Spectral rigidity for addition of random matrices at the regular edge

Zhigang Bao

HKUST

Random Matrices and Related Topics

KIAS, Seoul, Korea

May 6 -10, 2019

Joint with László Erdős and Kevin Schnelli

Addition of random matrices

Matrix model: Given real $A = \mathrm{diag}(a_1, \ldots, a_N)$ and $B = \mathrm{diag}(b_1, \ldots, b_N)$, consider the model

$$H = A + UBU^*$$

where U is a Haar unitary matrix.

Global spectral distribution [Voiculescu '91]:

Let $\quad \mu_A = \frac{1}{N}\sum_i \delta_{a_i} \qquad \mu_B = \frac{1}{N}\sum_i \delta_{b_i}$

When N is large, The empirical spectral distribution of H

$$\mu_H = \frac{1}{N}\sum_i \delta_{\lambda_i}, \qquad \lambda_1 \geqslant \ldots \geqslant \lambda_N : \text{ eigenvalues of } H$$

is close to the free additive convolution $\mu_A \boxplus \mu_B$.

We choose neither A nor B to be multiples of identity.

Our questions

Theorem [Voiculescu '91] For any fixed interval $\mathcal{I} \subset \mathbb{R}$,

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \xrightarrow{\text{a.s.}} 0, \quad N \to \infty. \qquad \mu_H(\mathcal{I}) = \frac{|\{i : \lambda_i \in \mathcal{I}\}|}{N}$$

Alternative proofs [Speicher'93, Biane'98, Collins'03, Pastur-Vasilchuk'00]

Question 1 (local law) Does the convergence still hold if $|\mathcal{I}| = o(1)$, and how small can $|\mathcal{I}|$ be? (**Answer**: $\frac{1}{N}$)

Question 2 (convergence rate) What is the convergence rate of

$$\sup_{\mathcal{I} \subset \mathbb{R}} \left| \mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I}) \right|$$

(**Answer**: $\frac{1}{N}$)

Question 3 (Spectral rigidity) What is the size of

$$|\lambda_i - \gamma_i|$$

where γ_i is the $N - i + 1$-th N-quantile of $\mu_A \boxplus \mu_B$.

Stieltjes transform

Definition: For any probability measure μ, its Stieltjes transform $m_\mu(z)$ is

$$m_\mu(z) = \int \frac{1}{\lambda - z} \mathrm{d}\mu(\lambda), \qquad z \in \mathbb{C}^+.$$

Inverse formula: one to one correspondence between measure and its Stieltjes transform: density of μ given by

$$\rho(E) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \mathrm{Im}\, m_\mu(E + \mathrm{i}\eta).$$

Notation: For $\alpha = A, B$, and $A \boxplus B$, we will use $m_\alpha(z)$ to denote the Stieltjes transfrom of μ_A, μ_B and $\mu_A \boxplus \mu_B$, respectively. Note that for $\mu_A = \frac{1}{N}\sum \delta_{a_i}$ and $\mu_B = \frac{1}{N}\sum \delta_{b_i}$, we have

$$m_A(z) = \frac{1}{N}\sum \frac{1}{a_i - z}, \qquad m_B(z) = \frac{1}{N}\sum \frac{1}{b_i - z}.$$

Analytic definition of free additive convolution

Theoerm [Belinschi-Bercovici '06, Chistyakov-Götze '05] There exist unique analytic $\omega_A, \omega_B : \mathbb{C}^+ \to \mathbb{C}^+$, s.t. $\Im\omega_k(z) \geqslant \Im z$ and $\lim_{\eta\uparrow\infty} \frac{\omega_k(\mathrm{i}\eta)}{\mathrm{i}\eta} = 1$ for $k = A, B$, such that

$$m_A(\omega_B(z)) = m_B(\omega_A(z)), \qquad -[m_A(\omega_B(z))]^{-1} = \omega_A(z) + \omega_B(z) - z.$$

- $\omega_A(z), \omega_B(z)$: subordination functions

Let $$m(z) := m_A(\omega_B(z)) = m_B(\omega_A(z)).$$

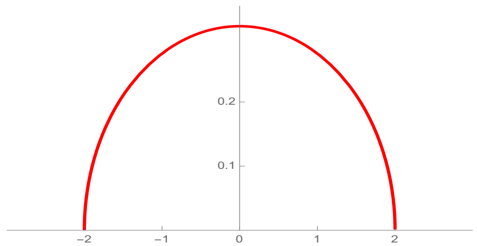
Claim: $m(z)$ is a Stieltjes transform of a probability measure: $\mu_A \boxplus \mu_B$.

