May 9, 2019

Seoul

Periodically weighted tilings and (matrix) orthogonal polynomials

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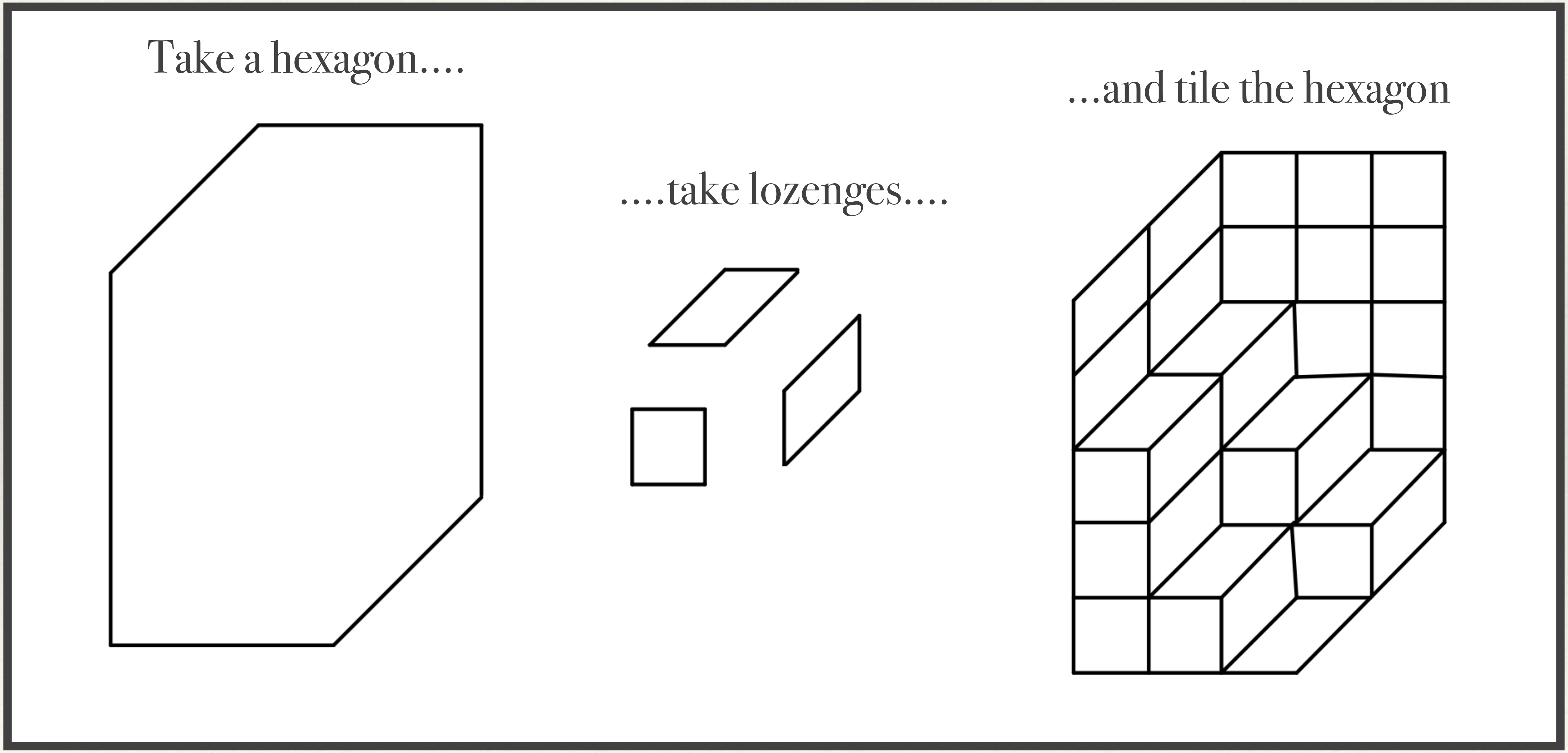
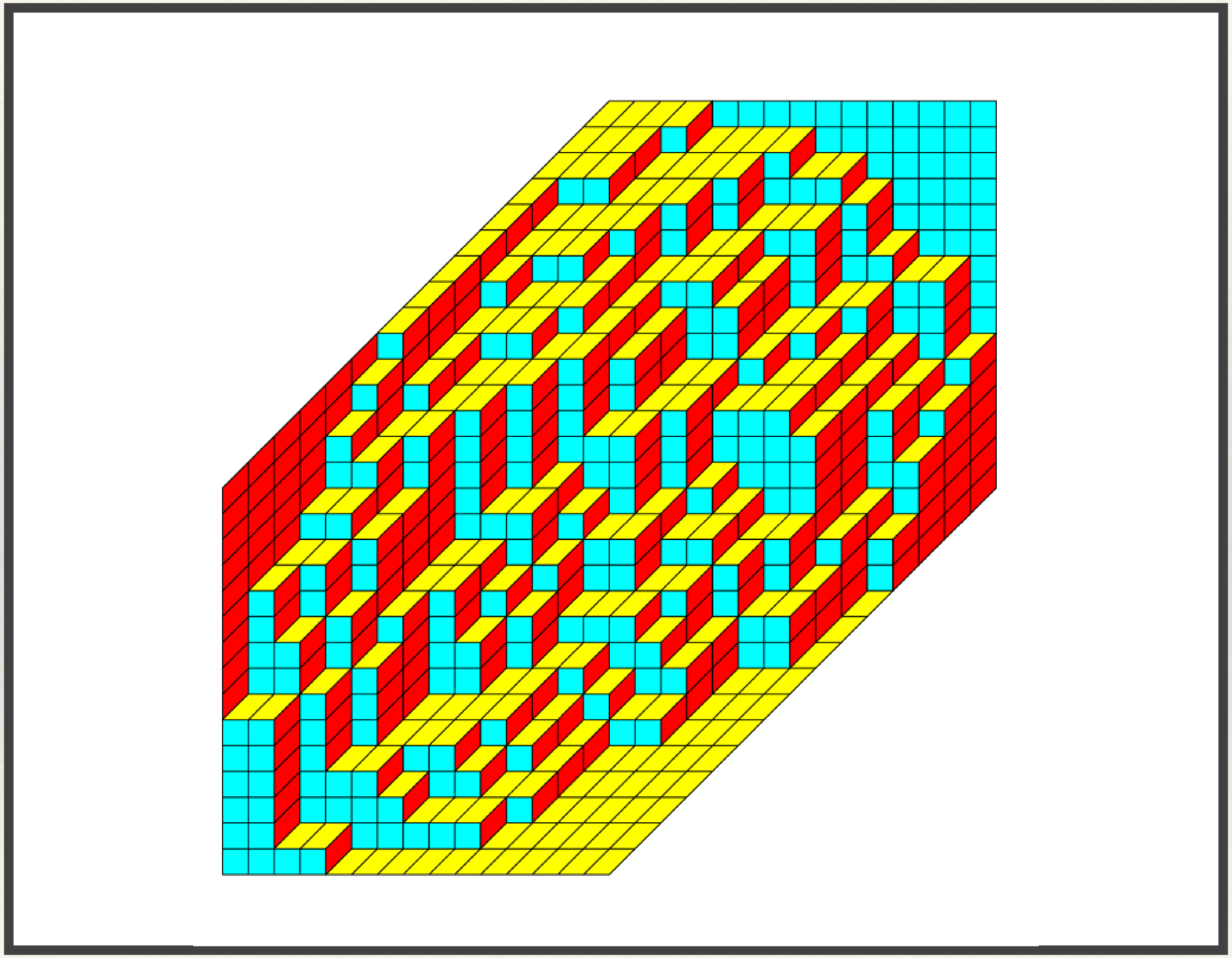
Based on joint works with :

- *A.B.J. Kuijlaars, The two periodic Aztec diamond and Matrix orthogonal polynomials, to appear in JEMS, arXiv:1712.05636*
- *C. Charlier, A.B.J. Kuijlaars and J. Lenells, A periodic hexagon tiling model and non-Hermitian orthogonal polynomials, (upcoming)*
- *T. Berggren, Correlation functions for determinantal point processes defined by infinite block Toeplitz minors, arXiv:1901.*

