Extreme gap problems in random matrix theory

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Joint density of eigenvalues of GβE

GβE: Given n point $\lambda_1, \cdots, \lambda_n$ ($\beta > 0$) with the joint density

$$J(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_{\beta,n}} \prod_{k=1}^{n} e^{-\frac{\beta n}{4}\lambda_k^2} \prod_{i<j} |\lambda_j - \lambda_i|^\beta \ ,$$

here, $Z_{\beta,n}$ is a norming constant which can be computed by the Selberg integral, $\beta = 1$ is corresponding to GOE, $\beta = 2$ for GUE, $\beta = 4$ for GSE.

Joint density of eigenvalues of CβE

CβE: Given n points on the unit circle $e^{i\theta_1}, \cdots, e^{i\theta_n}$ with joint density

$$J(\theta_1, \cdots, \theta_n) = \frac{1}{C_{\beta,n}} \prod_{i<j} \left| e^{i\theta_j} - e^{i\theta_i} \right|^{\beta} ,$$

$C_{\beta,n} = (2\pi)^n \frac{\Gamma(1+\beta n/2)}{(\Gamma(1+\beta/2))^n}$, $\beta = 1$ is corresponding to COE, $\beta = 2$ for CUE, $\beta = 4$ for CSE.

Extreme gaps I: smallest gaps for CUE

Let $e^{i\theta_1}, \cdots, e^{i\theta_n}$ be n eigenvalues of CUE, consider the 2-dimensional process of spacing of eigenangles and its position,

$$\chi_n = \sum_{i=1}^{n} \delta_{(n^{4/3}(\theta_{i+1}-\theta_i),\theta_i)}.$$

Theorem (Vinson, Soshinikov, Ben Arous-Bourgade)

χ_n tends to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A \times I) = \left(\frac{1}{24\pi}\int_A u^2 du\right)\left(\int_I \frac{du}{2\pi}\right).$$

Let $t_1^n < t_2^n \cdots < t_k^n$ be the first k smallest eigenangles gaps, denote $\tau_k^n = (72\pi)^{-1/3} t_k^n$, then as a consequence,

$$\lim_{n\to+\infty} \mathbb{P}(\tau_k^n \in [x, x+dx]) = \frac{3}{(k-1)!} x^{3k-1} e^{-x^3} dx.$$

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When β is an positive integer, consider 2-dimensional process

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Theorem (F.-Wei)

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$$\mathbb{E}\chi(A \times I) = \frac{A_\beta |I|}{2\pi} \int_A u^\beta du,$$

where $A_\beta = (2\pi)^{-1} \frac{(\beta/2)^\beta (\Gamma(\beta/2+1))^3}{\Gamma(3\beta/2+1)\Gamma(\beta+1)}$. In particular, the result holds for COE, CUE and CSE with

$$A_1 = \frac{1}{24}, \; A_2 = \frac{1}{24\pi}, \; A_4 = \frac{1}{270\pi}.$$

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Extreme gaps I: smallest gaps for CβE

Corollary

Let's denote t_k^n as the k-th smallest gap, and

$$\tau_k^n = n^{(\beta+2)/(\beta+1)} \times (A_\beta/(\beta+1))^{1/(\beta+1)} t_k^n,$$

then for any bounded interval $A \subset \mathbb{R}_+$, we have

$$\lim_{n\to+\infty} \mathbb{P}(\tau_k^n \in [x, x+dx]) = \frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} dx.$$

- No determinantal point process structure can be used as CUE (which is used by Soshinikov and Ben Arous-Bourgade, Figalli-Guionnet), we have to start from the Selberg integral
- Conjecture: The result must be true for any $\beta > 0$, but our method does not work other than integer β.

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Extreme gaps I: why such order heuristically?

We have the gap probability

$$\mathbb{P}(n(\theta_{j+1} - \theta_i) < x) \sim x^{\beta+1},$$

thus for a single gap

$$\mathbb{P}(s < x) = \mathbb{P}(ns < nx) \sim (nx)^{\beta+1}$$

if we treat the gaps 'independently', we have

$$\mathbb{E}(\#\{\text{gaps} < x\}) \sim n\mathbb{P}(s < x) \sim n(nx)^{\beta+1},$$

hence, we must have

$$x \sim n^{-\frac{\beta+2}{\beta+1}}$$

to get some nontrivial result.

Extreme gaps I: how we get A_β?