- Algebraic definition: Addition of freely independent random variables [Voiculescu '86].

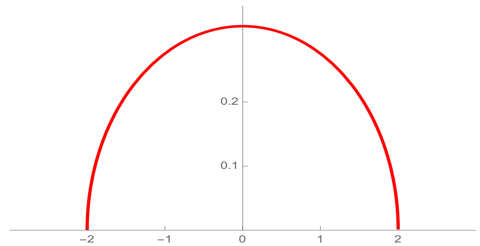
- Subordination phenomenon: [Voiculescu '93], [Biane '98].

examples

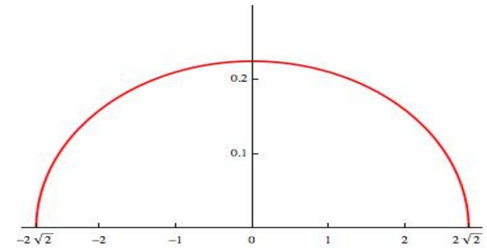
semicircle $\boxplus$ semicircle



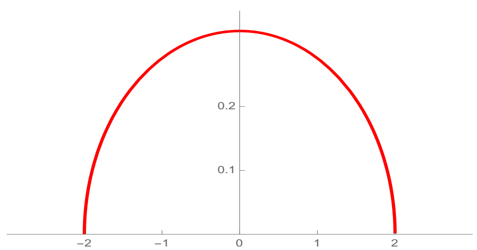
$\boxplus$



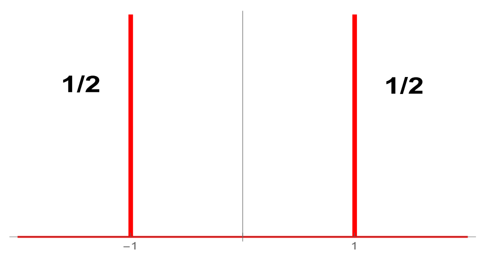
$=$



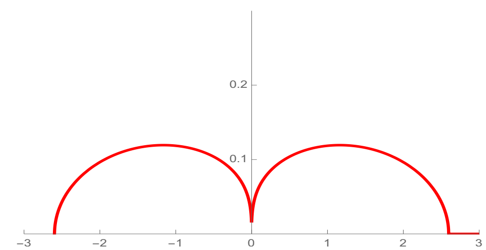
semicircle $\boxplus$ Bernoulli



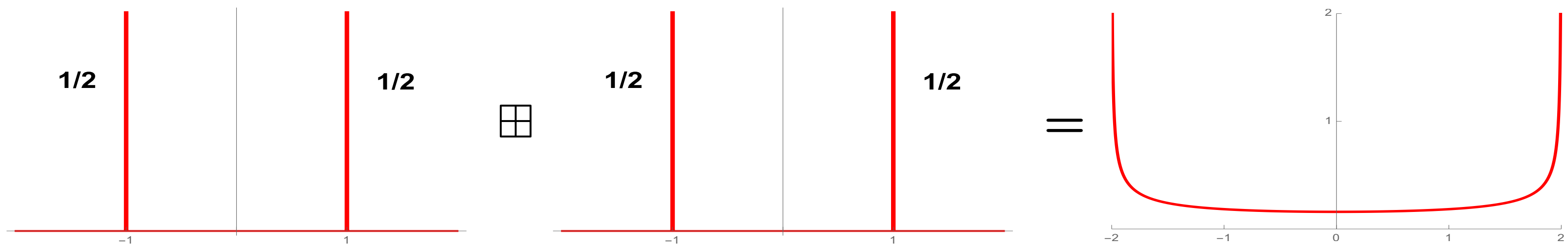
$\boxplus$



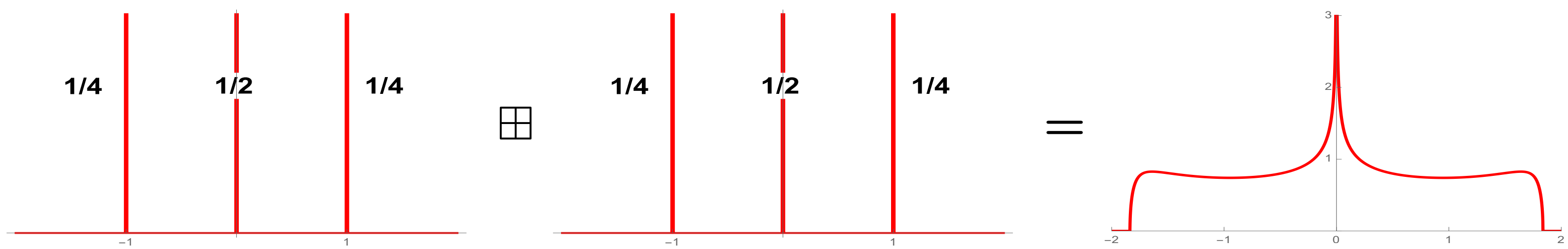
$=$



Bernoulli $\boxplus$ Bernoulli



three point masses $\boxplus$ three point masses



regular bulk: where the density is positive and finite

regular edge: where the density vanishes as a square root

Optimal local law for the regular bulk

Assumption: $\|A\|, \|B\| \leqslant C$; $\mu_A \Rightarrow \mu_\alpha$, $\mu_B \Rightarrow \mu_\beta$; μ_α, μ_β not one point mass

Theorem [B-Erdős-Schnelli '15b] local law for Stieltjes transform

$$\Big| m_H(E + \mathrm{i}\eta) - m_{A \boxplus B}(E + \mathrm{i}\eta) \Big| \prec \frac{1}{N\eta}, \qquad N^{-1+\gamma} \leqslant \eta \leqslant 1, \quad E \in \text{bulk},$$

where m_H is the Stieltjes transform of μ_H.

$$\Downarrow$$

Theorem [B-Erdős-Schnelli '15b] local law for spectral distribution

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \prec \frac{1}{N|\mathcal{I}|}, \qquad N^{-1+\gamma} \leqslant |\mathcal{I}| \leqslant 1, \qquad \mathcal{I} \subset \text{bulk}$$

