starting with $\mathcal{S}$ for $\tau = 0$.

Illustration of the obstacle problem

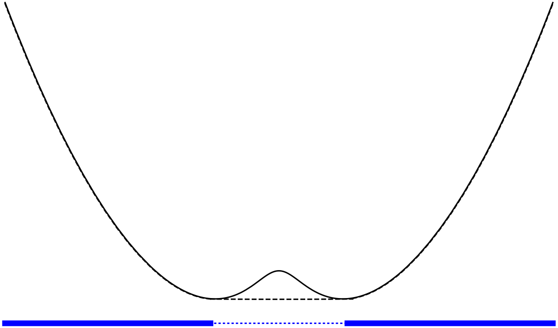

Figure: Illustration of the spectral droplet corresponding to the potential $Q(z) = |z|^2 - \log(a + |z|^2)$, with $a = 0.04$. The spectrum is illustrated with a thick line, and appears as the contact set between Q (solid) and the solution $\hat{Q}$ to the obstacle function (dashed).

illustration of a compact spectral droplet.

Figure: Illustration of a compact spectral droplet (shaded) with two simply connected holes. In this case there are three off-spectral components: the two holes as well as the unbounded component. If we think of this in the context of the Riemann sphere we may allow for the point at infinity to be inside the spectrum.

Admissible potentials Q

Admissibility

The C^2-smooth function $Q : \mathbb{C} \to \mathbb{R}$ is said to be admissible if
(i) $\tau_Q > 0$,
(ii) Q is real-analytically smooth and strictly subharmonic in a neighborhood of $\partial\mathcal{S}$,
(iii) there exists a bounded component Ω of the complement $\mathcal{S}^c = \mathbb{C} \setminus \mathcal{S}$ which is simply connected, with real-analytically smooth Jordan curve boundary.

In particular, we read off from (iii) that $\mathcal{S}^c$ must be nontrivial, and hence subharmonic Q are excluded from consideration under admissibility of Q.

(τ, w_0)-admissible potentials Q

Pinched admissibility

The C^2-smooth function $Q : \mathbb{C} \to \mathbb{R}$ is said to be (τ, w_0)-admissible if
(i) $0 \le \tau \le \tau_Q$,
(ii) Q is real-analytically smooth and strictly subharmonic in a neighborhood of $\partial\mathcal{S}_{\tau,w_0}$,
(iii) the point w_0 is an off-spectral point, i.e., $w_0 \notin \mathcal{S}_{\tau,w_0}$, and the component Ω_{τ,w_0} of the complement $\mathcal{S}^c_{\tau,w_0}$ containing w_0 is bounded and simply connected, with real-analytically smooth Jordan curve boundary.

For an interval $I \subset [0, +\infty[$, we speak of (I, w_0)-admissibility if we have (τ, w_0)-admissibility for each $\tau \in I$, while the associated domains Ω_{τ,w_0} change smoothly as τ moves in the interval I.
Here, subharmonic potentials Q are allowed, because $\mathcal{S}^c_{\tau,w_0}$ is automatically nontrivial for $\tau > 0$.

Some notation

Conformal mappings

Let φ_{w_0} denote the conformal mapping $\Omega \to \mathbb{D}$ with $\varphi_{w_0}(w_0) = 0$ and $\varphi'_{w_0}(w_0) > 0$, provided $w_0 \in \Omega$. In the pinched situation, we denote by φ_{τ,w_0} the conformal mapping $\Omega_{\tau,w_0} \to \mathbb{D}$ with $\varphi_{\tau,w_0}(w_0) = 0$ and $\varphi'_{\tau,w_0}(w_0) > 0$.

Complexification of Q

We let $\mathcal{Q}_{w_0}$ be the function which is bounded and holomorphic in Ω and whose real part equals Q along $\partial\Omega$, while $\operatorname{Im} \mathcal{Q}_{w_0}(w_0) = 0$. Analogously, in the pinched situation, we $\mathcal{Q}_{\tau,w_0}$ for the bounded holomorphic function in Ω_{τ,w_0} whose real part equals Q along $\partial\Omega_{\tau,w_0}$, while $\operatorname{Im} \mathcal{Q}_{w_0}(w_0) = 0$. These functions are tacitly extended holomorphically across the corresponding boundary curves.

Off-spectral expansion of the Bergman kernel

THEOREM I

Suppose Q an admissible potential. Then, given a positive integer κ and a positive real A, there exist a neighborhood $\Omega^{\langle\kappa\rangle}$ of the closure of Ω and bounded holomorphic functions $\mathcal{B}_{j,w_0}$ on $\Omega^{\langle\kappa\rangle}$ for $j = 0, \ldots, \kappa$, as well as domains $\Omega_m = \Omega_{m,\kappa,A}$ with $\Omega \subset \Omega_m \subset \Omega^{\langle\kappa\rangle}$ which meet

$$\mathrm{dist}_{\mathbb{C}}(\partial\Omega_m, \partial\Omega) \geq A\, m^{-\frac{1}{2}}(\log m)^{\frac{1}{2}},$$

such that the normalized Bergman kernel at the point w_0 enjoys the expansion

$$\mathrm{k}_m(z, w_0) = \frac{K_m(z, w_0)}{K_m(w_0, w_0)^{1/2}} = m^{\frac{1}{4}}(\varphi'_{w_0}(z))^{\frac{1}{2}}\mathrm{e}^{m\mathcal{Q}_{w_0}(z)} \times \left\{\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,w_0}(z) + \mathrm{O}(m^{-\kappa-1}\right\},$$

as $m \to +\infty$, where the error term is uniform on Ω_m.