The constant A_β is very meaningful, it appears when one studied the kth factorial moment of χ_n. To prove χ_n (ignoring the position) tends to Poisson, we may consider the process with k-pair of smallest gaps,

$$\rho_n = \sum \delta_{n^{\frac{\beta+2}{\beta+1}}(\theta_{i_2}-\theta_{i_1}),\cdots,n^{\frac{\beta+2}{\beta+1}}(\theta_{i_{2k}}-\theta_{i_{2k-1}})}.$$

We proved that

$$\mathbb{E}\rho_n(A^k) \to (A_\beta \int_A u^\beta du)^k,$$

where

$$A_\beta^k = \lim_{n\to+\infty} \frac{Z_{\beta,n-2k,k}}{Z_{\beta,n} n^{k\beta}}.$$

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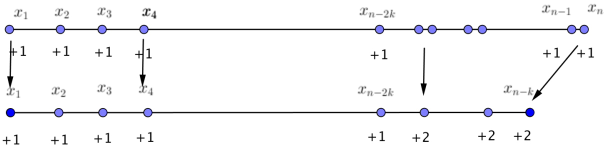

For one-component log-gas of n particles with charge $+1$,

$$Z_{\beta,n} = \int_{[0,2\pi]^n} \prod_{1\le i<j\le n} \left| e^{i\theta_j} - e^{i\theta_i} \right|^{\beta} d\theta_1 \ldots d\theta_n.$$

For two-component log-gas of $n-2k$ particles of charge $+1$ and k particles of charge $+2$,

$$Z_{\beta,n-2k,k} = \int_{[0,2\pi]^{n-k}} \prod_{1\le i<j\le n-k} \left| e^{i\theta_j} - e^{i\theta_i} \right|^{q_i q_j \beta} d\theta_1 \ldots d\theta_{n-k}$$

where $q_i = 1$ for $1 \le i \le n-2k$; $q_i = 2$ for $n-2k+1 \le i \le n-k$.

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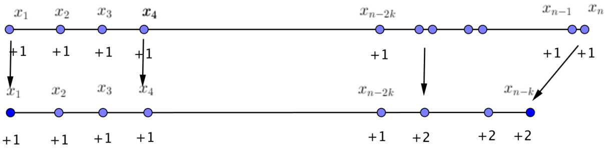

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Extreme gaps II: smallest gaps for GUE

Consider the 2-dimensional process of (interior) eigenvalues of GUE

$$\chi_n = \sum_{i=1}^{n} \delta_{(n^{\frac{4}{3}}(\lambda_{i+1}-\lambda_i),\lambda_i)} 1_{|\lambda_i|<2-\eta}$$

Theorem (Vinson, Soshinikov, Ben Arous-Bourgade)

χ_n tends to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A \times I) = \left(\frac{1}{48\pi^2}\int_A u^2 du\right)\left(\int_I (4-x^2)^2 dx\right),$$

where $A \subset \mathbb{R}_+$ and $I \subset (-2+\eta, 2-\eta)$.

The k-th smallest gaps $\tau_k^n = (\int_I (4-x^2)^2 dx/144\pi^2)^{1/3} t_k^n$ has the limiting density $\frac{3}{(k-1)!}x^{3k-1}e^{-x^3}$, same as CUE.

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Extreme gaps II: smallest gaps for GOE

Consider the 1-dimensional process of eigenvalues of GOE

$$\chi^{(n)} = \sum_{i=1}^{n-1} \delta_{n^{3/2}(\lambda_{(i+1)} - \lambda_{(i)})}$$

Theorem (F.-Tian-Wei)

$\chi^{(n)}$ converges to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A) = \frac{1}{4}\int_A u du.$$

Let's denote t_k as the k-th smallest gaps, and $\tau_k = 2^{-3/2} n^{3/2} t_k$, then the limiting density is

$$\frac{2}{(k-1)!} x^{2k-1} e^{-x^2}.$$

Extreme gaps II: conjectures for GβE

We conjecture that the smallest gaps of GβE and CβE are the same, i.e., there exists c_β such that $\tau_k^n = c_\beta n^{(\beta+2)/(\beta+1)} t_k$ has the limiting density

$$\frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}}.$$

The conjecture should be true for more general universal ensembles,

$$\frac{1}{Z_{n,\beta,V}} e^{-n\beta \sum_{i=1}^{n} V(\lambda_i)} \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta.$$

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Extreme gaps III: order of largest gaps

Let's denote m_k as the kth largest gap of eigenangles of CUE or the kth largest gap in the interior of the semicircle law of GUE, i.e., $m_1 > m_2 > m_3 \cdots$.

Theorem (Ben Arous-Bourgade, AOP 2013)

For any $p > 0$ and $l_n = n^{o(1)}$, one has

$$\frac{n m_{l_n}}{\sqrt{32 \ln n}} \xrightarrow{L^p} 1.$$

Extreme gaps III: why such order heuristically?