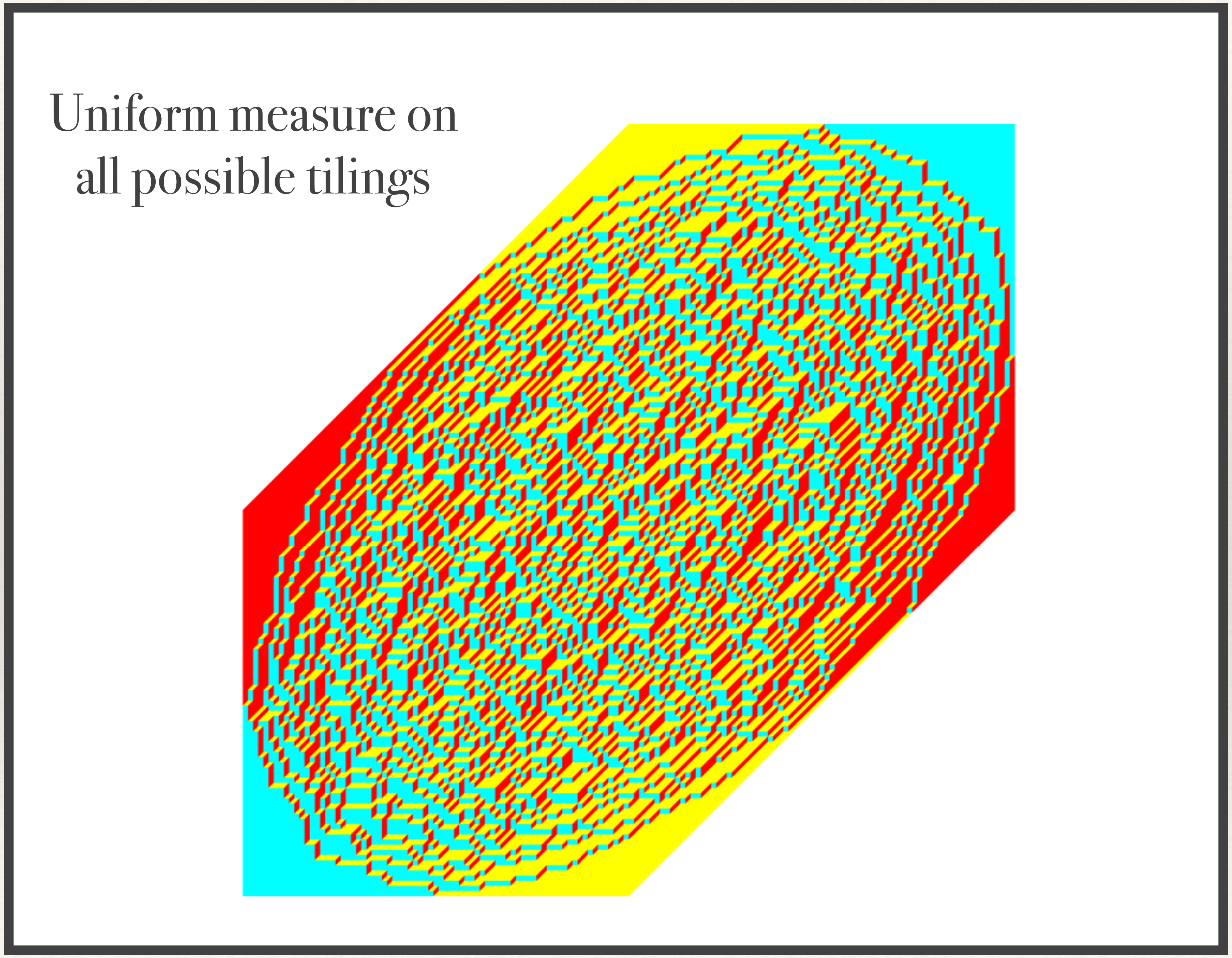
Lozenge tilings of the hexagon



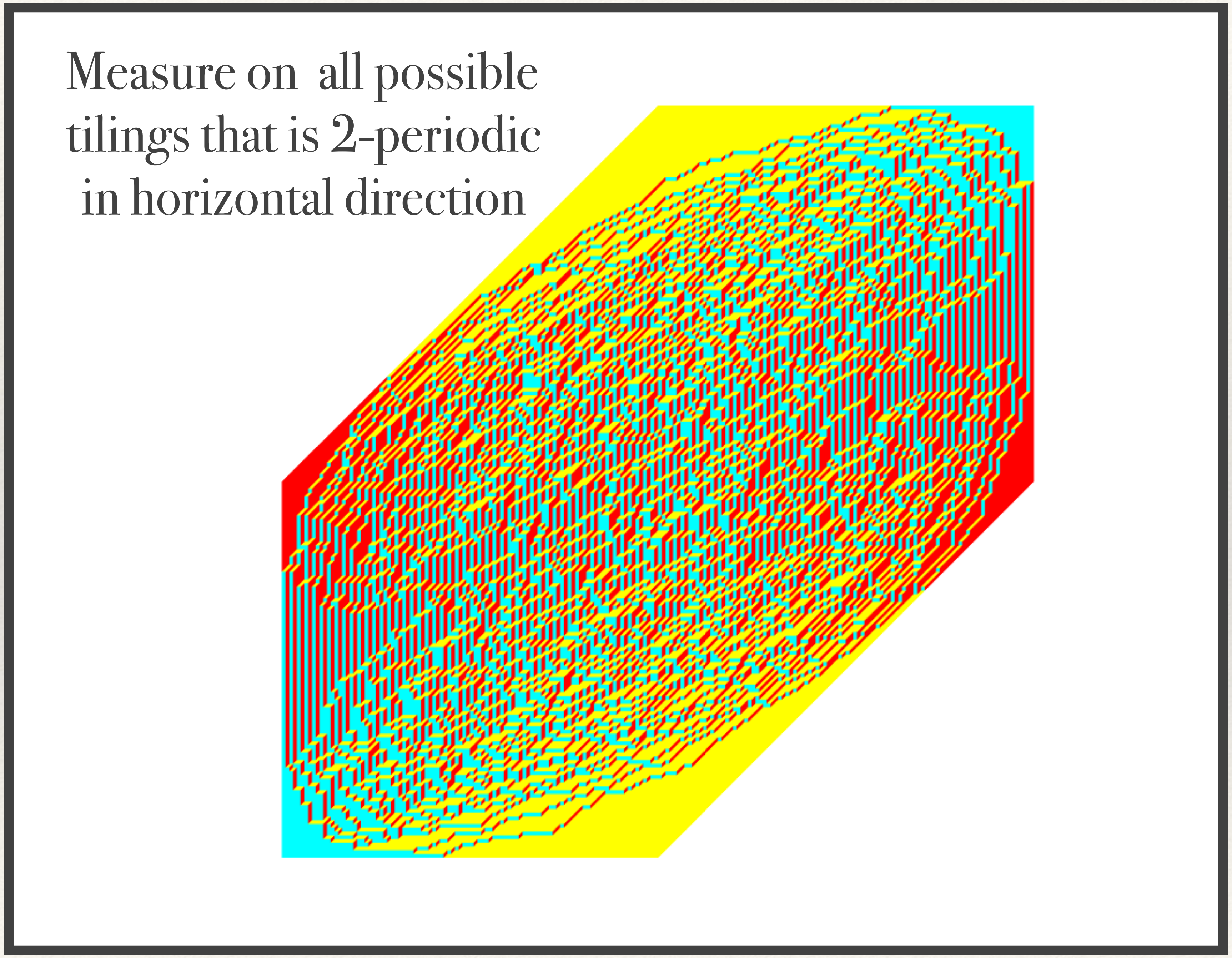
Lozenge tilings of the hexagon



Random lozenge tilings large hexagons

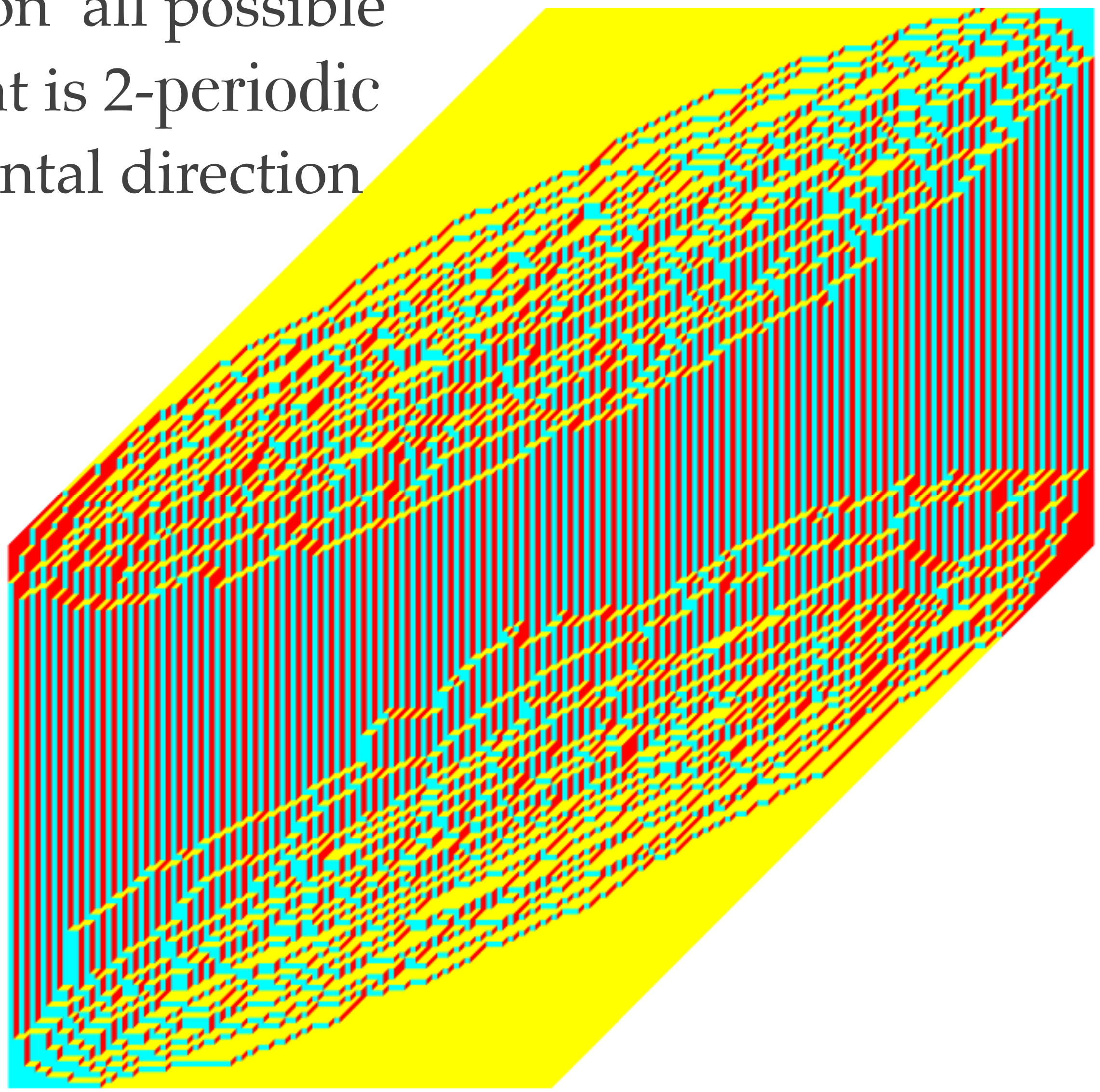


Lozenge tilings of the hexagon



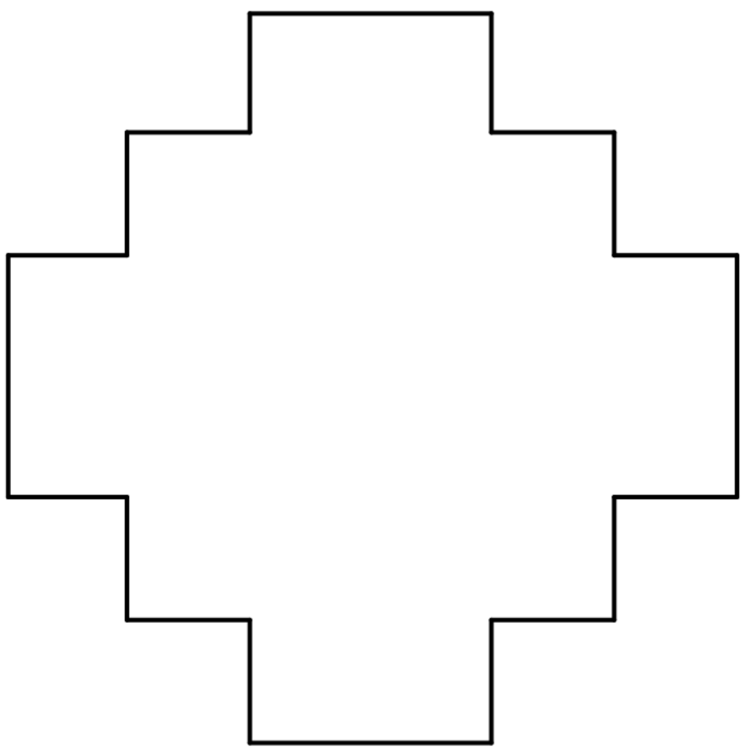
Lozenge tilings of the hexagon

Measure on all possible
tilings that is 2-periodic
in horizontal direction

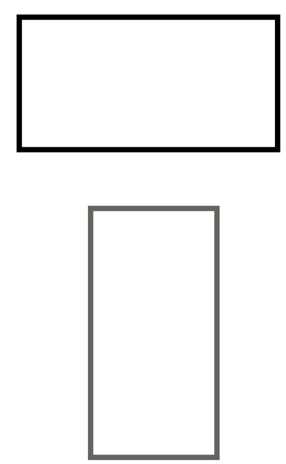


Domino tilings of an Aztec diamond

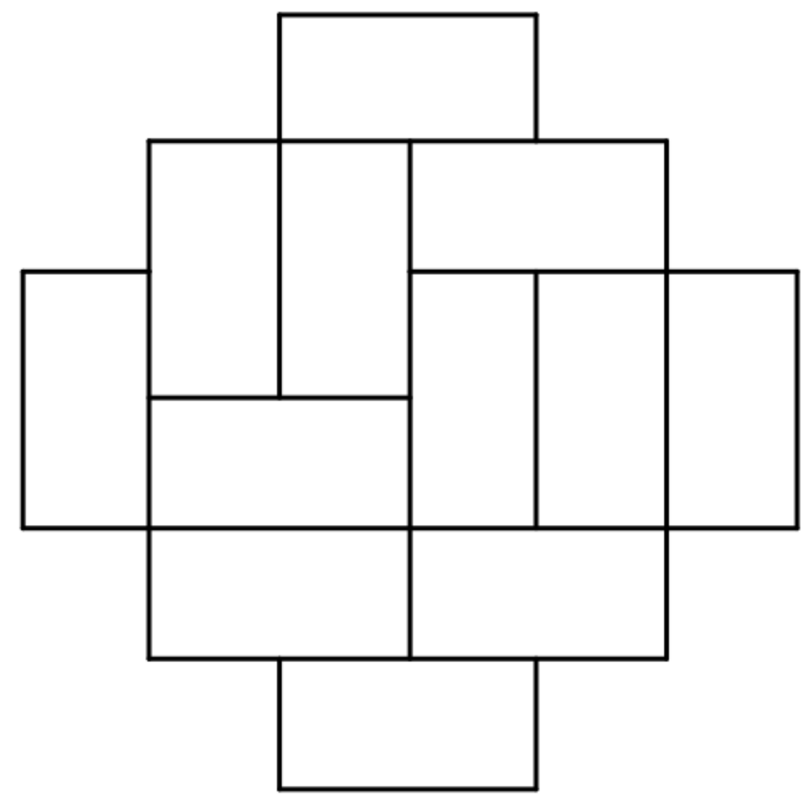
Take an Aztec diamond....



....take horizontal and vertical dominos....

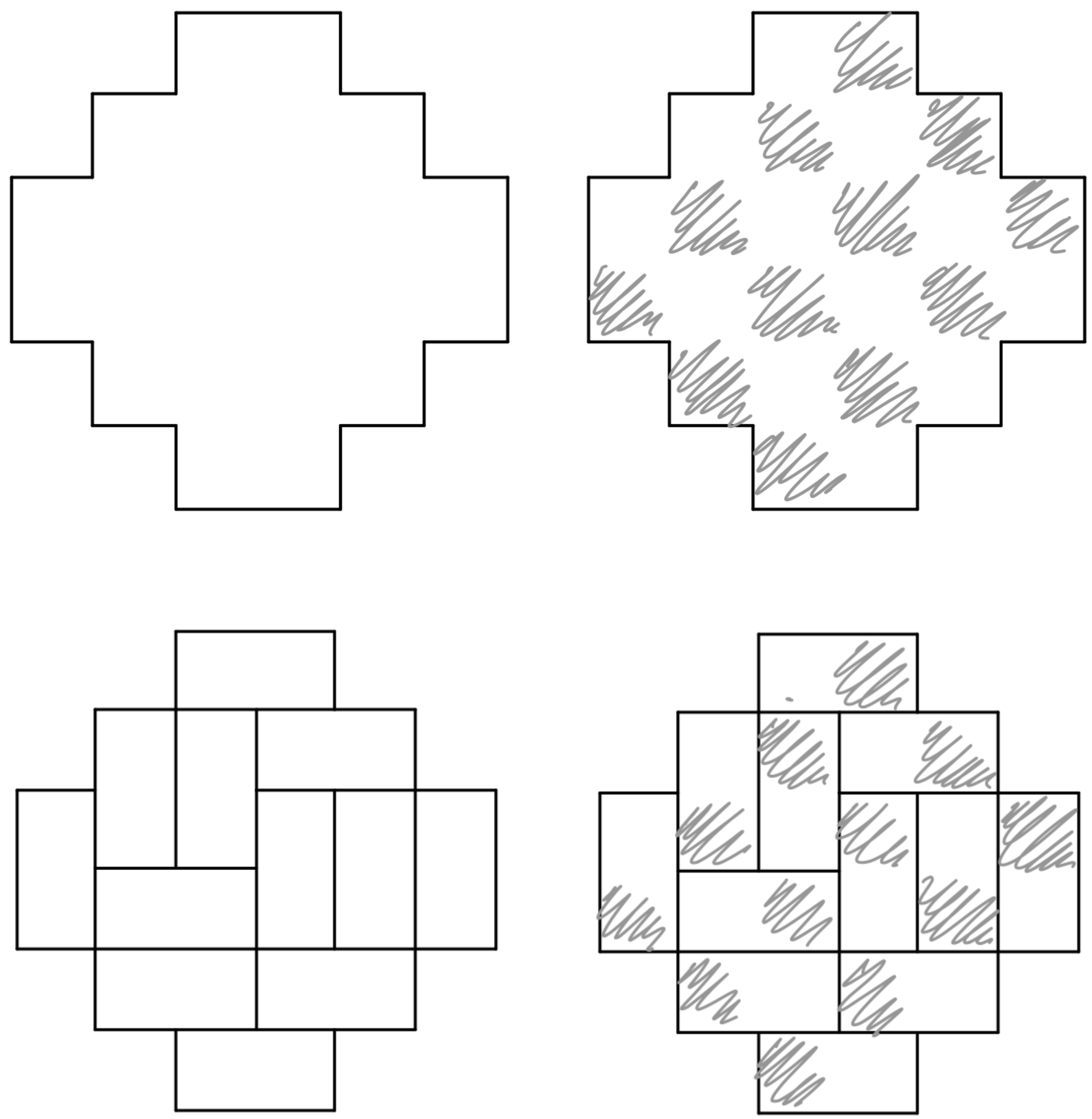


...and tile the Aztec diamond.

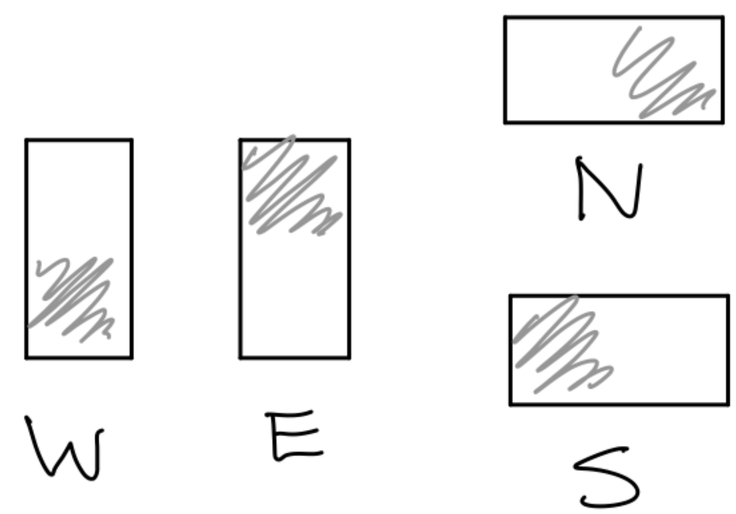


Domino tilings of the hexagon

Draw a checkerboard on the Aztec diamond…

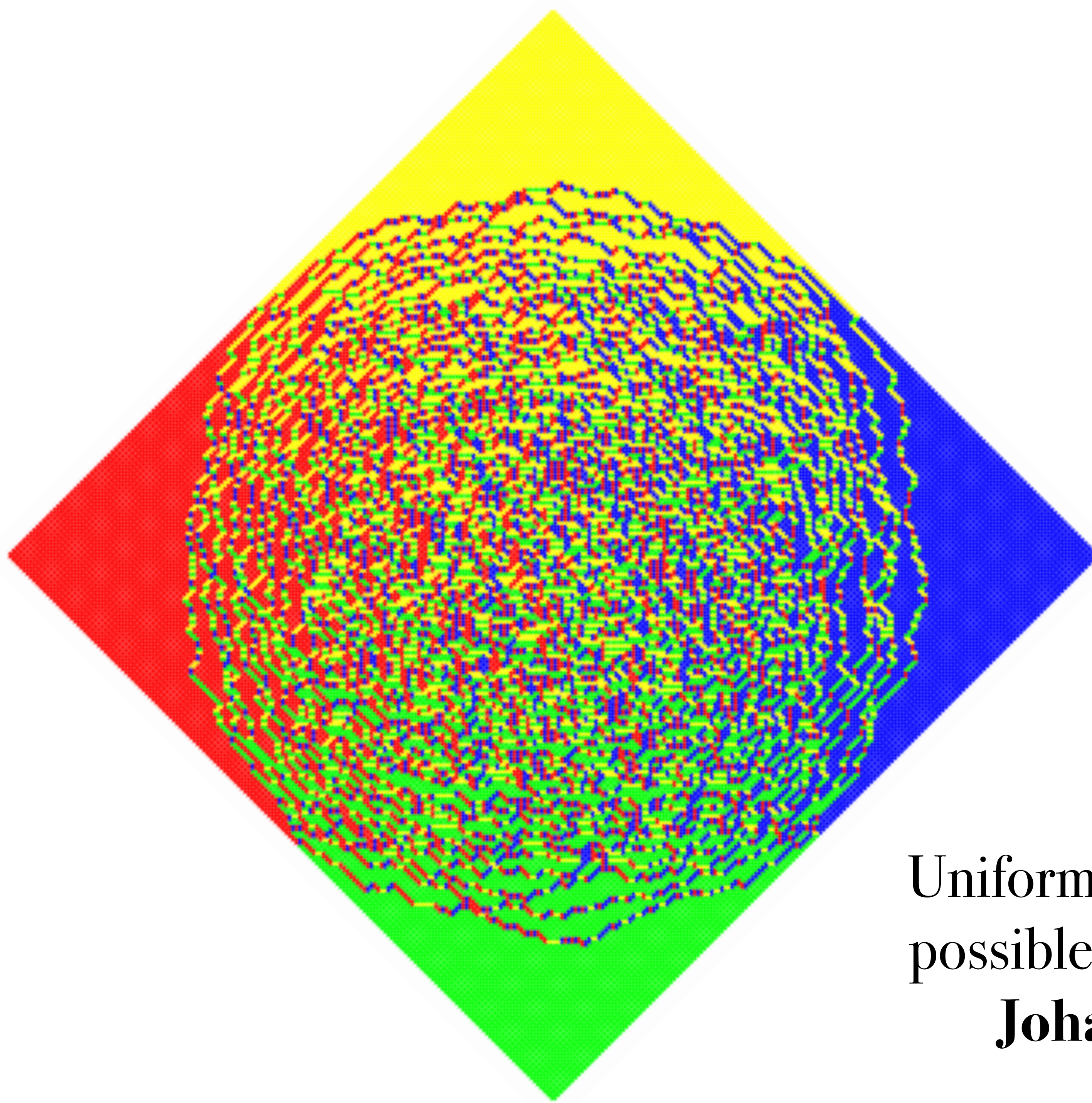


… giving four type of dominos…



….each will have its own color.

