Spherical Sherrington-Kirkpatrick model and random matrices

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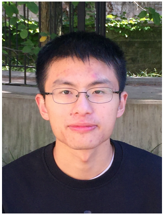

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The **Spherical Sherrington-Kirkpatrick (SSK)** model is defined by the random Gibbs measure

$$p(\sigma) = \frac{1}{Z_N} e^{\beta H(\sigma)} \quad \text{where } H(\sigma) = \frac{1}{2}\sigma^T M \sigma$$

with (traceless) GOE matrix M for

$$\sigma \in \mathbb{R}^N \text{ with } \|\sigma\| = \sqrt{N}$$

The inverse temperature is

$$\beta = \frac{1}{T}$$

When $T = 0$, the spin is concentrated on $\sigma = \pm u_1$. When $T = \infty$, the spin is uniformly distributed on the sphere.

If the sphere is replaced by the hypercube $\sigma \in \{-1, 1\}^N$, then it is called the **Sherrington-Kirkpatrick (SK) model**.

More generally, one may consider random symmetric polynomial $H(\sigma)$ on a manifold/graph. For example, 3-spin Hamiltonian is $H(\sigma) = \sum_{i,j,k} M_{ijk}\sigma_i\sigma_j\sigma_k$

We only consider SSK model in which $H(\sigma)$ is a quadratic function of σ

We define the **free energy** (per spin component) of the SSK by

$$F_N = \frac{1}{N\beta} \log Z_N = \frac{1}{N\beta} \log \left[\int_{\|\sigma\|=\sqrt{N}} e^{\frac{\beta}{2}\sigma^T M\sigma} \mathrm{d}\Omega(\sigma) \right]$$

For the zero-temperature case (i.e. $\beta = \infty$),

$$F_N = \frac{1}{2N} \max_{\|\sigma\|=\sqrt{N}} \sigma^T M\sigma = \frac{\lambda_1}{2}$$

Hence, F_N with $T > 0$ is a finite-temperature version of the largest eigenvalue.

We consider the spherical spin glass model as $N \to \infty$ for

(1) SSK $\qquad H(\sigma) = \frac{1}{2}\sigma^T M \sigma$

(2) SSK + external field $\qquad H(\sigma) = \frac{1}{2}\sigma^T M \sigma + h\sigma^T g$

We use random matrix theory to study the **fluctuations** of the free energy and the **spin distributions**. For simplicity, we assume that M is GOE. We assume that M is scaled to that $\lambda_1 \to 2$. For (1), RMT tells us that

$$F_N \overset{\mathcal{D}}{\simeq} 1 + \frac{\mathrm{TW}_1}{2N^{2/3}} \quad \text{at } T = 0$$

Outline

(1) **SSK model** $H(\sigma) = \frac{1}{2}\sigma^T M\sigma$

(i) Fluctuation results

(ii) History

(iii) Random single integral formula

(iv) Linear statistics vs largest eigenvalue

(2) SSK+external field

Fluctuations of free energy

Theorem [Baik and Lee 2016]

- For $T < 1$,

$$F_N \overset{\mathcal{D}}{\simeq} \left(1 - \frac{3T}{4} + \frac{T \log T}{2}\right) + \frac{1-T}{2N^{2/3}} \mathrm{TW}_1$$

- For $T > 1$,

$$F_N \overset{\mathcal{D}}{\simeq} \frac{1}{4T} + \frac{T}{2N}\mathcal{N}(-\alpha, 4\alpha)$$

  where $\alpha = -\frac{1}{2}\log(1 - T^{-2})$

$T = 1$ is open

History

The limiting free energy $F_N \to F$ was obtained for general spin glass models.

- For SSK, Kosterlitz, Thouless, Jones (1976), Guionnet and Maïda (2005), Panchenko and Talagrand (2007)
- For gpin glass with general Hamiltonian, **Parisi formula** (1980) for $\{-1,1\}^N$, Crisanti and Sommers formula (1992) for spherical case
- Rigorously proved by Guerra (2003), Talagrand (2006), Panchenko (2014)

The fluctuations of $F_N - F$ are not as well studied.

- SK for **high temperature**, $T > 1$: Gaussian, N^{-1} [Aizenmann, Lebowitz, Ruelle 1987, Fröhlich and Zegarliński 1987, Comets and Neveu 1995]
- pure p-spin spin glass **high temperature** $T > T_0$: Gaussian, $N^{-p/2}$ [Bovier, Kurkova, and Löwe 2002]
- pure p-spin **spherical** spin glass with $p \geq 3$ **zero temperature**, $T = 0$: Gumbel N^{-1} [Subag and Zeitouni 2017]
- Spin glass with external field: see below.

