Dynamics of gases of particles

with singular repulsion

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Random Matrices and Related Topics
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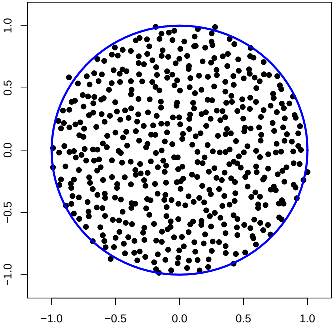
Outline

High dimensional phenomenon : random matrix spectrum




```
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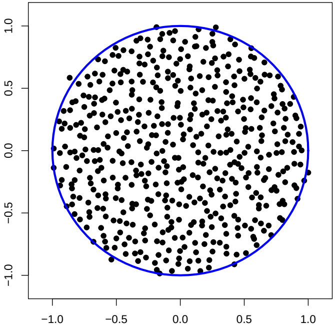

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Law of large numbers: orthonormal rows/columns for large *n*

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- Terence Tao (1975 –), Sylvia Serfaty (1975 –), Robert Berman (1976 –), ...

Gibbs measures: Ginibre versus Dyson

- Non-Hermitian (Ginibre) 2D

$$\mathrm{e}^{-n\mathrm{Trace}(MM^*)} = \prod_{j,k=1}^{n} \mathrm{e}^{-n|M_{jk}|^2}$$

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- Ginibre is not log-concave (contrary to Dyson)

Outline

Coulomb kernel in mathematical physics

- Coulomb kernel in $\mathbb{R}^d$, $d \geq 2$,

$$x \in \mathbb{R}^d \mapsto g(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2, \\ \dfrac{1}{|x|^{d-2}} & \text{if } d \geq 3. \end{cases}$$

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- For both Dyson and Ginibre: two dimensional repulsion

Coulomb energy and equilibrium measure

- Coulomb energy of probability measure μ on $\mathbb{R}^d$:

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- If V is smooth then μ_* has density

$$\frac{\Delta V}{2c_d}$$

Examples of equilibrium measures

d	Interaction g	Confinement V	Equilibrium μ_*
1	2	$\infty\mathbf{1}_{\text{interval}^c}(x)$	arcsine
1	2	x^2	semicircle (Dyson)
2	2	$\lvert x\rvert^2$	uniform on a disc (Ginibre)
≥ 3	d	$\lVert x\rVert^2$	uniform on a ball
≥ 2	d	radial	radial in a ring

Coulomb gas or one component plasma

- Energy of n Coulomb charges $\frac{1}{n}$ at positions $x_1, \ldots, x_n$ in $\mathbb{R}^d$:

$$H(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} V(x_i) + \frac{1}{n^2}\sum_{i \neq j} g(x_i - x_j)$$

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- $\propto \mathrm{e}^{-\beta_n(\langle V,\mu_n\rangle + \langle \Delta^{-1}\mu_n,\mu_n\rangle)} \rightsquigarrow$ CLT with Gaussian Free Field

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- Quantitative estimates? How to relate dist and $\mathcal{E}_V(\cdot) - \mathcal{E}_V(\mu_*)$?

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Theorem (Transport type inequality – C.-Hardy-Maïda)

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- Extends free transport inequality to any d

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- Electric field: $\nabla U^\mu(x)$. “Carré du champ”: $|\nabla U^\mu|^2$

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- Finally $\mathrm{W}_1(\mu,\nu)^2 \leq |D|c_d\mathscr{E}(\mu-\nu)$.