Previous works: [Kargin'12] ($\eta \geqslant (\log N)^{-1/2}$), [Kargin'15] ($\eta \geqslant N^{-1/7}$), [B.-Erdős-Schnelli'15a] ($\eta \geqslant N^{-2/3}$).

Notation $A \prec B$: $|A| \leqslant N^{\varepsilon}|B|$ with high probability for any given $\varepsilon > 0$.

Extension to the edge: Assumption

Assumption: $\|A\|, \|B\| \leqslant C$; $\mu_A \Rightarrow \mu_\alpha$, $\mu_B \Rightarrow \mu_\beta$ (sufficiently fast), with μ_α, μ_β Jacobi type, i.e., μ_α and μ_β are a.c. with densities ρ_α, ρ_β supported on $[E_-^\alpha, E_+^\alpha]$ and $[E_-^\beta, E_+^\beta]$, respectively, and such that for some $C \geqslant 1$,

$$C^{-1} \leqslant \frac{\rho_\alpha(x)}{(x - E_-^\alpha)^{\alpha_-}(E_+^\alpha - x)^{\alpha_+}} \leqslant C, \qquad a.e. \quad x \in [E_-^\alpha, E_+^\alpha]$$

$$C^{-1} \leqslant \frac{\rho_\beta(x)}{(x - E_-^\beta)^{\beta_-}(E_+^\beta - x)^{\beta_+}} \leqslant C, \qquad a.e. \quad x \in [E_-^\beta, E_+^\beta]$$

with exponents

$$-1 < \alpha_\pm, \beta_\pm < 1.$$

Theorem [B.-Erdős-Schnelli '18] Let μ_α and μ_β be of Jacobi type. Then $\mathrm{supp}\mu_\alpha \boxplus \mu_\beta = [E_-, E_+]$ for some $E_- < E_+ \in \mathbb{R}$, and the density $\rho_{\alpha\boxplus\beta}$ of $\mu_\alpha \boxplus \mu_\beta$ satisfies

$$C^{-1} \leqslant \frac{\rho_{\alpha\boxplus\beta}(x)}{\sqrt{x - E_-}\sqrt{E_+ - x}} \leqslant C, \qquad a.e. \quad x \in [E_-, E_+].$$

Similar problem was considered in [Olver-Nadakuditi '12].