Lemma [Kosterlitz, Thouless, Jones 1976]

$$Z_N = C_N \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}G(z)} dz \quad \text{where } G(z) = \beta z - \frac{1}{N}\sum_{k=1}^{N} \log(z-\lambda_k)$$

with $\gamma > \lambda_1$

Proof: By definition, $Z_N = \int_{\|\sigma\|=\sqrt{N}} e^{\frac{\beta}{2}\sigma^T M\sigma} \mathrm{d}\Omega(\sigma) = \int_{\|u\|=\sqrt{N}} e^{\frac{\beta}{2}\sum_i \lambda_i a_i^2} \mathrm{d}\Omega(a)$

- Let $f(\mathbf{r}) = \frac{1}{2} r^{N/2-1} \int_{\|u\|=1} e^{\mathbf{r}\sum_i \lambda_i a_i^2} \mathrm{d}\omega(a)$
- **Laplace transform** $L(z) = \int_0^\infty e^{-zr} f(r)\mathrm{d}r = \int_0^\infty e^{-zr^2} f(r^2) 2r\mathrm{d}r$
- By Gaussian integral, $L(z) = \int_{\mathbb{R}^N} e^{-z\sum y_i^2 + \sum \lambda_i y_i^2} \mathrm{d}^N y = \prod_{i=1}^N \sqrt{\frac{\pi}{z-\lambda_i}}$
- **Inverse Laplace transform** $f(r) = \frac{1}{2\pi \mathrm{i}} \int e^{rz} L(z) \mathrm{d}z$

Does the method of steepest-descent apply to random integrals? Yes, thanks to

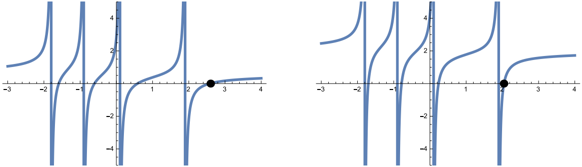
Rigidity of eigenvalues [Erdös, Yau and Yin (2012)]

$$|\lambda_k - \gamma_k| \leq \hat{k}^{-1/3} N^{-2/3+\epsilon} \quad \text{uniformly for } 1 \leq k \leq N \text{ **with high probability**}$$

where $\hat{k} = \min\{k, N+1-k\}$ and γ_k is the **classical location** (i.e. quantile of the semicircle law), $\int_{\gamma_k}^{2} \frac{\sqrt{4-x^2}}{2\pi} \mathrm{d}x = \frac{k}{N}$

Find z solving

$$G'(z) = \beta - \frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\lambda_k} = 0 \qquad \text{satisfying } \mathrm{Re}(z) > \lambda_1$$



Since the Stieltjes transform

$$s(z) = \int \frac{\mathrm{d}\sigma_{\mathrm{sc}}(x)}{z-x}$$

satisfies $s(2) = 1$ and $s(\infty) = 0$, the approximate equation

$$G'(z) \approx \beta - s(z) = 0$$

has a solution $z_c > 2$ only if $\beta < 1$.

We have

- For $\beta < 1$ (i.e. $T > 1$),

$$\beta - s(z) = 0 \text{ is a good approximate and } z_c \simeq \beta + \frac{1}{\beta}$$

- For $\beta > 1$ (i.e. $T < 1$),

$$z_c = \lambda_1 + O(N^{-1+\epsilon}) \text{ with high probability}$$

High temperature regime $T > 1$

We find that

$$F_N = \frac{1}{N\beta} \log Z_N = \frac{1}{N\beta} \log \left[C_N \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{N}{2} G(z)} dz \right] \approx \frac{1}{2\beta} G(z_c) + c_N$$

With $z_c = \beta + \frac{1}{\beta}$,

$$G(z_c) = \beta z_c - \frac{1}{N} \sum_{k=1}^{N} \log(z_c - \lambda_k)$$

For $T > 1$, a linear statistic gives fluctuations

$$F_N = \frac{1}{4T} + \frac{T}{2N}\left(\log(1 - T^{-2}) - L_N\right) + O(N^{-2+\epsilon})$$

with high probability where

$$L_N = \sum_{i=1}^{N} g(\lambda_i) - N\int_{-2}^{2} g(x)\mathrm{d}\sigma_{sc}(x)$$

with $g(x) = \frac{1}{2}\log\left(T + T^{-1} - x\right)$

Linear statistics [Johansson 1998, Bai-Silverstein 2004, Lytova-Pastur 2009]
For smooth g,

$$L_N \xrightarrow{D} \mathcal{N}(a, b)$$

- The critical point $z_c = \lambda_1 + O(N^{-1+\epsilon})$ and λ_1 is a branch point
- It still holds that $\log \left[\int e^{\frac{N}{2} G(z)} dz \right] \simeq \frac{N}{2} G(z_c)$
- Using $z_c = \lambda_1 + O(N^{-1+\epsilon})$ and noting $\lambda_1 = 2 + O(N^{-2/3+\epsilon})$