Coulomb transport inequality for equilibrium measures

Corollary (Transport type inequality – C.-Hardy-Maïda)

$$\mathrm{d}_{\mathrm{BL}}(\mu,\mu_*)^2 \leq C_{\mathrm{BL}}\Big(\mathscr{E}_V(\mu)-\mathscr{E}_V(\mu_*)\Big).$$

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For any probability measure μ on $\mathbb{R}^d$ with $\mathcal{E}(\mu) < \infty$

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Moreover if V has at least quadratic growth then

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- Growth condition is optimal for W_1

Concentration of measure for Coulomb gases

Theorem (Concentration inequality – C.-Hardy-Maïda)

If V has reasonable growth then for every β, n, r

$$\mathbb{P}\Big(\mathrm{d}_{\mathrm{BL}}(\mu_n,\mu_*) \geq r\Big) \leq \mathrm{e}^{-a\beta n^2 r^2} \quad .$$

Moreover if V has at least quadratic growth then W_1 instead of d_{BL}.

- Optimal order in n

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- Implies speed of convergence:

$$\mathbb{P}\Big(\mathrm{W}_1(\mu_n, \mu_*) \geq r\Big) \leq \mathrm{e}^{-cn^2 r^2}, \quad r \geq \begin{cases} \sqrt{\frac{\log n}{n}} & \text{if } d = 2, \\ n^{-1/d} & \text{if } d \geq 3. \end{cases}$$

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- See also Rougerie & Serfaty

Idea of proof of concentration

- Starting point

$$\mathbb{P}(\mathrm{W}_1(\mu_n,\mu_*) \geq r) = \frac{1}{Z}\int_{\mathrm{W}_1(\mu_n,\mu_*)\geq r} \mathrm{e}^{-\frac{\beta}{2}n^2\mathcal{E}_V^{\neq}(\mu_n)}\mathrm{d}x.$$

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$$\frac{1}{Z} \leq \exp\left\{n^2\frac{\beta}{2}\mathscr{E}_V(\mu_*) - n\left(\frac{\beta}{2}\mathscr{E}(\mu_*) - \mathrm{S}(\mu_*)\right)\right\}.$$

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- Regularization: g superharmonic, $\mu_n^{(\varepsilon)} = \mu_n * \lambda_\varepsilon$,

$$-n^2\mathscr{E}_V^{\neq}(\mu_n) \leq -n^2\mathscr{E}_V(\mu_n^{(\varepsilon)}) + n\mathscr{E}(\lambda_\varepsilon) + n\sum_{i=1}^{n}(V * \lambda_\varepsilon - V)(x_i).$$

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- Coulomb transport $-\mathcal{E}_V(\mu_n^{(\varepsilon)}) + \mathcal{E}_V(\mu_*) \leq -\frac{1}{C}\mathrm{W}_1^2(\mu_n^{(\varepsilon)},\mu_*)$.

Concentration for spectrum of Ginibre matrices

Corollary (Ginibre Random Matrices)

If M is $n \times n$ with iid Gaussian entries of variance $\frac{1}{n}$ in $\mathbb{C}$

- Eigenvalues of $M \propto \exp\left(-n\sum_{i=1}^{n}|x_i|^2\right)\prod_{i<j}|x_i - x_j|^2$