Extension to the edge: Results

Assumption: $\|A\|, \|B\| \leqslant C$; $\mu_A \Rightarrow \mu_\alpha$, $\mu_B \Rightarrow \mu_\beta$ (sufficiently fast), with μ_α, μ_β Jacobi type

Theorem [B.-Erdős-Schnelli '16-'18] Under the above assumption

(i)(local law) For any fixed $\gamma > 0$, and any compact interval $\mathcal{I} \subset \mathbb{R}$ with $|\mathcal{I}| \geqslant N^{-1+\gamma}$,

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \prec \frac{1}{N|\mathcal{I}|}$$

(ii) (convergence rate)

$$\sup_{\mathcal{I} \subset \mathbb{R}} \left| \mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I}) \right| \prec \frac{1}{N}$$

(iii) (rigidity) For any $i = 1, \dots, N$,

$$|\lambda_i - \gamma_i| \prec \max\{i^{-\frac{1}{3}}, (N - i + 1)^{-\frac{1}{3}}\} N^{-\frac{2}{3}}$$

Local law of Green function

Green function: $G(z) := (H - z)^{-1}$, note

$$m_H(z) = \frac{1}{N}\sum \frac{1}{\lambda_i - z} = \mathrm{tr}\, G(z) = \frac{1}{N}\sum G_{ii}(z), \qquad \mathrm{tr} = \frac{1}{N}\mathrm{Tr}.$$

Theorem [B.-Erdős-Schnelli '16-'18] Let $z = E + \mathrm{i}\eta$. Under the previous assumption, for any $N^{-1+\gamma} \leqslant \eta \leqslant 1$ with any small $\gamma > 0$

(i) (Green function subordination)

$$\max_{i,j}\left|G_{ij}(z) - \frac{\delta_{ij}}{a_i - \omega_B(z)}\right| \prec \frac{1}{\sqrt{N\eta}}$$

(ii) (Local law for Stieltjes transform)

$$\left|m_H(z) - m_{A\boxplus B}(z)\right| \prec \frac{1}{N\eta}$$

(iii) (Improvement of (ii) outside the support)

$$\left|m_H(z) - m_{A\boxplus B}(z)\right| \prec \frac{1}{N(\kappa + \eta)}, \qquad \kappa := \mathrm{dist}(E, \partial\, \mathrm{supp}(\mu_A \boxplus \mu_B))$$

when $E \in \mathbb{R} \setminus \mathrm{supp}(\mu_A \boxplus \mu_B)$ and $\kappa \geqslant N^{-\frac{2}{3}+\varepsilon}$.

Local laws in RMT

Local laws for Wigner type matrices were widely studied in the last ten years. A key difference for the additive model is the complicated dependence structure of the entries of the Haar unitary.

For the model discussed: Universality of local bulk eigenvalue statistics was proved in [Che-Landon '17]

Some reference (on optimal scale)

• (Wigner type) [Erdös-Schlein-Yau '07-'09], [Tao-Vu '09-'12], [Erdös-Yau-Yin '10-'12], [Erdös-Knowles-Yau-Yin '13], [Götze-Naumov-Tikhomirov-Timushev '16], [Götze-Naumov-Tikhomirov '15-'19],...

• (Addition of Wigner type) [Lee-Schnelli '13], [Knowles-Yin '14], [He-Knowles-Rosenthal '16], [Ajanki-Erdös-Krüger '16], [Erdös, Krüger, Schröder, '18]...

• (Random d-regular graph) [Bauerschmidt-Knowles-Yau '15]...

Perturbed subordination equation for random matrix

Subordination equation: $\Phi_{\mu_A,\mu_B}(\omega_A(z),\omega_B(z),z)=0$, where

$$\Phi_{\mu_A,\mu_B}(\omega_1,\omega_2,z):=\begin{pmatrix} -(m_A(\omega_2))^{-1}-\omega_1-\omega_2+z \\ -(m_B(\omega_1))^{-1}-\omega_1-\omega_2+z \end{pmatrix}$$

Approximate subordination functions

$$\omega_A^c(z):=z-\frac{\mathrm{tr}AG(z)}{m_H(z)},\qquad \omega_B^c(z):=z-\frac{\mathrm{tr}UBU^*G(z)}{m_H(z)}.$$

By $(A+UBU^*-z)G=I$, we have

$$(m_H(z))^{-1}=-\omega_A^c(z)-\omega_B^c(z)+z.$$

Observe that

$$\begin{pmatrix} \big(m_H(z)\big)^{-1}-\big(m_A(\omega_B^c(z))\big)^{-1} \\ \\ \big(m_H(z)\big)^{-1}-\big(m_B(\omega_A^c(z))\big)^{-1} \end{pmatrix}=\Phi_{\mu_A,\mu_B}(\omega_A^c,\omega_B^c,z)$$

Denote by

$$\Lambda_i(z) := \omega_i^c(z) - \omega_i(z), \quad i = A, B,$$

In order to estimate $\Lambda_i(z)$, we need two ingredients:

(i): A stability analysis of the equation $\Phi_{\mu_A,\mu_B}(\omega_A(z), \omega_B(z), z) = 0$.