Domino tilings of the hexagon

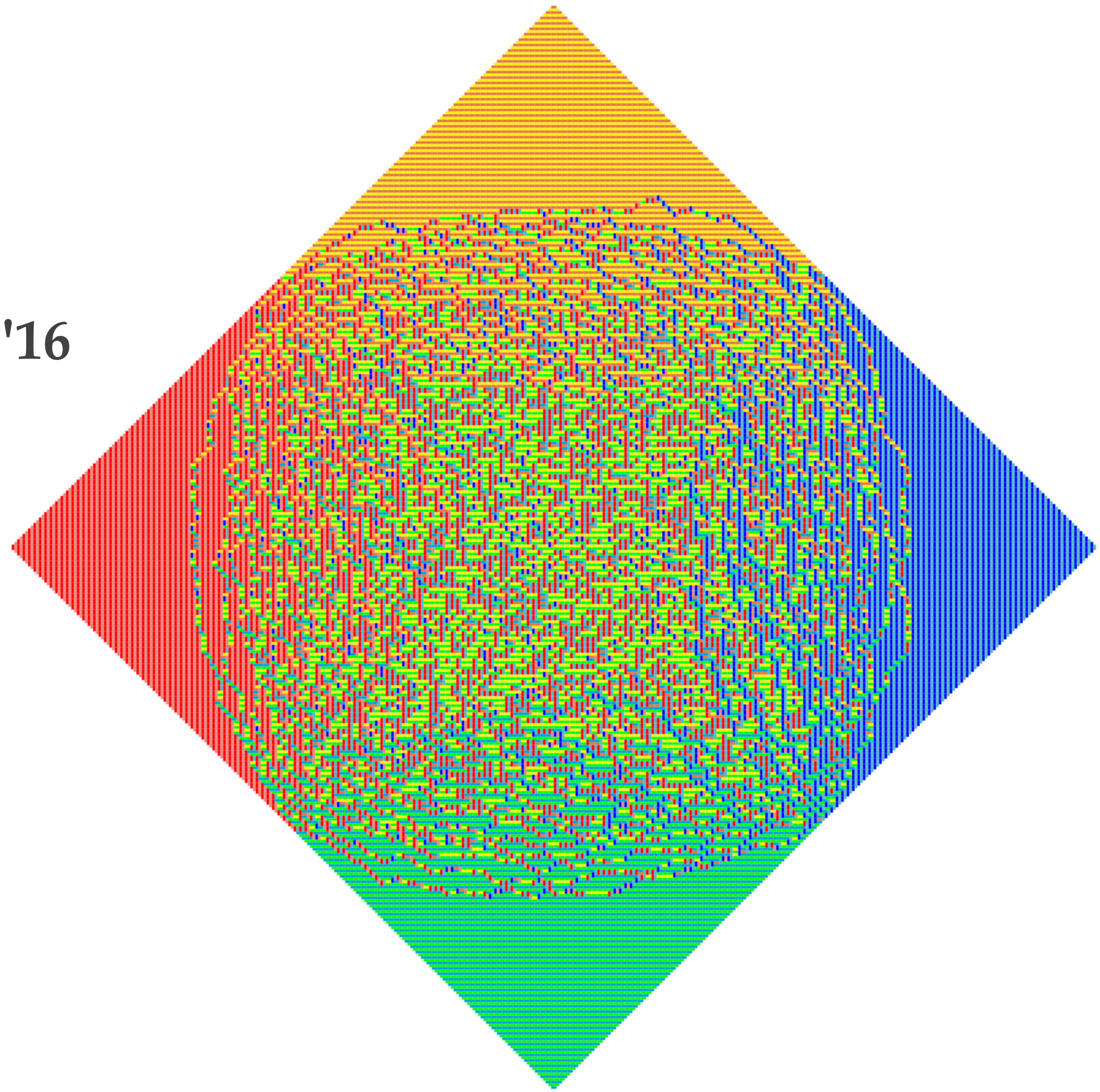


Uniform measure on all possible domino tilings
Johansson '03

Domino tilings of the Aztec Dimoand

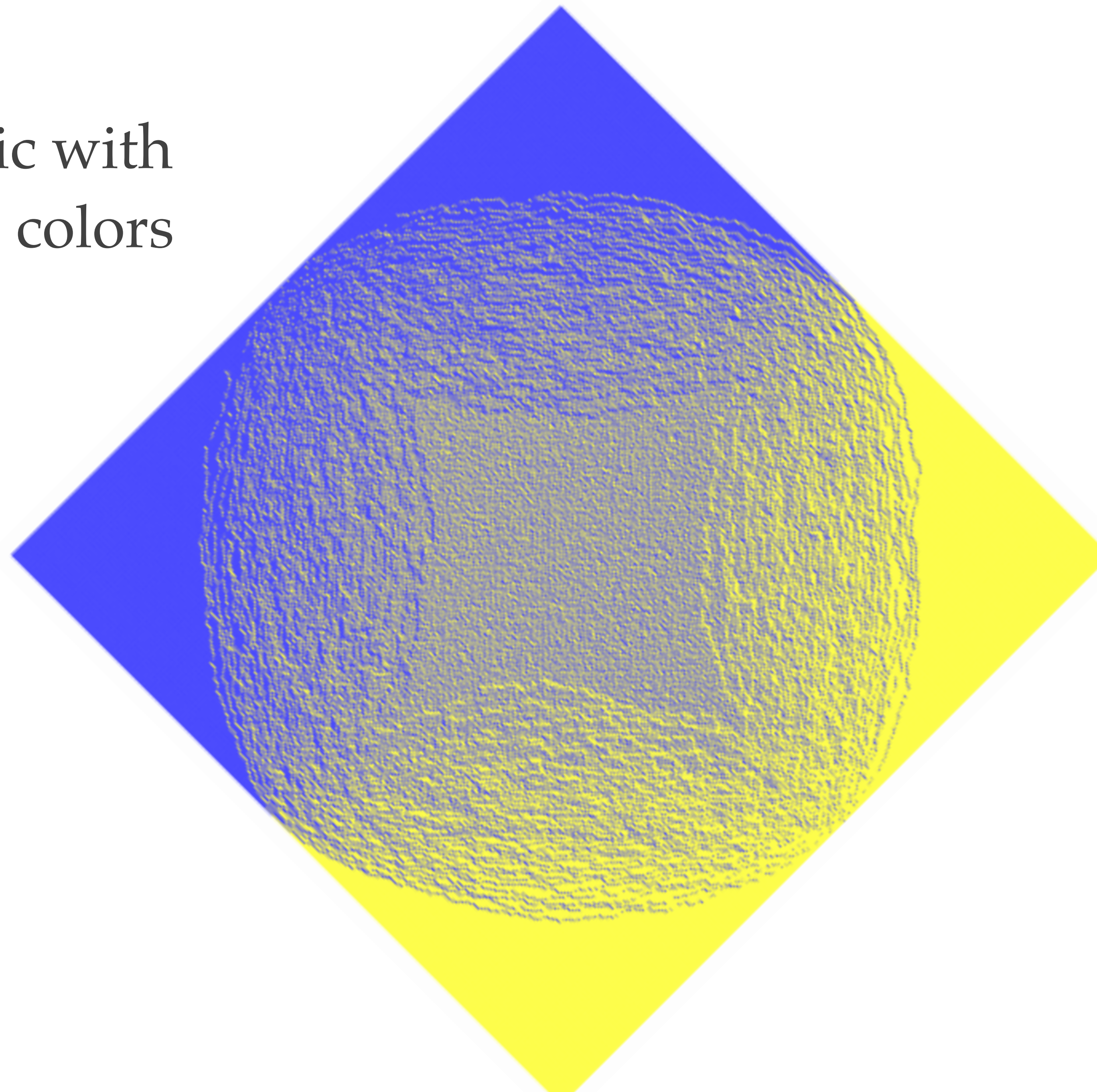
2-periodic weighting

Chhita-Young '14
Chhita-Johansson ′16
Beffara-Chhita-Johansson '16
D-Kuijlaars ′17



Domino tilings of the Aztec Diamond

2-periodic with
only two colors

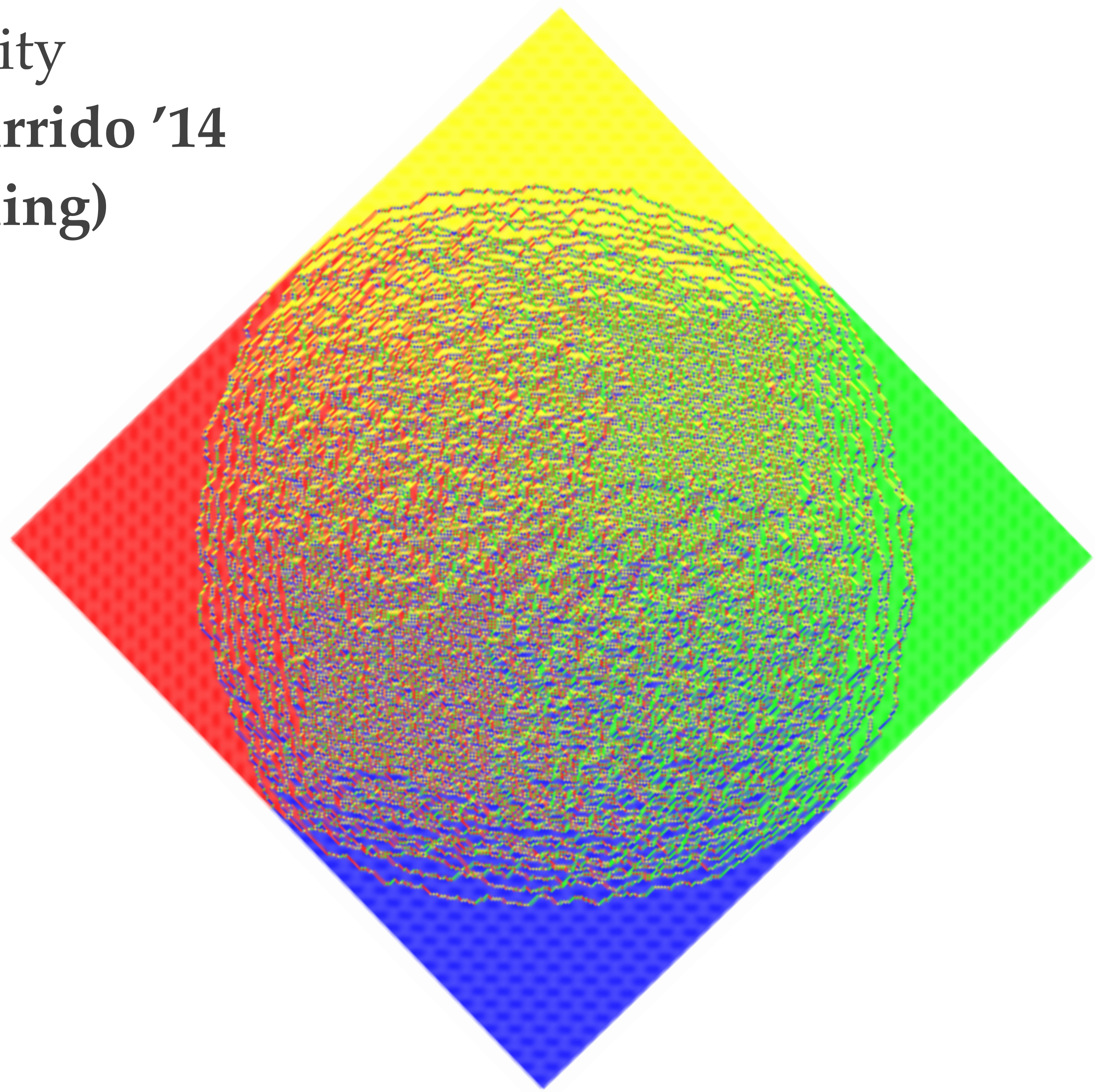


Domino tilings of the Aztec Diamond

Higher periodicity

Di Francesco Soto-Garrido '14

Berggren (upcoming)

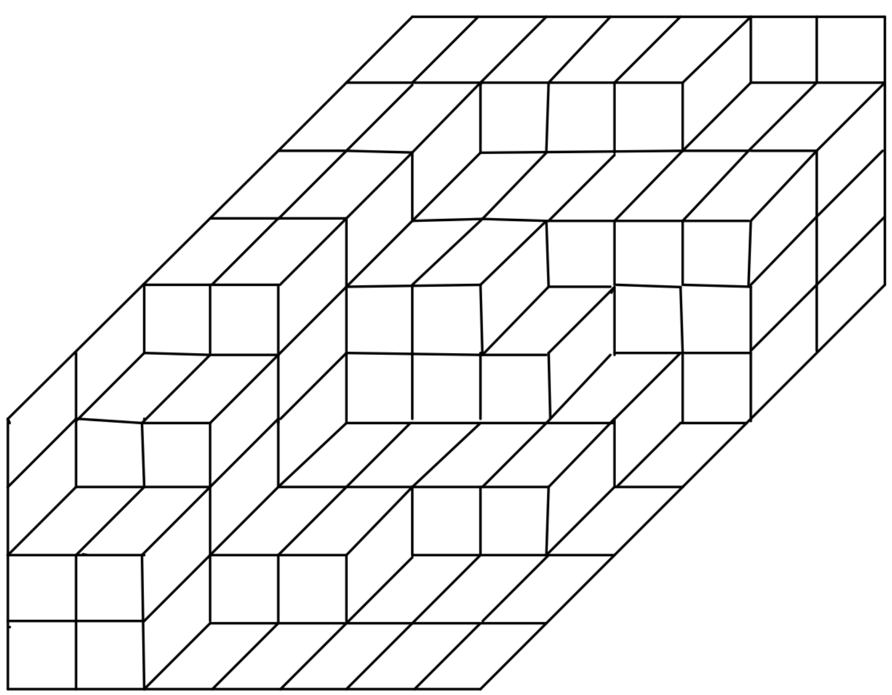


Tilings of planar domains

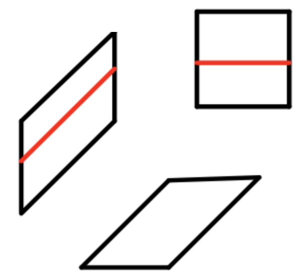
- There is a very large amount of studies for random tilings of planar domains in the past two decades.
- Limit shapes are described by the complex Burger's equation **Kenyon-Okounkov '07** (and many other works). Shape fluctuations are expected to be described terms of the Gaussian free field.
- For doubly periodic weightings **Kenyon-Okounkov-Sheffield '06** not much results the fine asymptotic properties of such models are known. First results are by **Chhita-Johansson '16** and **Beffara-Chhita-Johansson '16**
- In **D-Kuijlaars '17** we introduced a new approach to study tiling models, using (matrix-valued) polynomials that satisfy orthogonality relations on curves in the complex plane. A tandem of Riemann-Hilbert techniques and classical stationary phase methods can be used for asymptotic studies.
- In particular, this approach also gives an alternative studying random tilings of hexagons, which typically are not in the Schur class.