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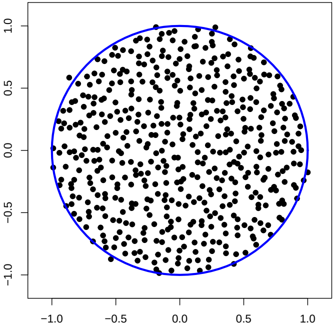
- Bernoulli ± 1 random matrices (universality)
  $\mu_n \to \mu_\bullet$ (Tao-Vu 2010) but $\mathbb{P}(W_1(\mu_n, \mu_\bullet) \geq r)$ is not known

Concentration for spectrum of Ginibre random matrices



`plot(eig(rand(n,n)+i*randn(n,n))/sqrt(2*n)))`

Dynamics leaving invariant in law this picture

Outline

Ginibre process

- Ginibre energy $H(x_1,\ldots,x_n) = \frac{1}{n}\sum_{i=1}^{n}|x_i|^2 + \frac{1}{n^2}\sum_{i\neq j}\log\frac{1}{|x_i - x_j|}$

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$$\mathrm{d}X_t^n=\sqrt{\frac{2}{\beta_n}}\mathrm{d}B_t^n-\nabla H(X_t^n)\mathrm{d}t.$$

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  - Optimal Poincaré and log-Sobolev constants (C.-Lehec)

Second moment dynamics

Theorem (Second moment dynamics – Bolley-C.-Fontbona)

$(R_t)_{t\geq 0} = (\frac{1}{n}\|X_t\|^2)_{t\geq 0}$ *is an ergodic Cox–Ingersoll–Ross process:*

$$\mathrm{d}R_t = \sqrt{\frac{8\alpha_n}{n\beta_n}R_t}\,\mathrm{d}B_t + 4\frac{\alpha_n}{n}\left[\frac{n}{\beta_n} + \frac{n-1}{2n} - R_t\right]\mathrm{d}t.$$

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In particular, with $\gamma_n = \mathrm{Gamma}(n + \frac{n-1}{2n}\beta_n, \beta_n)$*, for any* $t \geq 0$

$$\mathrm{W}_1(\mathrm{Law}(R_t), \gamma_n) \leq \mathrm{e}^{-4\frac{\alpha_n}{n}t}\,\mathrm{W}_1(\mathrm{Law}(R_0), \gamma_n).$$

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Furthermore for any $x \in D$ and $t \geq 0$, we have

$$\mathbb{E}(R_t \mid R_0 = r) = r\mathrm{e}^{-\frac{4\alpha_n}{n}t} + \left(\frac{1}{2} + \frac{n}{\beta_n} - \frac{1}{2n}\right)\left(1 - \mathrm{e}^{-\frac{4\alpha_n}{n}t}\right).$$

Eigenvectors: $\sum_{i=1}^N \Re(z_i)$, $\sum_{i=1}^N \Im(z_i)$, $\sum_{i=1}^N |z_i|^2 + c_N$.

McKean–Vlasov mean-field limit

Theorem? (MKV Mean-field limit – Bolley-C.-Fontbona)

If $\sigma = \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \in [0,\infty)$ *then* $\lim_{n\to\infty} \mu_t^n = \mu_t$ *with*

$$\partial_t \mu_t = \sigma \Delta \mu_t + \nabla \cdot ((\nabla V + \nabla g * \mu_t)\mu_t).$$

- $\mathrm{d}X_t^{i,n} = \sqrt{2\frac{\alpha_n}{\beta_n}}\mathrm{d}B_t^{i,n} - 2\frac{\alpha_n}{n} X_t^{i,n}\mathrm{d}t - 2\frac{\alpha_n}{n}\sum_{j\neq i} \frac{X_t^{i,n} - X_t^{j,n}}{|X_t^{j,n} - X_t^{i,n}|^2}\mathrm{d}t.$

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- Carrillo–McCann–Villani, Fournier–Hauray–Mischler, Serfaty–Duerinckx, ...