(ii): An estimate of $\Phi_{\mu_A,\mu_B}(\omega_A^c, \omega_B^c, z) = (\Phi_1^c, \Phi_2^c)^T$, where

$$\Phi_1^c = (m_H)^{-1} - (m_A(\omega_B^c))^{-1}, \qquad \Phi_2^c = (m_H)^{-1} - (m_B(\omega_A^c))^{-1}.$$

Local stability for subordination equation

Expansion of the perturbed subordination eq. around $(\omega_A(z), \omega_B(z), z)$ gives

$$\mathcal{S}\Lambda_A + \mathcal{T}_A\Lambda_A^2 + \cdots = \Phi_1^c + (F_A'(\omega_B) - 1)\Phi_2^c$$
$$\mathcal{S}\Lambda_B + \mathcal{T}_B\Lambda_B^2 + \cdots = \Phi_2^c + (F_B'(\omega_A) - 1)\Phi_1^c$$

where $F_i(\cdot) = -1/m_i(\cdot), i = A, B$ are the negative reciprocal Stieltjes transforms, and

$$\mathcal{S} = \big(F_A'(\omega_B(z)) - 1\big)\big(F_B'(\omega_A(z)) - 1\big) - 1$$
$$\mathcal{T}_A = \frac{1}{2}\Big(F_A''(\omega_B(z))\big(F_B'(\omega_A(z)) - 1\big)^2 + F_B''(\omega_A(z))\big(F_A'(\omega_B(z)) - 1\big)\Big)$$

and $\mathcal{T}_B$ is defined analogously.

Basic facts:

$$\mathcal{S}(z) \sim \sqrt{\kappa + \eta}, \qquad \mathcal{T}_A(z) \sim 1, \qquad \mathcal{T}_B(z) \sim 1$$

Estimate of the random Φ

Roughly, our aim is to show that

$$|\Phi_1^c + (F_A'(\omega_B) - 1)\Phi_2^c| \prec \frac{\Im m_{A\boxplus B}}{N\eta},$$
$$|\Phi_2^c + (F_B'(\omega_A) - 1)\Phi_1^c| \prec \frac{\Im m_{A\boxplus B}}{N\eta}, \qquad \eta = \Im z.$$

Basic facts:

$$\Im m_{A\boxplus B}(z) \sim \Im \omega_A(z) \sim \Im \omega_B(z) \sim \begin{cases} \sqrt{\kappa+\eta}, & E \in \mathrm{supp}(\mu_A \boxplus \mu_B) \\ \frac{\eta}{\sqrt{\kappa+\eta}}, & E \in \mathbb{R}\setminus \mathrm{supp}(\mu_A \boxplus \mu_B) \end{cases}$$

Recall

$$\Phi_1^c = \big(m_H(z)\big)^{-1} - \big(m_A(\omega_B^c(z))\big)^{-1}, \qquad \Phi_2^c = \big(m_H(z)\big)^{-1} - \big(m_B(\omega_A^c(z))\big)^{-1}$$

Hence, essentially, one needs to bound

$$m_H - m_A(\omega_B^c) = \frac{1}{N}\sum_i \Big(G_{ii} - \frac{1}{a_i - \omega_B^c}\Big).$$

and its analogue via switching the role of A and B.

Heuristic of Green function subordination

Goal:

$$G_{ii} \sim \frac{1}{a_i - \omega_B^c}, \qquad \omega_B^c = z - \frac{\mathrm{tr}\widetilde{B}G(z)}{\mathrm{tr}G(z)}, \quad \widetilde{B} := UBU^*$$

By $(A + \widetilde{B} - z)G(z) = I$, we have $(a_i - z)G_{ii} + (\widetilde{B}G)_{ii} = 1$, so that

$$G_{ii} = \frac{1}{a_i - z + \frac{(\widetilde{B}G)_{ii}}{G_{ii}}}.$$

We shall show

$$\left|(\widetilde{B}G)_{ii}\mathrm{tr}G - G_{ii}\mathrm{tr}\widetilde{B}G\right| \prec \frac{1}{\sqrt{N\eta}},$$

and

$$\left|\frac{1}{N}\sum_i d_i\big((\widetilde{B}G)_{ii}\mathrm{tr}G - G_{ii}\mathrm{tr}\widetilde{B}G\big)\right| \prec \frac{\Im m_H}{N\eta}$$

for some specifically chosen (random) d_i's.