Non-Intersecting paths

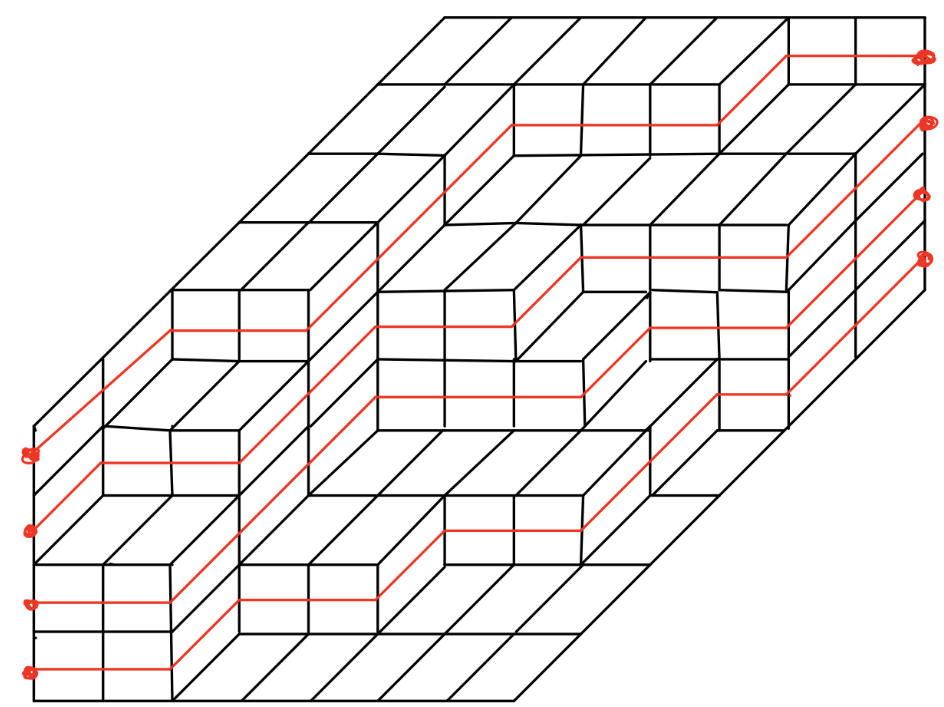
Start with a tiling....



....draw red lines on the lozenges as...

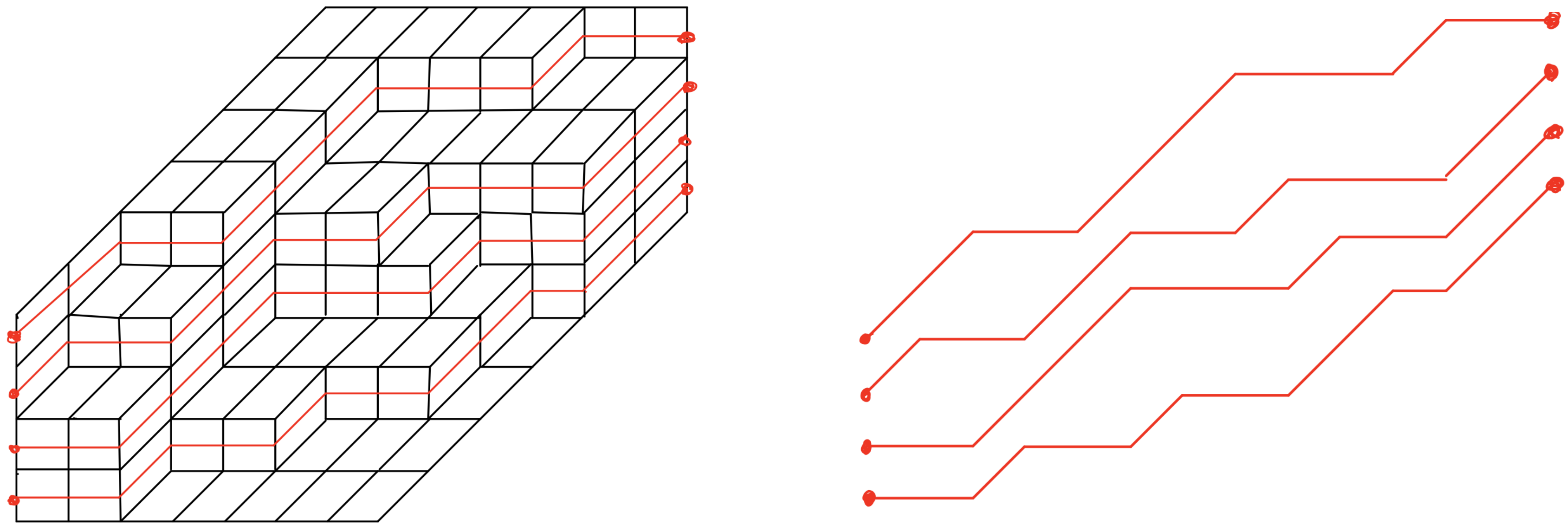


....and non-intersecting up-right paths appear.



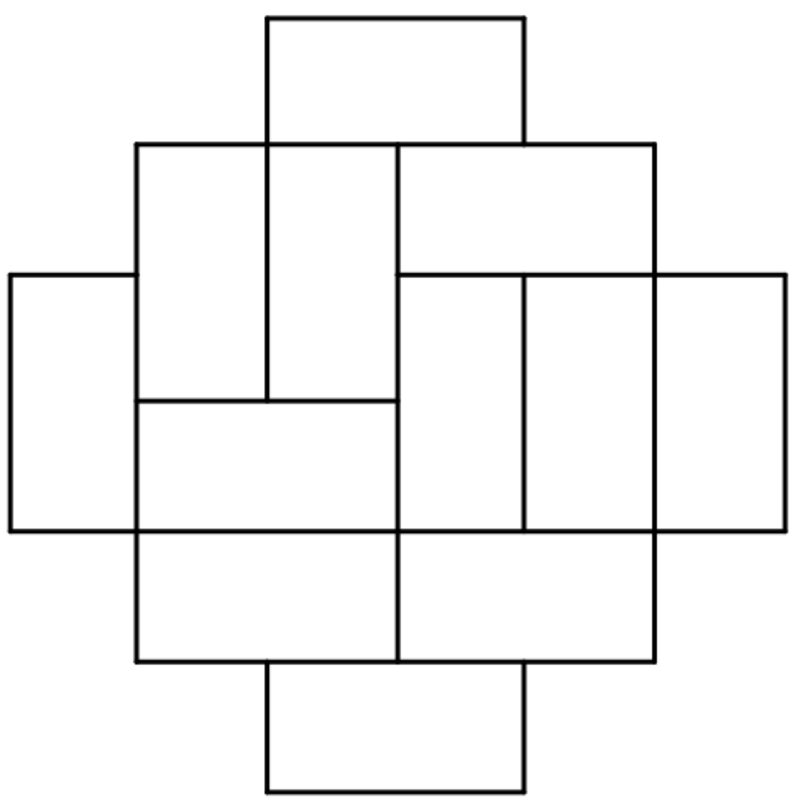
Non-Intersecting paths

The two pictures are in fact equivalent.....

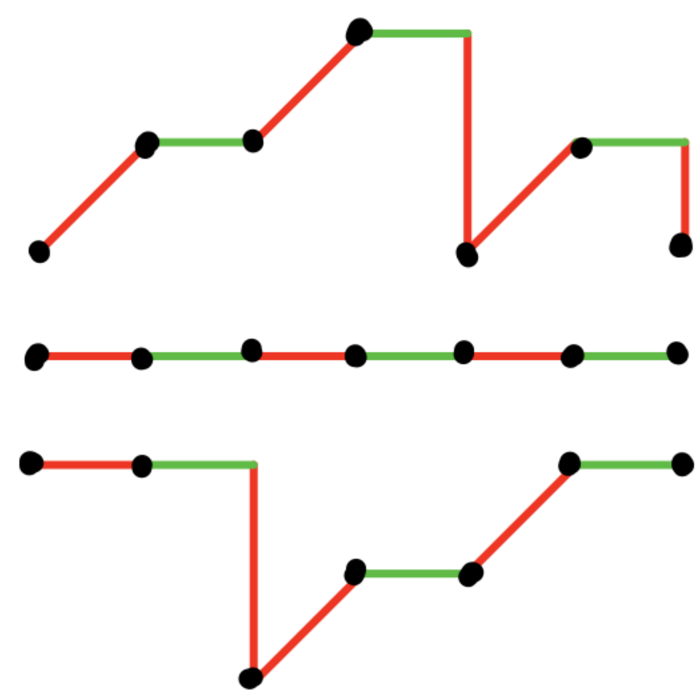


Non-Intersecting paths

A slightly more complicated collection of paths can be found for the Aztec diamond.....

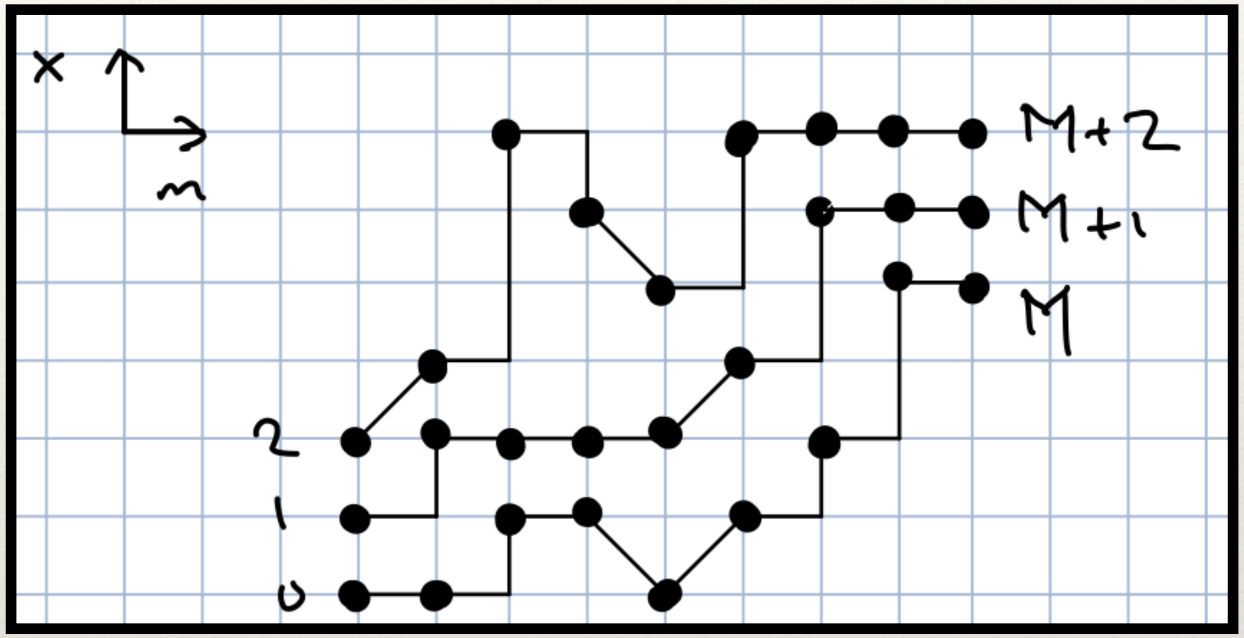


.... leading to paths that end at the same points as they started, and are up-right for odd steps and go down on the even steps



Products of determinants

- The probability measure on the tilings induces a probability measure on the non-intersecting path



- Denote the position of the *j*-th path after step m by x_j^m
- LGV Theorem: probability measure can be written as :

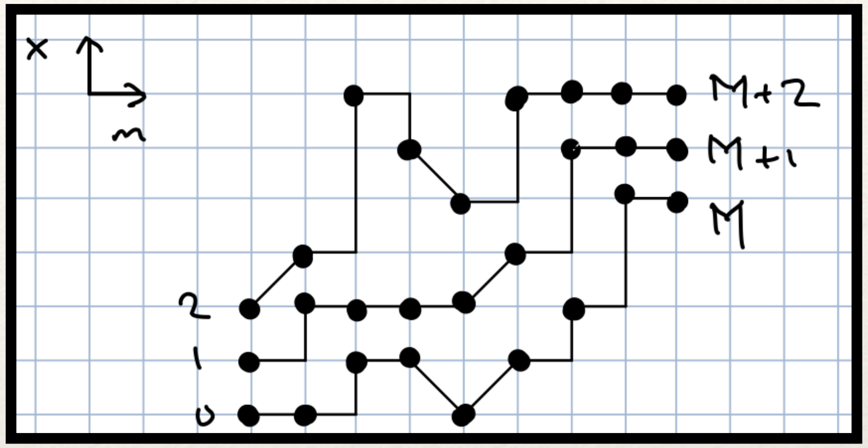
$$\sim \prod_{m=1}^{N} \det \left(T_m(x_j^{m-1}, x_k^m) \right)_{j,k=1}^{n}$$

where for $j = 1,\ldots,n$ we have as initial and endpoints:

$$x_j^0 = j - 1 \qquad x_j^N = M + j - 1$$

n = number of paths
N = number of steps
M = the shift at endpoints
$T_m(x, y)$ = Transition probability at step m to jump from x to y

Toeplitz matrices



- The first class of models is when the transition matrices in

$$\sim \prod_{m=1}^{N} \det\left(T_m(x_j^{m-1}, x_k^m)\right)_{j,k=1}^{n}$$

are **Toeplitz matrices**

$$T_m(x,y) = \hat{\phi}_m(y-x) = \frac{1}{2\pi i}\oint \phi_m(z)\frac{dz}{z^{y-x+1}}$$

That is, the step probability from x to y depends only on the size y-x.