The gap probability of CUE is Toeplitz determinant

$$D_n(\alpha) := \mathbb{P}(\text{no eigenangle in } (-\alpha, \alpha)) = \det_{1 \le j,k \le n} \left(\frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} e^{i(j-k)\theta} d\theta \right).$$

One has asymptotic expansion (proved by Deift et al)

$$\ln D_n(\alpha) = n^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} \ln \left(n \sin \frac{\alpha}{2} \right) + c_0 + O\left(\frac{1}{n \sin(\alpha/2)} \right)$$

where

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1).$$

Substituting $u = 2\alpha = \sqrt{\lambda \log n}/n$, the expectation of the number of gaps greater than u is

$$n\mathbb{P}(\theta_2 - \theta_1 > u) = -\frac{1}{2} \frac{dD(\alpha)}{d\alpha} = n^{1-\lambda/32+o(1)},$$

thus one may expect the constant $\lambda = 32$.

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Extreme gaps III: fluctuation of largest gaps

Theorem (F.-Wei)

Let's denote m_k as the k-th largest gap of CUE, and

$$\tau_k = (2\ln n)^{\frac{1}{2}}(nm_k - (32\ln n)^{\frac{1}{2}})/4 - (3/8)\ln(2\ln n),$$

then $\{\tau_k^*\}$ will tend to a Poisson distribution and we have the limit of the Gumbel distribution,

$$\lim_{n\to+\infty} \mathbb{P}(\tau_k \in I) = \int_I \frac{e^{k(c_1-x)}}{(k-1)!} e^{-e^{c_1-x}} dx.$$

Here, $c_1 = c_0 + \ln\frac{\pi}{2}$. In particular, the limiting density for the largest gap τ_1 is,

$$e^{c_1-x}e^{-e^{c_1-x}}.$$

Wigner's semicircle law

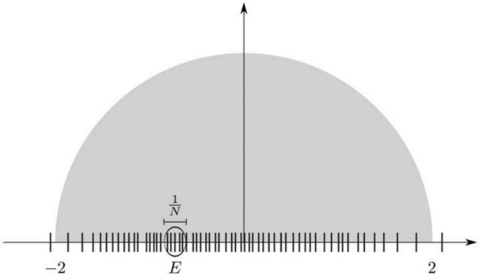

Figure: Density of eigenvalues of GUE

Globally, the largest gap is on the edge of the semicircle law which is indicated by Tracy-Widom law, so one has to look at the bulk regime.

Extreme gaps III: fluctuation of largest gaps

Theorem (F.-Wei)

Let's denote m_k as the k-th largest gap in the interior of GUE, $S(I) = \inf_I \sqrt{4-x^2}$ and

$$\tau_k^* = (2\ln n)^{\frac{1}{2}}(nS(I)m_k - (32\ln n)^{\frac{1}{2}})/4 + (5/8)\ln(2\ln n),$$

then $\{\tau_k^*\}$ will tend to a Poisson distribution and we have the limit of the Gumbel distribution,

$$\lim_{n\to+\infty} \mathbb{P}(\tau_k^* \in I_1) = \int_{I_1} \frac{e^{k(c_2-x)}}{(k-1)!} e^{-e^{c_2-x}} dx.$$

Here, $c_2 = c_0 + M_0(I)$ depending on I, where
$M_0(I) = (3/2)\ln(4-a^2) - \ln(4|a|)$ if $a+b<0$,
$M_0(I) = (3/2)\ln(4-b^2) - \ln(4|b|)$ if $a+b>0$,
$M_0(I) = (3/2)\ln(4-a^2) - \ln(2|a|)$ if $a+b=0$.

Extreme gaps III: fluctuation of largest gaps

- In both proofs, one of the essential parts is to find the correct rescaling factors.
- The most essential part is to show that the rescaling largest gaps are asymptotic to some Poisson processes, i.e., they are asymptotically independent.
- We do not know how to work for COE/GOE, CSE/GSE, but we can guess the order, it's $\sqrt{\frac{64}{\beta}\ln n}/n$ for CβE/GβE, but how to prove?

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Extreme gaps IV: universality of extreme gaps

Recently, our results are generalized for Hermitian/symmetric Wigner matrices with mild assumptions.

- P. Bourgade, Extreme gaps between eigenvalues of Wigner matrices, arXiv:1812.10376.
- B. Landon, P. Lopatto, J. Marcinek, Comparison theorem for some extremal eigenvalue statistics, arXiv:1812.10022.

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- Small gaps of circular beta-ensemble, arXiv:1806.01555
- Small gaps of GOE, arXiv:1901.01567.

Thank you for your attention!