Ward Identity

For $t \in \mathbb{R}$, set

$$G_t(z) := \frac{1}{A + \mathrm{e}^{\mathrm{i}tX} UBU^* \mathrm{e}^{-\mathrm{i}tX} - z}, \qquad X = X^*$$

Left invariance of Haar measure implies

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathbb{E} G_t(z) = -\mathrm{i}\mathbb{E}\big(G_0(z)[X, UBU^*]G_0(z)\big),$$

which further implies

$$\mathbb{E} G \otimes (\widetilde{B} G) = \mathbb{E}(G \widetilde{B}) \otimes G$$

Therefore,

$$\mathbb{E}(\widetilde{B} G)_{ii} \mathrm{tr} G = \mathbb{E} G_{ii} \mathrm{tr} \widetilde{B} G$$

First try for concentration: use the randomness of U at once: Gromov-Milman, can only reach $\eta \gg N^{-\frac{1}{4}}$.

Recursive moment estimate

Let $$Q_i := (\widetilde{B}G)_{ii}\mathrm{tr}G - G_{ii}\mathrm{tr}\widetilde{B}G$$

Set for $k,\ell \in \mathbb{N}$, and some specifically chosen d_i's,

$$\mathfrak{m}_i^{(k,\ell)} := (Q_i)^k(\overline{Q_i})^\ell, \qquad \mathfrak{m}^{(k,\ell)} := \Big(\frac{1}{N}\sum_i d_iQ_i\Big)^k\Big(\frac{1}{N}\sum_i \overline{d_iQ_i}\Big)^\ell$$

Proposition For any $N^{-1+\gamma} \leqslant \eta \leqslant 1$, and $k \geqslant 2$,

$$\mathbb{E}\big[\mathfrak{m}_i^{(k,k)}\big] = \mathbb{E}\Big[O_\prec(\frac{1}{\sqrt{N\eta}})\mathfrak{m}_k^{(k-1,k)}\Big] + \mathbb{E}\Big[O_\prec(\frac{1}{N\eta})\mathfrak{m}_i^{(k-2,k)}\Big] + \mathbb{E}\Big[O_\prec(\frac{1}{N\eta})\mathfrak{m}_i^{(k-1,k-1)}\Big]$$

$$\mathbb{E}\Big[\mathfrak{m}^{(k,k)}\Big] = \mathbb{E}\Big[O_\prec(\frac{\Im m_H}{N\eta})\mathfrak{m}_k^{(k-1,k)}\Big] + \mathbb{E}\Big[O_\prec\Big((\frac{\Im m_H}{N\eta})^2\Big)\mathfrak{m}_i^{(k-2,k)}\Big] + \mathbb{E}\Big[O_\prec\Big((\frac{\Im m_H}{N\eta})^2\Big)\mathfrak{m}_i^{(k-1,k-1)}\Big]$$

Then the desired estimates of Q_i and $\frac{1}{N}\sum d_iQ_i$ follow by using Young and Markov inequalities.

Householder reflection as partial randomness

Proposition [Diaconis-Shahshahani '87] U: Haar on $\mathcal{U}(N)$,

$$U = -\mathrm{e}^{\mathrm{i}\theta_1}(I - \mathbf{r}_1\mathbf{r}_1^*)\begin{pmatrix} 1 & \\ & U^1 \end{pmatrix}, \qquad \mathbf{r}_1 := \sqrt{2}\frac{\mathbf{e}_1 + \mathrm{e}^{-\mathrm{i}\theta_1}\mathbf{v}_1}{\|\mathbf{e}_1 + \mathrm{e}^{-\mathrm{i}\theta_1}\mathbf{v}_1\|_2}$$

$\mathbf{v}_1 \in \mathcal{S}_{\mathbb{C}}^{N-1}$: uniform, $U^1 \in \mathcal{U}(N-1)$: Haar, $\mathbf{v}_1$, U^1 independent.

Remark: Analogously, we have independent pair $\mathbf{v}_i$ and U^i for all i. Actually, $-\mathrm{e}^{\mathrm{i}\theta_i}(I - \mathbf{r}_i\mathbf{r}_i^*)$ is the Householder reflection sending $\mathbf{e}_i$ to $\mathbf{v}_i$.

THANK YOU!