Bernoulli up: $\phi_m(z) = 1 + a_m z$

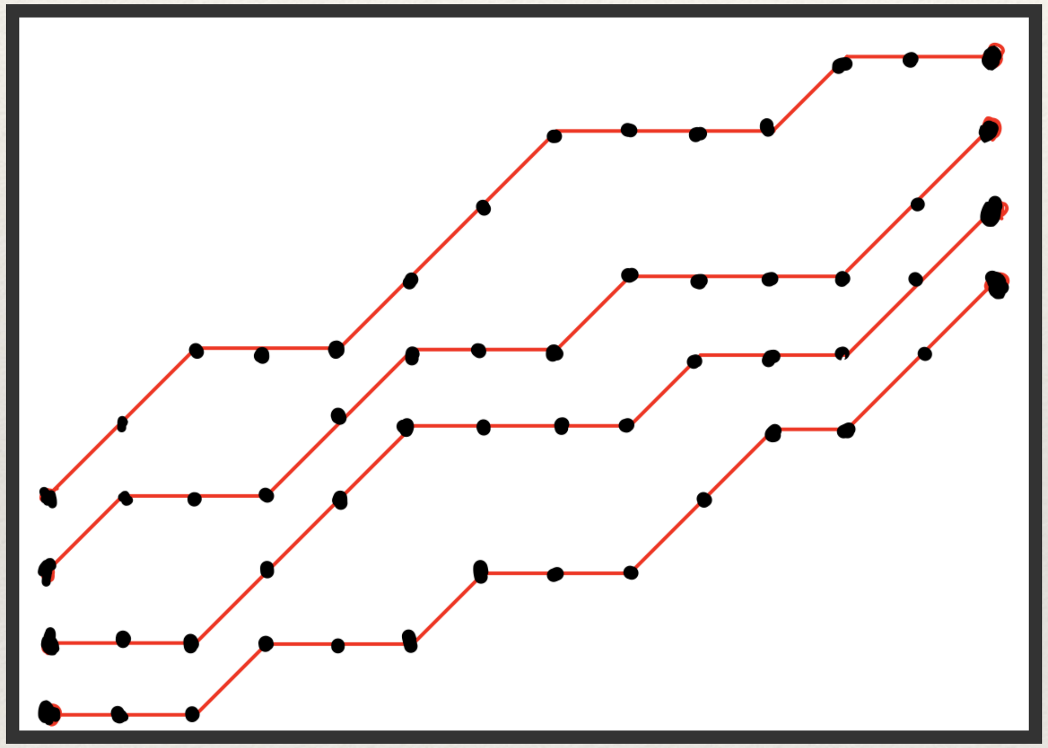
Bernoulli down: $\phi_m(z) = 1 + \dfrac{a_m}{z}$

Geometric up: $\phi_m(z) = \dfrac{1}{1 - a_m z}$

Geometric down: $\phi_m(z) = \dfrac{1}{1 - \frac{a_m}{z}}$

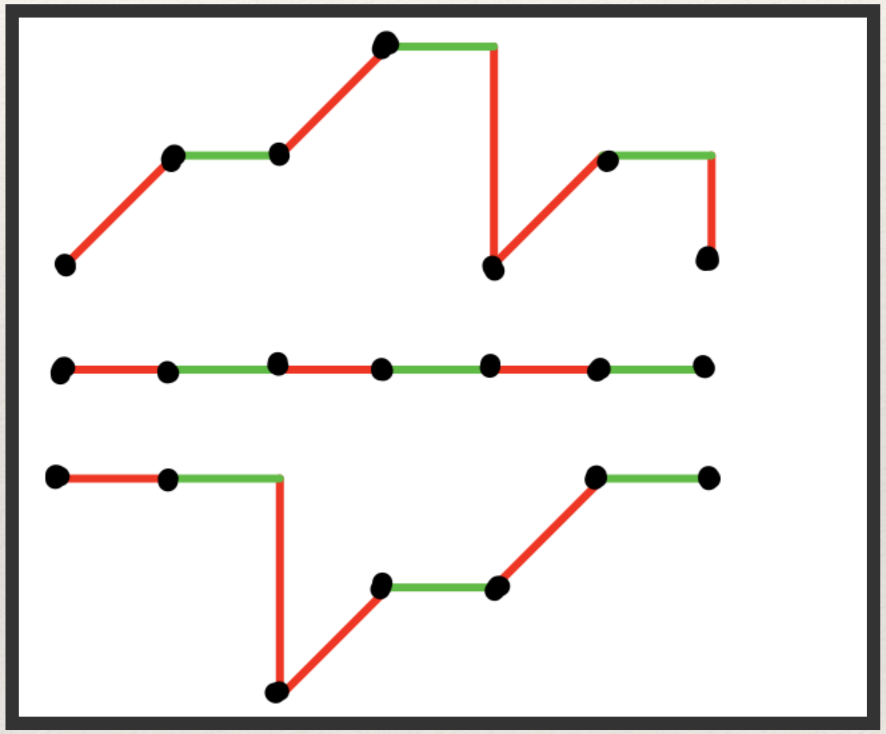
Examples

Uniform lozenge tilings of the hexagon



$$\phi_m(z) = 1 + z$$

Uniform domino tilings of the Aztec diamond



$$\phi_m(z) = \begin{cases} 1 + qz, & m \text{ odd} \\ (1 - \frac{q}{z})^{-1}, & m \text{ even} \end{cases}$$

....and take the limit $q \uparrow 1$

Orthogonal polynomials

- In **D-Kuijlaars '17** we used a biorthogonalization procedure using orthogonal polynomials in the complex plane to describe the k-point correlations.
- Let $p_k(z)$ be the monic polynomial of degree k such that

$$\oint_\gamma p_k(z)\ z^j\ \frac{\prod_{m=1}^{N} \phi_m(z) dz}{z^{M+n}} = 0, \qquad j = 0,1,\ldots,k-1$$

- Orthogonality relations is with respect to contour in the complex plane and non-hermitian. **The existence is not guaranteed!**

The idea of biorthogonalization is a standard trick for determinantal point processes. However, there are many ways to do it. The way we choose here is very different from the more common one, that would lead to Discrete Orthogonal Polynomials. **Baik-Deift-Kriechenbauer-McLaughlin** The relation between the two is not obvious.

Determinantal point process

By the Eynard-Mehta Theorem the process is determinantal.

$$\mathbb{P}\left(\text{ points at } (m_1, x_1), \ldots, (m_k, x_k)\right) = \det\left(K(m_j, x_j, m_\ell, x_\ell\right)_{j,\ell=1}^{n}$$

<u>Theorem</u> D-Kuijlaars '17

$$K(m, x, m', y) = -\frac{\chi_{m>m'}}{2\pi i}\oint_\gamma \prod_{\ell=m'+1}^{m} \phi_\ell(z) z^{y-x}\frac{dz}{z}$$

$$+\frac{c_n}{(2\pi i)^2}\oint_\gamma\oint_\gamma \prod_{\ell=m'+1}^{N} \phi_\ell(w)\ \frac{p_n(z)p_{n-1}(w) - p_n(w)p_{n-1}(z)}{z-w}\ \prod_{\ell=1}^{m}\phi_\ell(w)\ \frac{w^y}{z^{x+1}w^{M+n}} dzdw$$

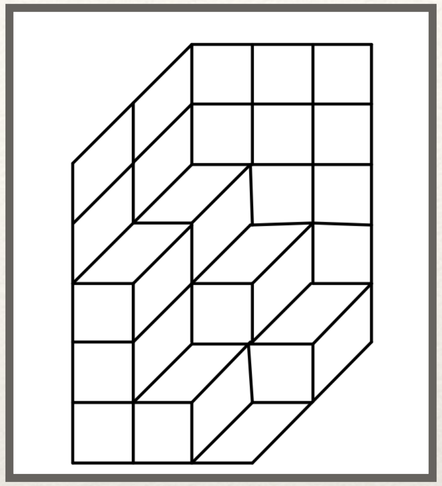
Strategy for asymptotic analysis

- To study the asymptotic behavior $n, N \to \infty$ we
  - First find the asymptotic behavior of the Orthogonal Polynomials. In particular for the Christoffel-Darboux kernel
  - Insert the asymptotics into the double integral formula and perform a steepest descent analysis.
- The asymptotic for the orthogonal polynomials can be done by a Riemann-Hilbert analysis.
- In certain special cases, like uniform lozenge tilings of the hexagon and domino tilings of the Aztec diamond, the orthogonal polynomials are "classical."
- **Schur processes**: when only $n \to \infty$ then the asymptotics of the polynomials is easy.

Jacobi polynomials

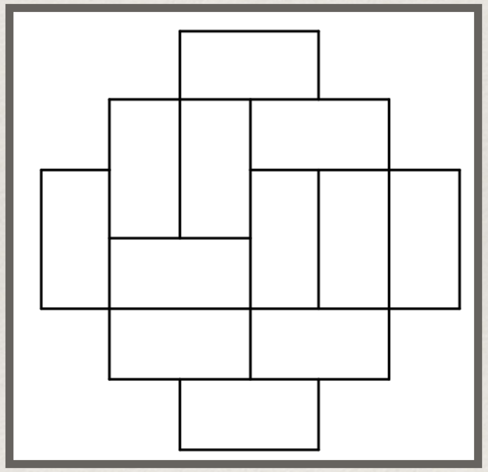
- In case of **uniform lozenge tilings of a hexagon** we obtain the "orthogonality measure"

$$\frac{(1+z)^N}{z^M}dz$$



- In case of **domino tilings of the Aztec diamond** we obtain the "orthogonality measure"

$$\left(\frac{1+qz}{1-qz}\right)^N dz$$



- In both cases, this means that the orthogonal polynomials are in fact **Jacobi polynomials** where one of the parameter is negative. In the Aztec diamond the choice is even degenerate and the Christoffel-Darboux kernel is explicit and we retrieve the Krawtchouk kernel from **Johansson '03**

Periodic weighting

- In **Charlier-D-Kuijlaars-Lenells** (upcoming) we consider lozenge tilings of the regular hexagon with the probability of having T_0 given by

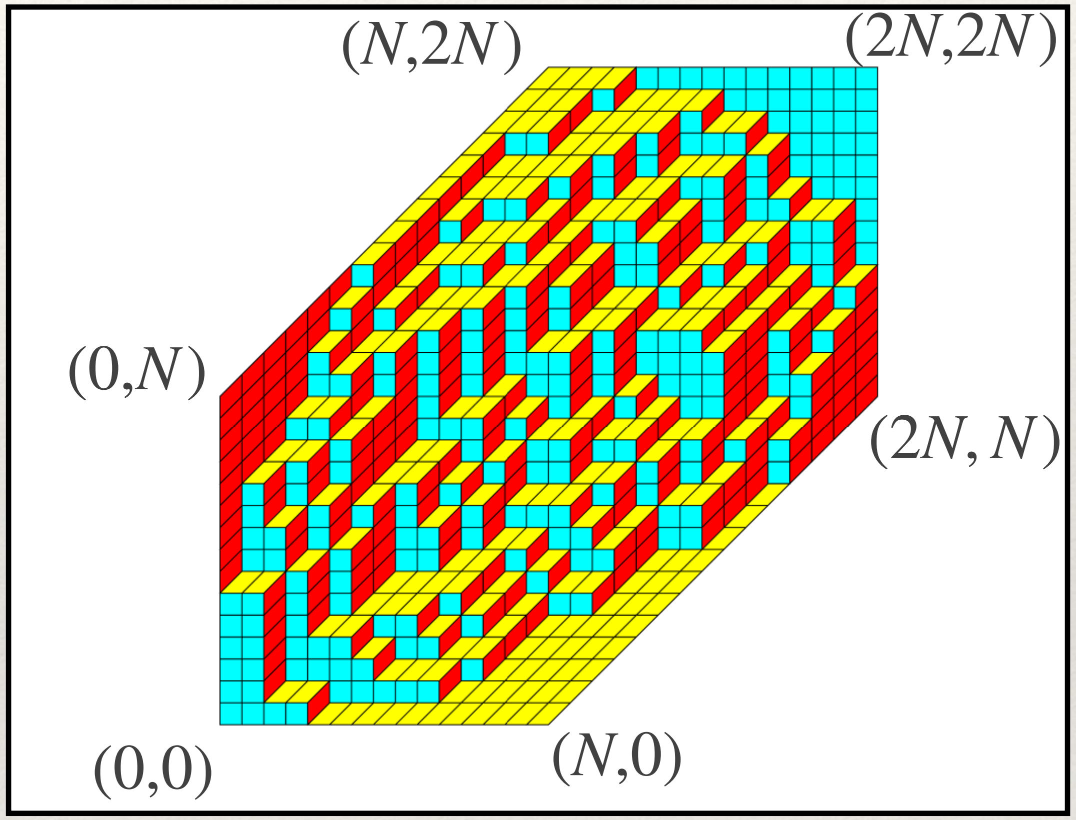
$$\mathbb{P}(T_0) = \frac{W(T_0)}{\sum_T W(T)}$$

where the weight of a tiling is given by

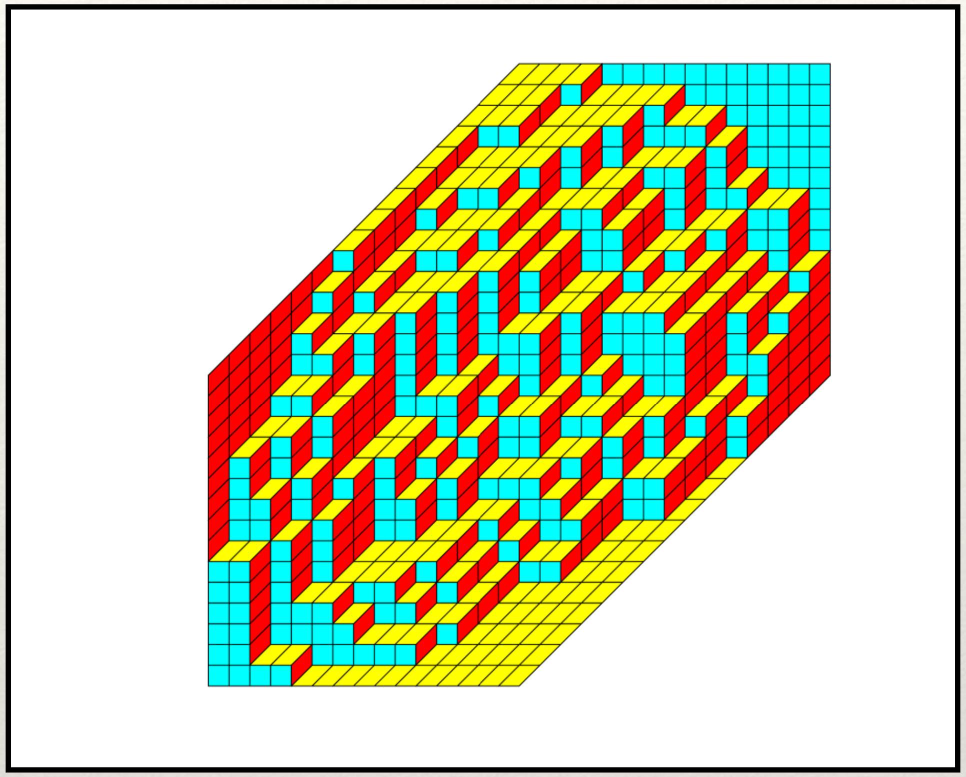
$$W(T) = \prod_{\square \in T} w(\square)$$

and

$$w(\square) = \begin{cases} 1, & \text{if } \square \text{ in an odd column} \\ \alpha, & \text{if } \square \text{ in an even column} \end{cases} \qquad 0 \leq \alpha \leq 1$$



Periodic weighting



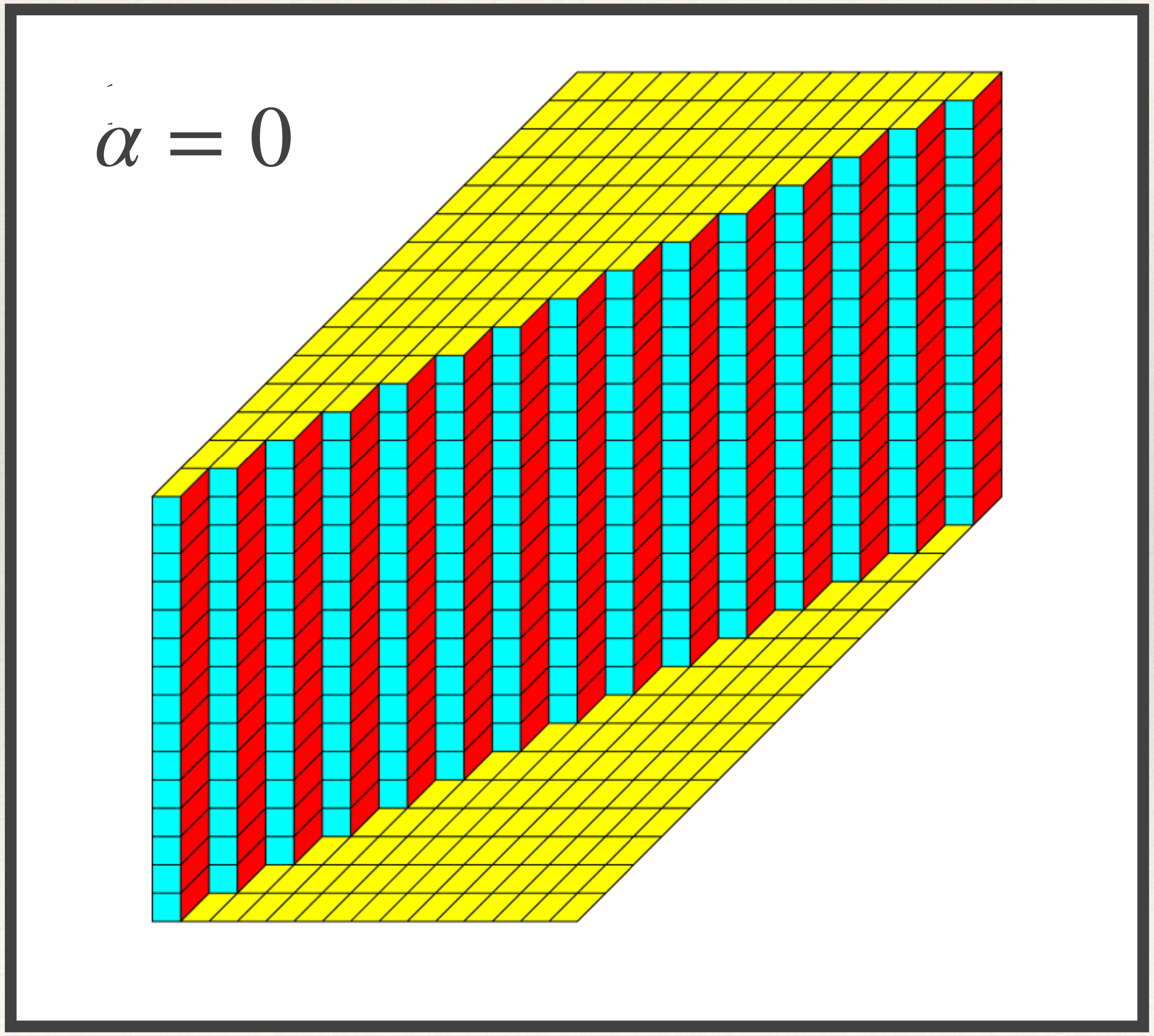
- This can be rewritten as

$$W(T) = \exp\left(-\log \alpha^{-1} \cdot \# \square \text{ in even columns}\right)$$