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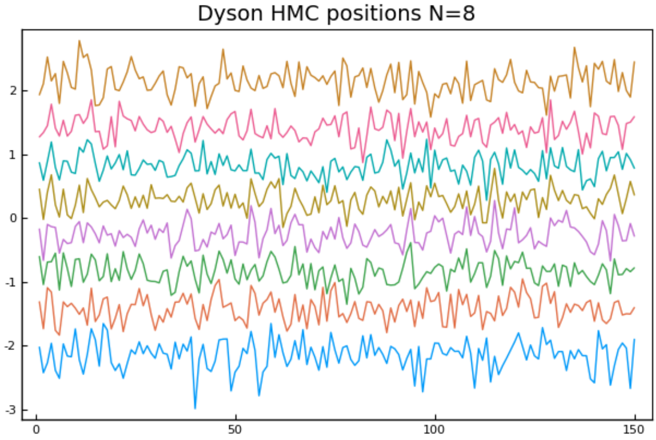
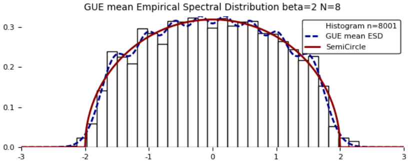
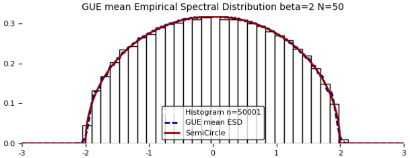
Eight trajectories for a Dyson Ornstein-Uhlenbeck HMC



Equilibrium measures





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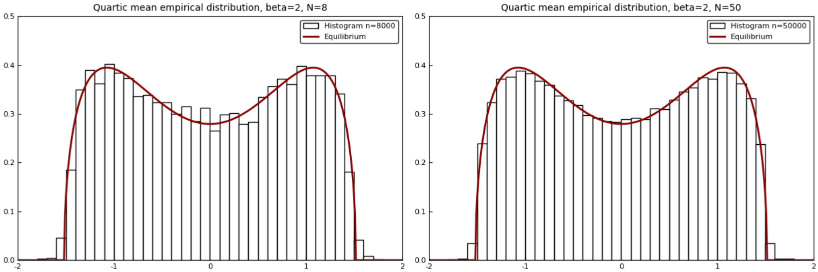

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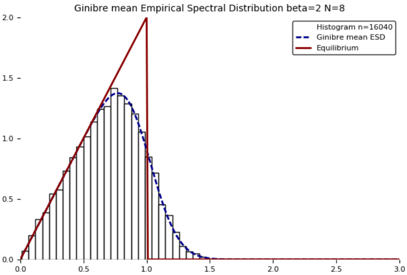

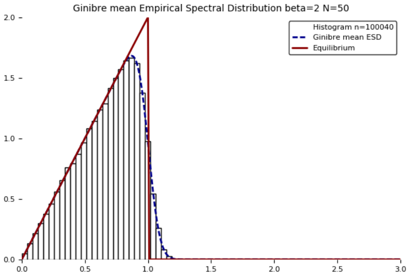

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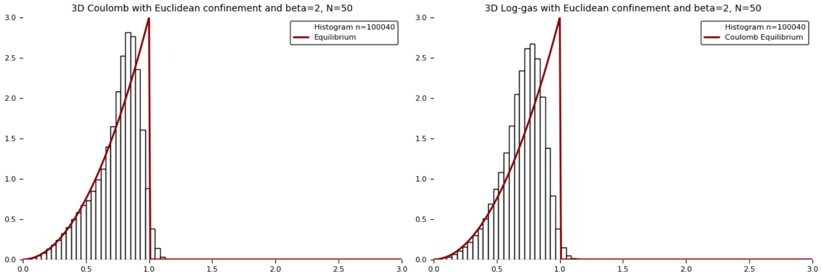
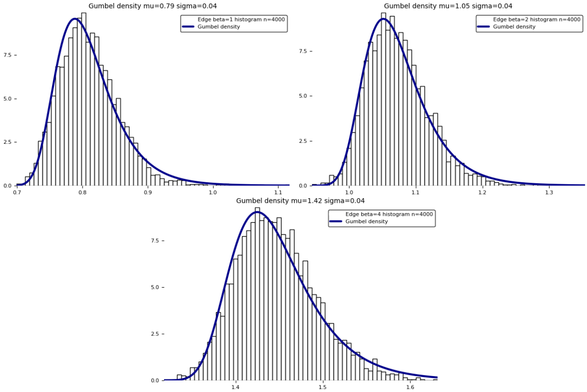


Universal Gumbel fluctuation for edge of beta Ginibre?



$$\sqrt{4n\gamma_n}\left(|\lambda|_{\max} - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{\text{law}} \text{Gumbel}.$$

$$\gamma_n = \log(n) - 2\log(\log(n)) - \log(2\pi).$$