The first term of the expansion

In the theorem, the first term $\mathcal{B}_{0,w_0}$ is obtained as the unique zero-free holomorphic function on Ω which is smooth up to the boundary, positive at w_0, with prescribed modulus on the boundary

$$|\mathcal{B}_{0,w_0}(z)| = (4\pi)^{-\frac{1}{4}} |\Delta Q(z)|^{\frac{1}{4}}, \qquad z \in \partial\Omega.$$

As for the later terms $\mathcal{B}_{j,w_0}$, with $j = 1, 2, 3, \ldots$, they may be obtain algorithmically. The expressions do get a bit large though.

The Gaussian wave associated with the normalized Bergman kernel

Associated with the normalized Bergman kernel we have the probability wave

$$|\mathrm{k}_m(z, w_0)|^2 \mathrm{e}^{-2mQ(z)}.$$

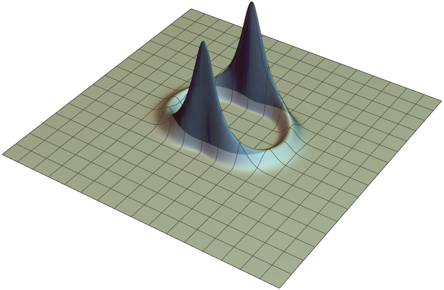

Figure: Illustration of the the probability wave of the Bergman kernel for $w_0 = 0$ and $Q(z) = \frac{1}{2}|z|^{-2} - \frac{1}{8}\,\mathrm{Re}(z^{-2}) + (1 + \frac{2}{m})\log|z|$.

Off-spectral expansion of root functions

THEOREM I

Suppose Q is (I_0, w_0)-admissible, where I_0 is compact. Then, given a positive integer κ and a positive real A, there exist a neighborhood $\Omega_{\tau,w_0}^{\langle\kappa\rangle}$ of the closure of Ω_{τ,w_0} and bounded holomorphic functions $\mathcal{B}_{j,w_0}$ on $\Omega_{\tau,w_0}^{\langle\kappa\rangle}$ for $j = 0, \ldots, \kappa$, as well as domains $\Omega_{\tau,w_0,m} = \Omega_{\tau,w_0,m,\kappa,A}$ with $\Omega_{\tau,w_0} \subset \Omega_{\tau,w_0,m} \subset \Omega_{\tau,w_0}^{\langle\kappa\rangle}$ which meet

$$\mathrm{dist}_{\mathbb{C}}(\Omega_{\tau,w_0,m}^{c}, \Omega_{\tau,w_0}) \geq A\, m^{-\frac{1}{2}}(\log m)^{\frac{1}{2}},$$

such that the root function of order n at w_0 enjoys the expansion

$$\mathrm{k}_{m,n,w_0}(z) = m^{\frac{1}{4}}(\varphi_{\tau,w_0}'(z))^{\frac{1}{2}}[\varphi_{\tau,w_0}(z)]^{n}\mathrm{e}^{m\mathcal{Q}_{w_0}(z)} \times \left\{ \sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau,w_0}(z) + \mathrm{O}(m^{-\kappa-1} \right\},$$

on $\Omega_{\tau,w_0,m}$ as $n = \tau m \to +\infty$ while $\tau \in I_0$, where the error term is uniform.

The first term in the expansion of root functions

In the theorem, the first term $\mathcal{B}_{0,\tau,w_0}$ is obtained as the unique zero-free holomorphic function on Ω_{τ,w_0} which is smooth up to the boundary, positive at w_0, with prescribed modulus on the boundary

$$|\mathcal{B}_{0,w_0}(z)| = (4\pi)^{-\frac{1}{4}}|\Delta Q(z)|^{\frac{1}{4}}, \qquad z \in \partial\Omega_{\tau,w_0}.$$

As for the later terms $\mathcal{B}_{j,w_0}$, with $j = 1, 2, 3, \ldots$, they may be obtain algorithmically. The expressions do get a bit large.

Boundary rescaled kernel and correlation kernel

We consider the rescaled density profile ("1-point function")

$$\varrho_m(\xi) = \varrho_{m,n,w_0}(\xi) = \frac{2}{m\Delta Q(z_m(\xi))} K_{m,n,w_0}(z_m(\xi), z_m(\xi))\,\mathrm{e}^{-2mQ(z_m(\xi))},$$

where

$$z_m(\xi) = z_0 + \frac{\nu\xi}{\sqrt{\frac{m}{2}\Delta Q(z_0)}},$$

is a way to blow up around the point z_0. Here, $n = \tau m$, and $\nu \in \mathbb{C}$ is a fixed unit vector, and ξ measures the deviation away from z_0.