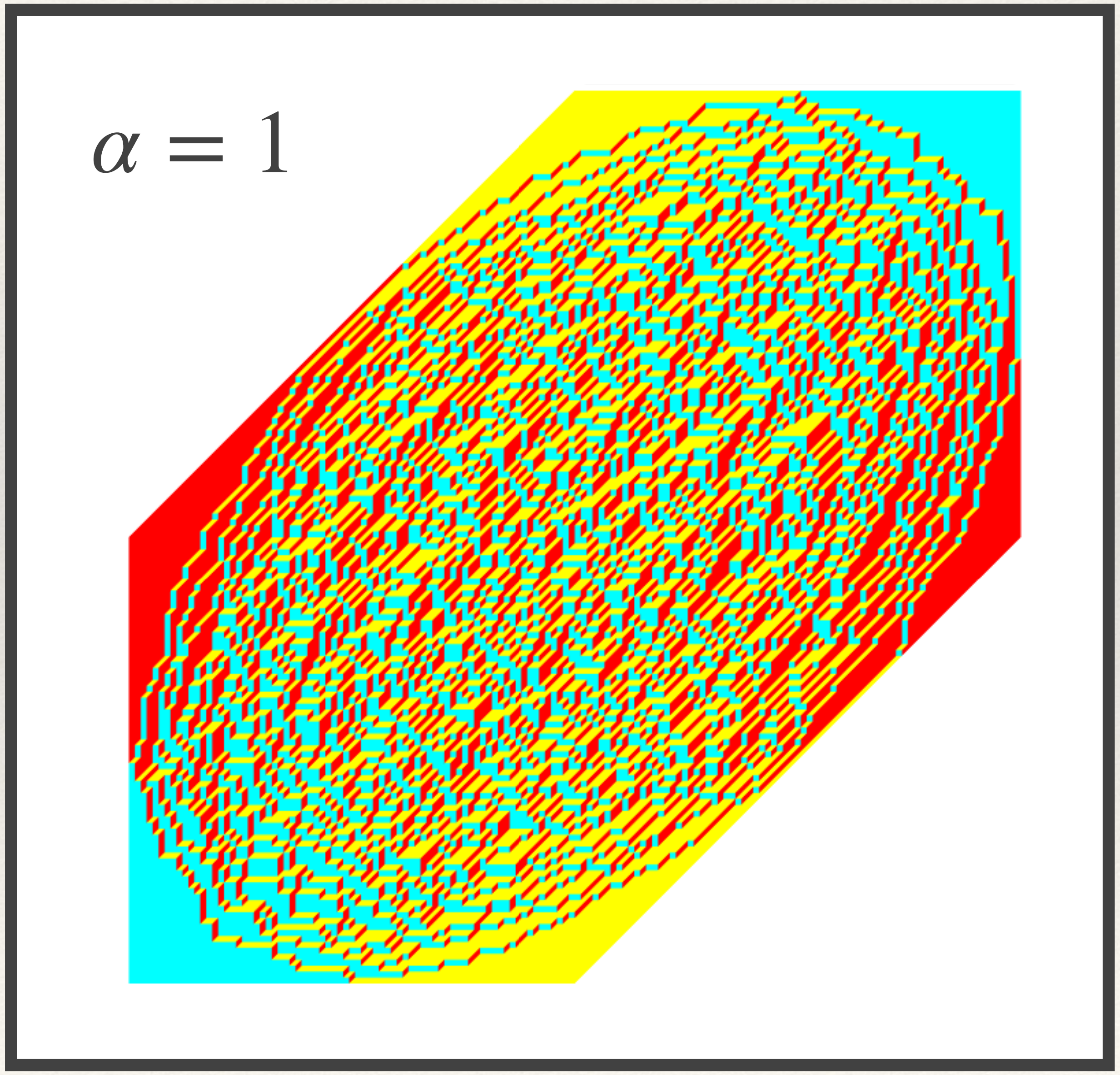
so we think of $\log \alpha^{-1}$ as an inverse temperature parameter.

Periodic weighting

Low temperature

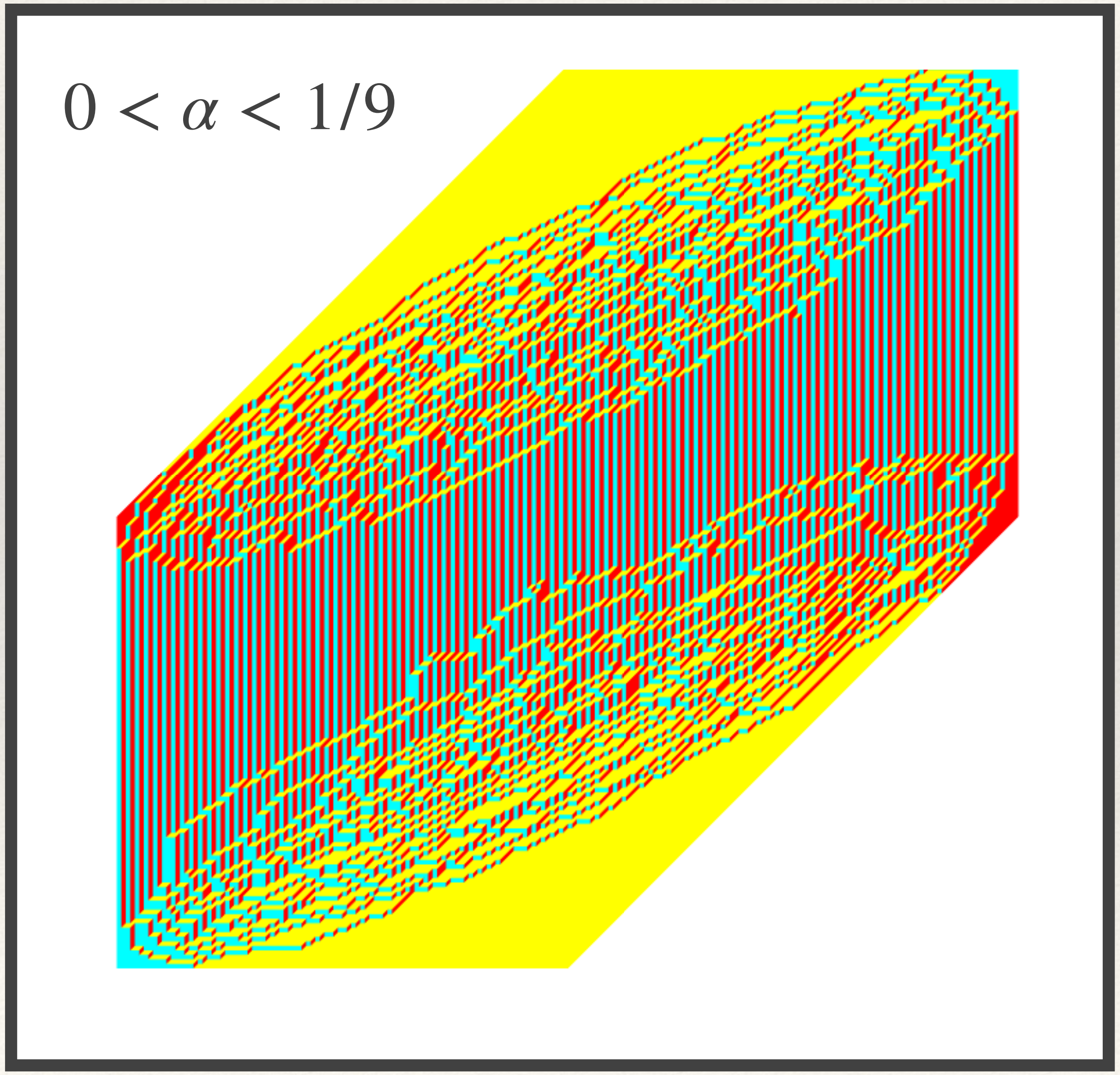


High temperature

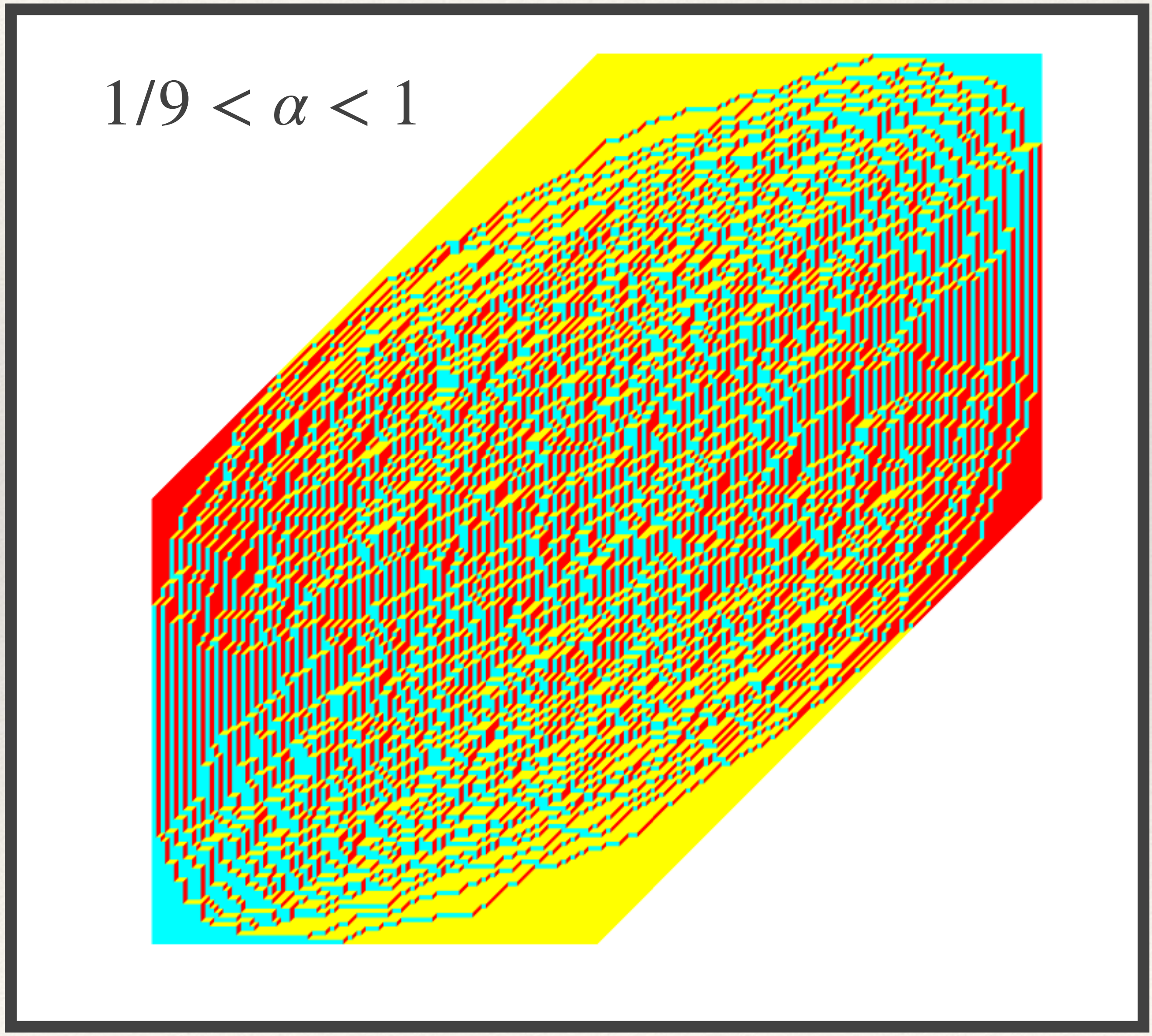


Periodic weighting

Low temperature

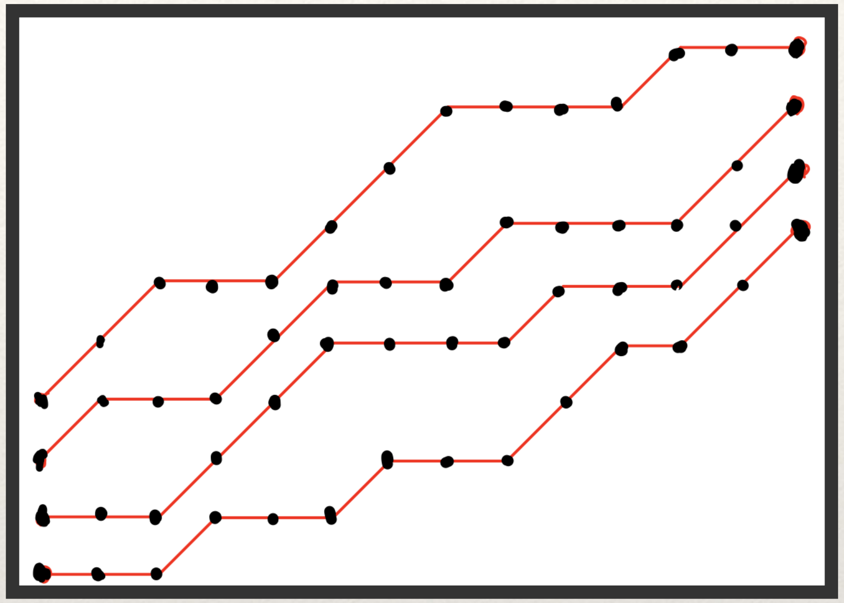


High temperature



Periodic weighting

- In terms of the non-intersecting paths, this means we look at N paths with 2N step given by



$$\phi_m(z) = \begin{cases} 1 + z, & m \text{ odd} \\ \alpha + z, & m \text{ even} \end{cases}$$

Meaning that the orthogonality weight is given by

$$\frac{(1+z)^N(\alpha+z)^N}{z^{2N}}dz \qquad 0 \leq \alpha \leq 1$$

- By steepest descent analysis on the **Riemann-Hilbert problem** for the polynomials we find the asymptotic behavior of these polynomials. By inserting that in the double integral formula and then performing a classical steepest descent analysis we can compute the thermodynamical limit.

Periodic weighting

The liquid region is described by the algebraic function $\zeta(z)$ defined by

$$\left(\zeta - \frac{\xi}{2}\left(\frac{1}{z+1} + \frac{1}{z+\alpha}\right) + \frac{\eta}{z}\right)^2 = Q_\alpha(z)\,.$$

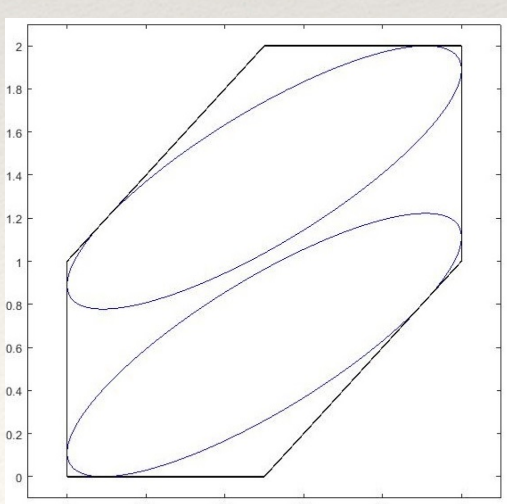
Here $Q_\alpha(z)$ is an explicit polynomial depending on α but not on (ξ,η)

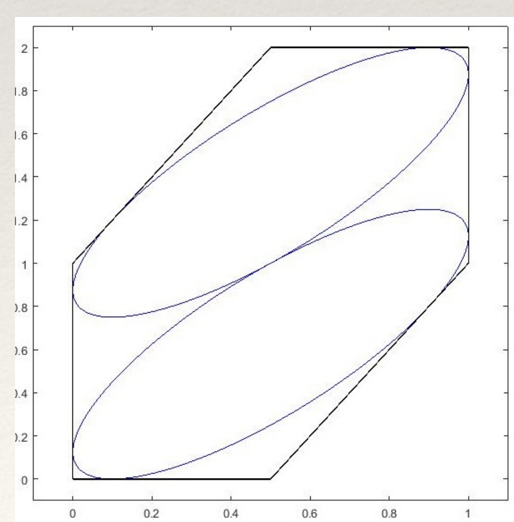
<u>Theorem (Charlier-D-Kuijlaars-Lenells '19)</u>

The **liquid region** consists of all (ξ,η) at most one zero $\zeta(s)=0$ with $Im\, s > 0$

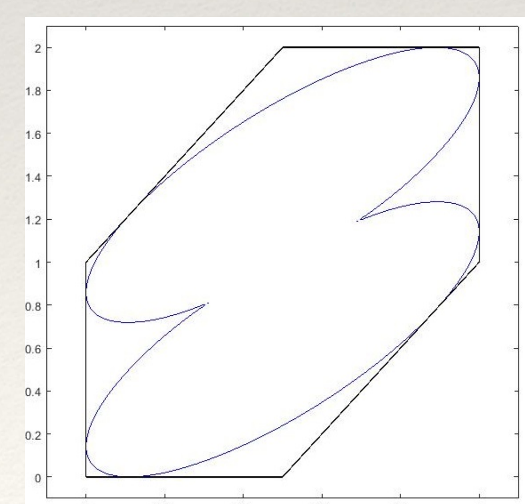
If it exists it is unique and denoted by $s(\xi,\eta)$

$0 < \alpha < \frac{1}{9}$





$\alpha = 1$



$\frac{1}{9} < \alpha < 1$

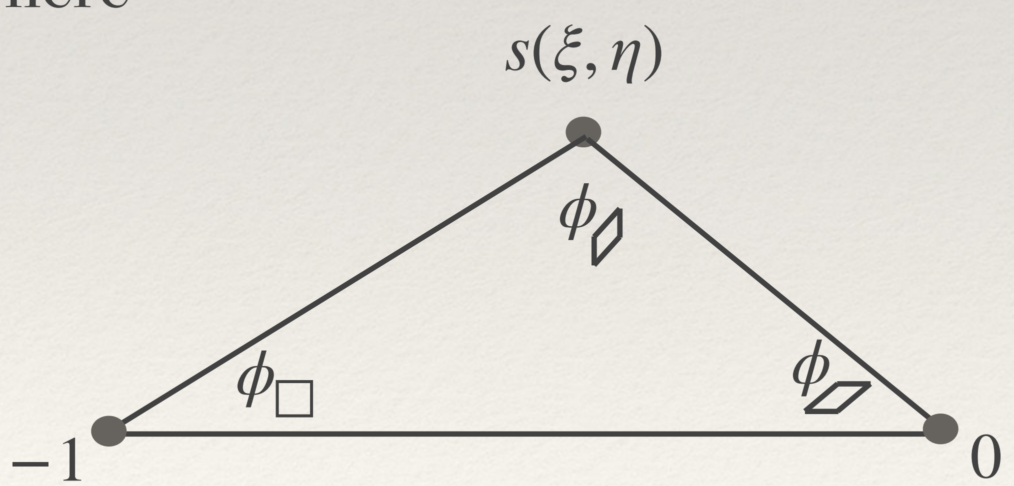
Periodic weighting

Theorem (Charlier-D-Kuijlaars-Lenells '19)
Take (x, y) such that $(x/N, y/N) \to (\xi, \eta)$ is a point in the liquid region.

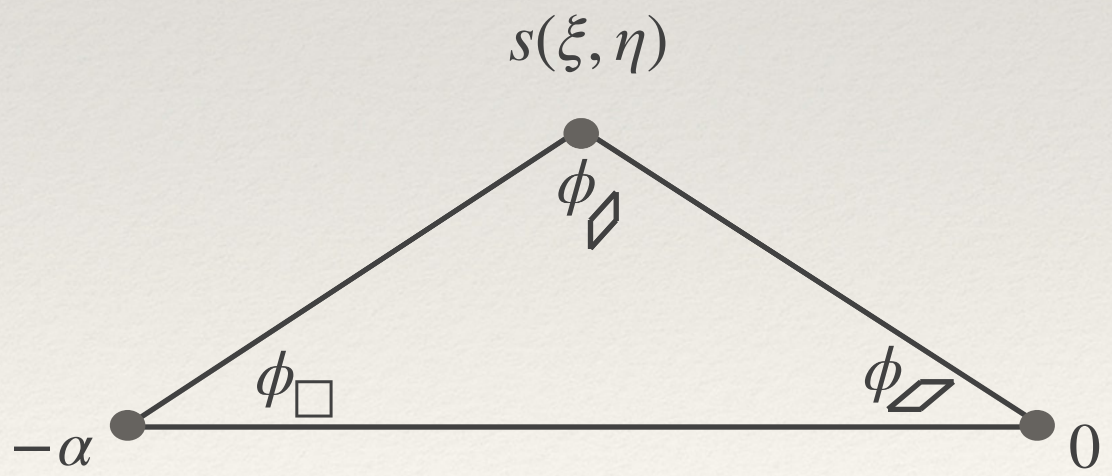
Then

$$\lim_{N\to\infty} \mathbb{P}\left(\square_{(x,y)}\right) = \frac{\phi_\square}{\pi} \qquad \lim_{N\to\infty} \mathbb{P}\left(\lozenge_{(x,y)}\right) = \frac{\phi_\lozenge}{\pi} \qquad \lim_{N\to\infty} \mathbb{P}\left(\lozenge_{(x,y)}\right) = \frac{\phi_\lozenge}{\pi}$$

where



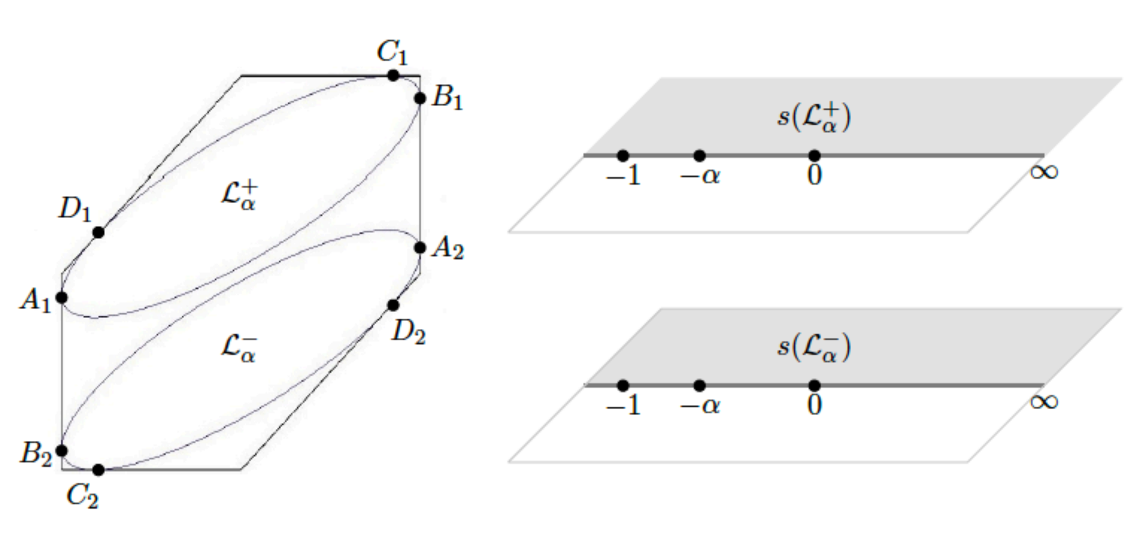
$x = odd$



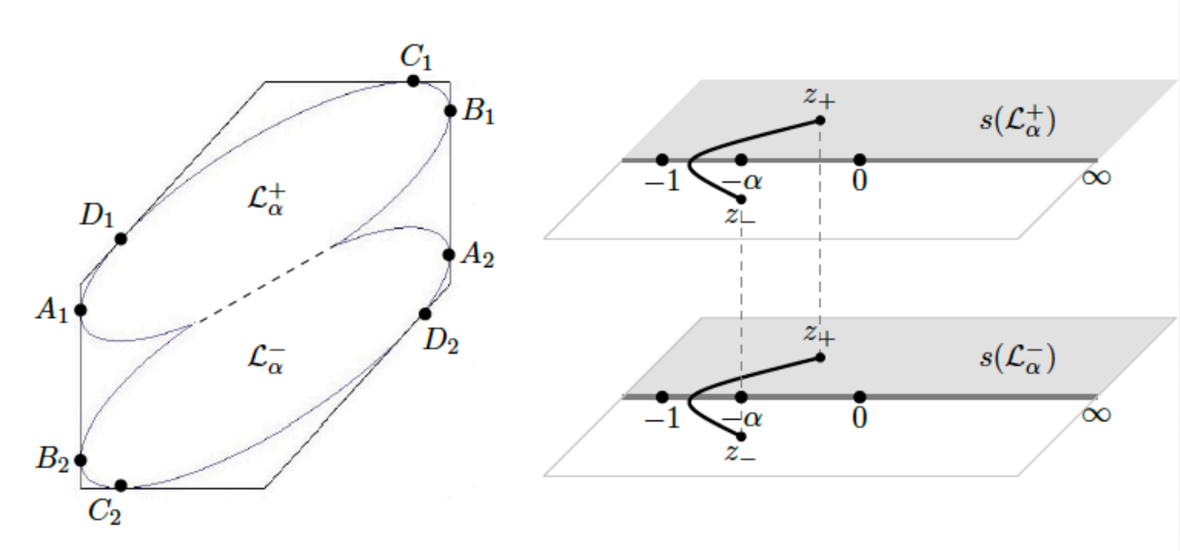
$x = even$

Periodic weighting

Theorem (Charlier-D-Kuijlaars-Lenells '19)
The map $(\xi, \eta) \mapsto s(\xi, \eta)$ is a diffeomorphism from the liquid region to two copies of the upper half plane that in the high temperature regime are glued together



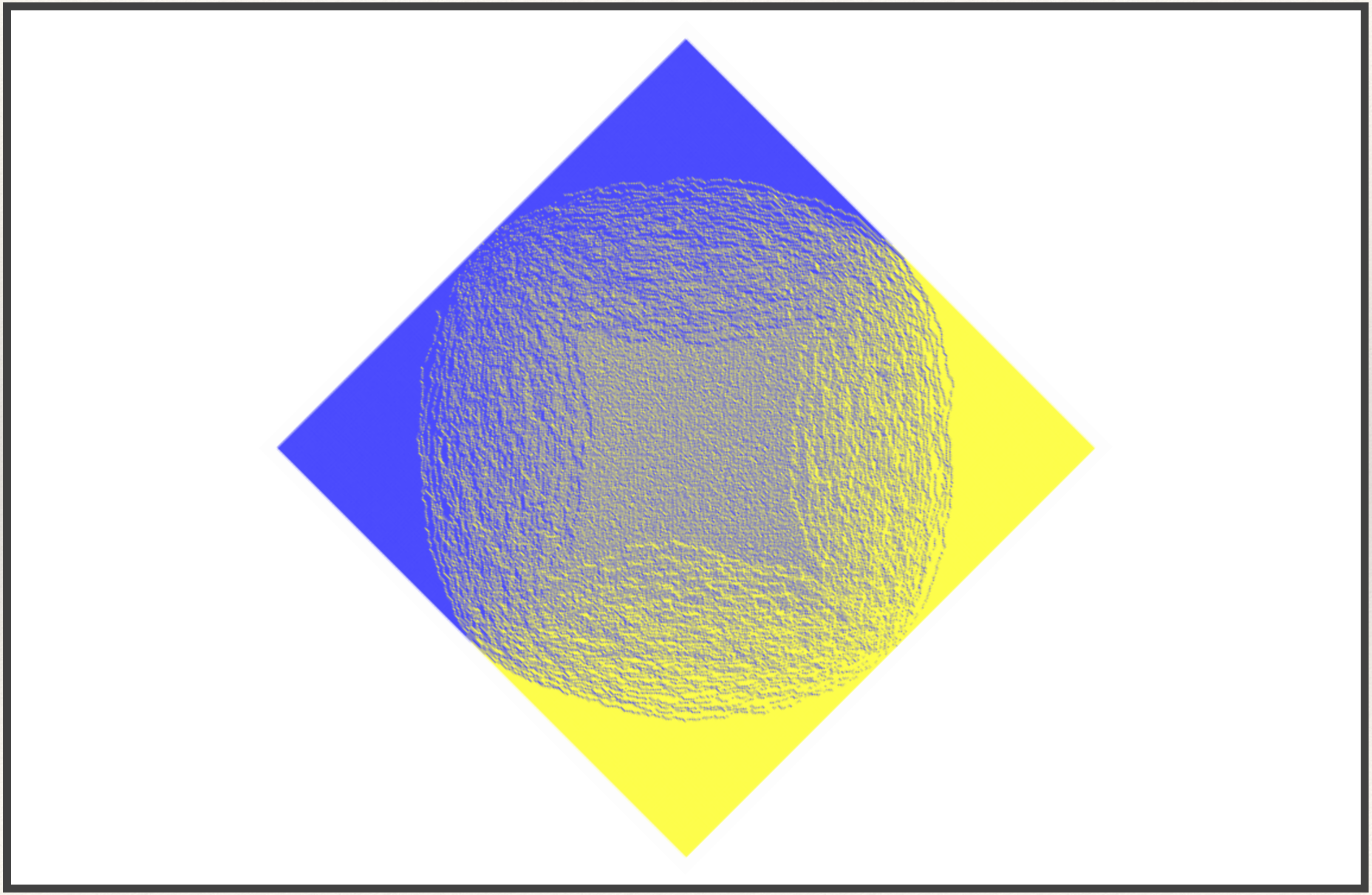
$0 < \alpha < \frac{1}{9}$



$\frac{1}{9} < \alpha < 1$

Work in progress: The fluctuations of the corresponding height function are described by the pull back of the Gaussian free field on the image of the liquid region of the map s.
Interesting transition at $\alpha = \frac{1}{9}$