BOUNDARY UNIVERSALITY

If z_0 is a boundary point for the spectrum and the associated hole Ω is smooth, while ν is the inward-pointing unit normal to $\partial\Omega$, then

$$\lim_{m\to+\infty} \varrho_m(\xi) = \mathrm{erf}(2\,\mathrm{Re}\,\xi) = (2\pi)^{-1/2} \int_{2\,\mathrm{Re}\,\xi}^{+\infty} \mathrm{e}^{-t^2/2}\mathrm{d}t,$$

with uniform convergence on compact subsets.

Some comments

Not just the density profile gets determined but the whole blow-up process as well. The key approach is (1) the expansion of the Bergman kernel in terms of root functions, and (2) the asymptotic expansion of root functions.

As for the asymptotics of the root functions, there is a non-local effect, coming from the fact that we begin at an off-spectral point. Compare the picture with the typical Gaussian probability wave associated with a bulk point.

The methods involve modifications of our earlier work of the asymptotics of planar orthogonal polynomials [HW1]. To be more precise, the Laplace method for area integral is used in the direction perpendicular to the boundary curve, and a foliation near $\partial\Omega$ is constructed in order to prove that the expansion suggested by Laplace's method is in fact correct.

The missing link: an elementary approach to local Bergman kernel expansions

We observe that if we consider the problem of maximizing $|f(z_0)|$ given that

$$\|f\|_{mQ}^2 = \int_{\mathbb{C}} |f(z)|^2 \mathrm{e}^{-2mQ(z)} \mathrm{d}A(z) = 1,$$

the answer is that the maximum is $K_m(z_0, z_0)^{\frac{1}{2}}$, and it is attained for $f = K_m(z_0, z_0)^{-\frac{1}{2}} K_m(\cdot, z_0)$. Without loss of generality, we put $z_0 = 0$, and we try to see how big we can get $|f(0)|$ given the integral condition, where (by locality) we may if necessary integrate instead over a fixed neighborhood of 0. By polar coordinates,

$$1 = \|f\|_{mQ}^2 = \int_0^{+\infty} \int_{-\pi}^{\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{e}^{-2mQ_r(\mathrm{e}^{\mathrm{i}\theta})} \mathrm{d}\theta \, r\mathrm{d}r,$$

where we write for convenience $Q_r(\mathrm{e}^{\mathrm{i}\theta}) := Q(r\mathrm{e}^{\mathrm{i}\theta})$.

The missing link II

To convert this into an estimate of $|f(0)|$, we write $\tilde{Q}_r$ for the harmonic extension of Q_r to the interior disk $\mathbb{D}$. The function

$$\zeta \mapsto |f(r\zeta)|^2 \mathrm{e}^{-2m\tilde{Q}_r(\zeta)}$$

is logarithmically subharmonic, hence subharmonic, so that by the sub-mean value property

$$|f(0)|^2 \mathrm{e}^{-2m\tilde{Q}_r(0)} \le \int_{-\pi}^{\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{e}^{-2mQ_r(\mathrm{e}^{\mathrm{i}\theta})} \frac{\mathrm{d}\theta}{2\pi}.$$

Integrating with respect to r as well we obtain

$$|f(0)|^2 \int_0^{+\infty} \mathrm{e}^{-2m\tilde{Q}_r(0)} r\mathrm{d}r \le \int_0^{+\infty} \int_{-\pi}^{\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{e}^{-2mQ_r(\mathrm{e}^{\mathrm{i}\theta})} \frac{\mathrm{d}\theta}{2\pi} r\mathrm{d}r = \frac{1}{2\pi},$$

and hence

$$|f(0)|^2 \le \frac{(2\pi)^{-1}}{\int_0^{+\infty} \mathrm{e}^{-2m\tilde{Q}_r(0)} r\mathrm{d}r}.$$

The missing link III

This estimate is rather crude but still is in the right ballpark. It is possible to show that

$$\tilde{Q}_r(0) \sim Q(0) + a_1 r^2 \Delta Q(0) + a_2 r^4 \Delta^2 Q(0) + \cdots$$

for some coefficients $a_1, a_2, \ldots$, which in its turn shows that our estimate obtains the first term in the expansion of $K_m(0,0)$. To do better, we introduce a local conformal mapping γ with $\gamma(0) = 0$ and replace Q by $Q \circ \gamma$ in the above argument. This is a surprisingly good idea, and it permits us to recover the next few terms in the expansion. However, to recover all terms we need to do even better. But what? We just need γ to be such that the restriction to a circle $|z| = r$ is the restriction of a conformal mapping $\tilde{\gamma}_r$ which fixes the origin, $\tilde{\gamma}_r(0) = 0$. So, γ need not be conformal itself, it suffices to glue it together from a continuous family of conformal mappings.

This was first worked out with Shimorin and later with Gaunard [GHS]. It is hoped that we may conclude this line of investigation with the assistance of Wennman.

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