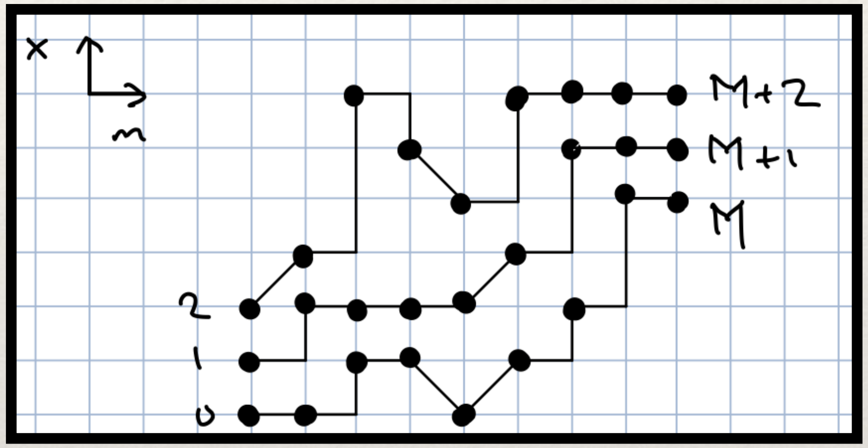
Doubly periodic tiling models



Block Toeplitz transition matrices

- The original motivation for **D-Kuijlaars '17** was to analyze the 2-periodic Aztec diamond (see also **Chhita-Young '14**, **Chhita-Johansson ′14**, **Beffara-Chhita-Johansson '15)**

- In a more general setup, we considered measures of the type



$$\sim \prod_{m=1}^{N} \det\left(T_m(x_j^{m-1}, x_k^m)\right)_{j,k=1}^{n}$$

where the transition matrices are **block Toeplitz matrices with blocks of size** $p \times p$

$$T_m(px+r, py+s) = \left(\hat{\phi}_m(y-x)\right)_{r+1,s+1} = \frac{1}{2\pi i}\oint \left(\phi_m(z)\right)_{r+1,s+1} \frac{dz}{z^{y-x+1}},$$

$$r, s = 0, \ldots, p-1 \qquad x, y \in \mathbb{Z}$$

Block Toeplitz transition matrices

- In the case $p = 2$ the following matrix symbols are canonical:

$$\phi_m(z) = \begin{pmatrix} a_m & b_m z \\ c_m & d_m \end{pmatrix}$$

"Bernoulli up"

$$\phi_m(z) = \frac{1}{1 - qz} \begin{pmatrix} a_m & b_m z \\ c_m & d_m \end{pmatrix}$$

"Geometric up"

$$\phi_m(z) = \begin{pmatrix} a_m & b_m \\ c_m/z & d_m \end{pmatrix}$$

"Bernoulli up"

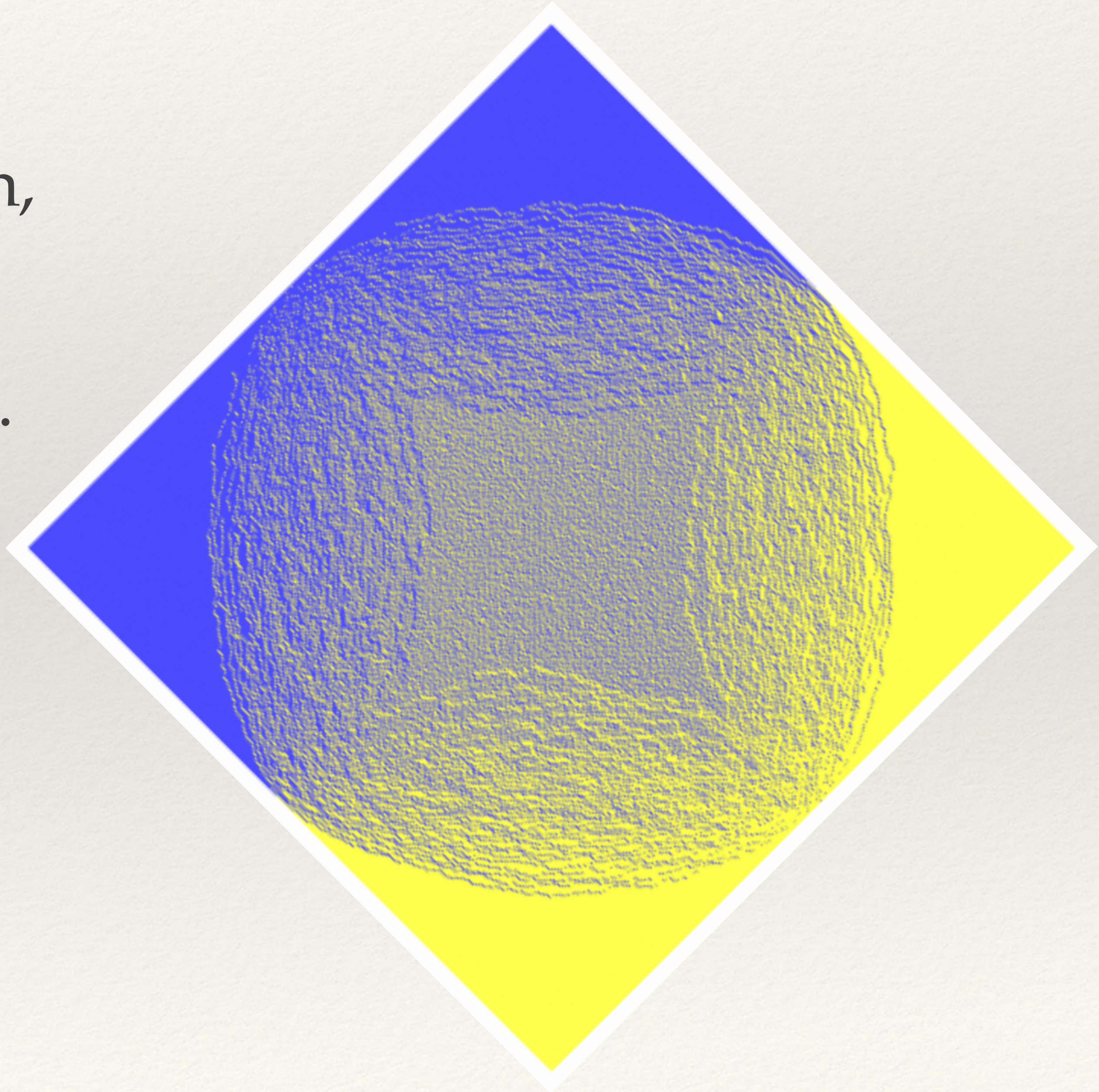
$$\phi_m(z) = \frac{1}{1 - q/z} \begin{pmatrix} a_m & b_m \\ c_m/z & d_m \end{pmatrix}$$

"Geometric down"

2 periodic Aztec diamond

- The 2-periodic Aztec diamond has the weight

$$\phi_m(z) = \begin{cases} \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} 1 & az \\ a & 1 \end{pmatrix}, & m \text{ even}, \\ \frac{1}{1 - a^2/z} \begin{pmatrix} 1 & a \\ a/z & 1 \end{pmatrix}, & m \text{ odd}. \end{cases}$$



- Here $\alpha > 1$
- where we also need $a \uparrow 1$

Matrix Orthogonal Polynomials

- In **D-Kuijlaars '17** we used **matrix orthogonal polynomials** in the complex plane to describe the k-point correlations.
- Let $p_k(z) = I_p z^k + \dots$ be the monic polynomial of degree k such that

$$\oint_\gamma p_k(z)\ z^j\ \frac{\prod_{m=1}^N \phi_m(z) dz}{z^{M+n}} = 0, \qquad j = 0,1,\dots,k-1$$

- Orthogonality relations is with respect to contour in the complex plane and non-hermitian.
- **The weight is matrix valued. Order in the product is important!**

Correlation kernel

<u>Theorem</u> D-Kuijlaars '17 The point process is determinantal with orrelation kernel:

$$\left[K(m,px+j;m',py+i)\right]_{i,j=0}^{p-1} = -\frac{\chi_{m>m'}}{2\pi i}\oint_\gamma \prod_{\ell=m'+1}^{m} \phi_\ell(z) z^{y-x}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\oint_\gamma\oint_\gamma \prod_{\ell=m'+1}^{N} \phi_\ell(w)\ R(z,w)\ \prod_{\ell=1}^{m}\phi_\ell(z)\frac{w^y}{z^{x+1}w^{M+n}}dzdw$$

where $R_n(z,w)$ is the Christoffel-Darboux kernel for the matrix orthogonal polynomials

- Due to non-commutativity, the order in the product is important!

2 periodic Aztec diamond

- For the 2-periodic the Riemann-Hilbert problem can be solved explicitly. This was done in **D-Kuijlaars ′17** and reproved in a different way in **Berggren-D ’19**

$$\left[\mathbb{K}_N(2m+r,n;2m'+s,n')\right]_{r,s=0}^{1} = -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma_{0,1}} A^{m-m'}(z) z^{(m'+n')/2-(m+n)/2}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\oint_{\gamma_{0,1}}\frac{dz}{z}\oint_{\gamma_1}\frac{dw}{z-w}A^{N-m'}(w)F(w)A^{-N+m}(z)\frac{z^{N/2}(z-1)^N}{w^{N/2}(w-1)^N}\frac{w^{(m'+n')/2}}{z^{(m+n)/2}}$$

where

$$A(z)=\frac{1}{z-1}\begin{pmatrix} 2\alpha z & \alpha(z+1) \\ \beta z(z+1) & 2\beta z\end{pmatrix}$$

and

$$F(z)=\frac{1}{2}I_2+\frac{1}{2\sqrt{z(z+\alpha^2)(z+\beta^2)}}\begin{pmatrix}(\alpha-\beta)z & \alpha(z+1)\\ \beta z(z+1) & -(\alpha-\beta)z\end{pmatrix},$$

2 periodic Aztec diamond

- In **D-Kuijlaars '17** we analyzed this double integral formula asymptotically
- An important role in the analysis is defined by the spectral curve

$$\det\left(A(z)-\lambda\right)=0 \qquad\qquad A(z)=\frac{1}{z-1}\begin{pmatrix} 2\alpha z & \alpha(z+1) \\ \beta z(z+1) & 2\beta z \end{pmatrix}$$

which is an important input for finding the saddle point in the steepest descent analysis.

- An important feature of the spectral curve is that it leads to a Rieman-surface with genus 1. The presence of a gas phase seems intrinsic to a non-zero genus.

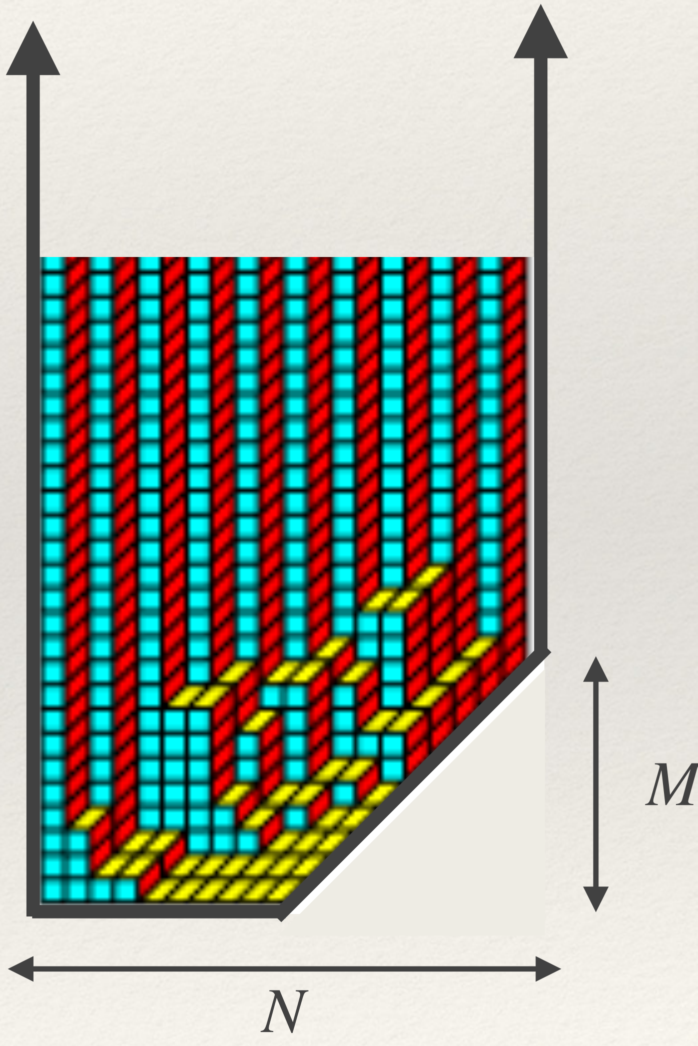
Products of infinite minors

- In **Berggren-D '19** we follow the approach of Schur processes and found a general statement for the kernel in case of infinite systems of paths.
- Think of lozenge tilings of the hexagon. Schur processes arise when vertical size of the hexagon tends to infinity.
- That is, instead of keeping the number of paths n finite, we can also define the process for $\mathbf{n} \to \infty$.

$$\sim \prod_{m=1}^{n} \det \left(T_m(x_j^{m-1}, x_k^m) \right)_{j,k=1}^{\infty}$$

where

$$T_m(x,y) = \hat{\phi}_m(y-x) = \frac{1}{2\pi i} \oint \phi_m(z) \frac{dz}{z^{y-x+1}}$$



- NOTE: There can be two interesting limits: at the **top** and **bottom** of the hexgaon

Matrix analogue of the Schur process

- We assume that the orthogonality weight has a matrix Wiener-Hopf type factorization

$$\prod_{m=1}^{N} \phi_m(z) = \phi_+(z)\phi_-(z) = \widetilde{\phi}_-(z)\widetilde{\phi}_+(z)$$

where

  - $\phi_+^{\pm 1}(z), \widetilde{\phi}_+^{\pm 1}(z)$ are analytic for $|z| < 1$ and continuous for $|z| \leq 1$
  - $\phi_-^{\pm 1}(z), \widetilde{\phi}_-^{\pm 1}(z)$ are analytic for $|z| > 1$ and continuous for $|z| \geq 1$
  - $\phi_-(z), \widetilde{\phi}_-(z) \sim z^M I_p$ as $z \to \infty$

- In **Beggren-D '19** we prove the following statement that is the analogue of the correlation kernels for the Schur process.

Matrix Analogue of the Schur process

- The bottom part of the paths converge to a DPP with kernel

$$\left[K_{bottom}(m,px+r;m',py+s)\right]_{r,s=0}^{p-1} = -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma}\prod_{\ell=m'+1}^{m}\phi_\ell(z)z^{y-x}\frac{dz}{z}$$

$$-\frac{1}{(2\pi i)^2}\iint_{|z|<|w|}\prod_{\ell=m'+1}^{N}\phi_\ell(z)\phi_-^{-1}(w)\phi_+^{-1}(z)\prod_{\ell=1}^{m}\phi_\ell(z)\frac{w^y}{z^{x+1}(z-w)}dzdw$$

- The top part of the paths converge to a DPP with kernel

$$\left[K_{top}(m,xp+r;m',yp+s)\right]_{r,s=0}^{p-1} = -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma}\prod_{\ell=m'+1}^{m}\phi_\ell(z)z^{y-x}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\iint_{|w|<|z|}\prod_{\ell=m'+1}^{N}\phi_\ell(z)\widetilde{\phi}_+^{-1}(w)\widetilde{\phi}_-^{-1}(z)\prod_{\ell=1}^{m}\phi_\ell(z)\frac{w^y}{z^{x+1}(z-w)}dzdw$$

Matrix Wiener-Hopf factorization

- These results are of course only meaningful if we can find a Matrix Wiener-Hopf factorization.
- The existence of such is a classical problem and many results are known. Existence results apply to the typical cases that we are interested in
- Still, existence is not enough. We want an explicit form of the factorization that is useful for an asymptotic study.
- So far we have been able to do several cases:
    - 2 periodic Aztec diamond
    - Higher periodic Aztec diamonds **(Berggen, upcoming)**
    - 2 periodic tilings of the infinite hexagons
- As a result, all of these example can be analyzed asymptotically…