$$
\begin{aligned}
G(z_c) &= \beta z_c - \frac{1}{N}\sum_{i=2}^{N} \log(z_c - \lambda_i) - \frac{1}{N}\log(z_c - \lambda_1) \\
&= \beta\lambda_1 - \frac{1}{N}\sum_{i=2}^{N} \log(\lambda_1 - \lambda_i) + O(N^{-1+\epsilon}) \\
&\simeq \beta\lambda_1 - \frac{1}{N}\sum_{i=2}^{N} \left[\log(2 - \lambda_i) + \frac{1}{2-\lambda_i}(\lambda_1 - 2) \right] \\
&\simeq \beta\lambda_1 - \int_{-2}^{2} \log(2-s)\,\mathrm{d}\sigma_{sc}(s) - (\lambda_1 - 2)\int_{-2}^{2} \frac{\mathrm{d}\sigma_{sc}(s)}{2-s}
\end{aligned}
$$

For $T < 1$, the largest eigenvalue gives the fluctuations

$$F_N = \left(1 - \frac{3T}{4} + \frac{T \log T}{2}\right) + \frac{1-T}{2}(\lambda_1 - 2) + O(N^{-1+\epsilon})$$

with high probability

(1) SSK model

(2) **SSK + external field** $H(\sigma) = \frac{1}{2}\sigma^T M\sigma + h\sigma^T g$

  (i) Free energy

  (ii) Spin distribution; overlaps

$$H(\sigma) = \frac{1}{2}\sigma^T M\sigma + h\sigma^T g$$

- $g = (g_1, \cdots, g_N)$ is a standard normal vector
- h is a coupling constant
- (Chen, Dey, Panchenko 2017) The free energy has the Gaussian fluctuations with $N^{-1/2}$ scale **for all $T > 0$ and $h > 0$**
- On the other hand, if $h = 0$, there is a transition at $T = 1$
- (Fyodorov and le Doussal 2014) For $T = 0$, the number of local max/min of $H(\sigma)$ has a transition when $h = O(N^{-1/6})$
- Goal: (a) Recover [CDP] result when $h > 0$ and (b) study the case when $h = HN^{-1/6}$

Let u_i be a unit **eigenvector** associated to λ_i. We still have the random integral formula with

$$G(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i) + \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i} \quad \text{where } n_i = u_i^T g$$

We have

$$G'(z) = \beta - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} - \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^2} \approx \beta - s(z) + h^2\beta s'(z)$$

Since $s'(2) = -\infty$ unlike $s(2) = 1$, the approximate equation has a solution $z_c > 2$ for **all** $\beta > 0$ when $h > 0$.

For the fluctuations, consider

$$\sum_{i=1}^{N} \frac{n_i^2}{z_c - \lambda_i} = \sum_{i=1}^{N} \frac{1}{z_c - \lambda_i} + \sum_{i=1}^{N} \frac{n_i^2 - 1}{z_c - \lambda_i}$$

The first sum has fluctuations of $O(1)$ from **linear statistics**. The second sum has fluctuations of $O(\sqrt{N})$ by **usual CLT**. We recover [CDP] result:

$$F_N \simeq F(T,h) + \frac{\mathcal{N}(a,b)}{N^{1/2}}$$

Now, if we take $h \to 0$ while $N \to \infty$?

Conjecture [Baik, le Doussal, Wu 2019]

For $T < 1$ and $h = HN^{-1/6}$, with high probability,

$$F_N \simeq \left(1 - \frac{3T}{4} + \frac{T \log T}{2} + \frac{h^2}{2}\right) + \frac{\mathcal{F}}{N^{2/3}}$$

Let $\{\alpha_i\}$ be a **GOE Airy point process** ($\alpha_i \sim -(3\pi i/2)^{2/3}$ as $i \to \infty$) and let $\{\nu_i\}$ be independent **standard normal** random variables. Let $\mathbf{s} > 0$ be the solution of the equation

$$\frac{1-T}{H^2} = \sum_{i=1}^{\infty} \frac{\nu_i^2}{(\mathbf{s} + \alpha_1 - \alpha_i)^2}$$

Set (cf. [Landon and Sosoe 2019])

$$\mathcal{E}(\mathbf{s}) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\nu_i^2}{\mathbf{s} + \alpha_1 - \alpha_i} - \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right)$$

Then,

$$\mathcal{F} \stackrel{\mathcal{D}}{=} \frac{(1-T)(s+\alpha_1) + H^2 \mathcal{E}(s)}{2}$$

Overlap I.

For a given realization of M and g, the spin is distributed as

$$p(\sigma) = \frac{1}{Z_N} e^{\beta(\frac{1}{2}\sigma^T M\sigma + h\sigma^T g)}$$

The **overlap with the external field** is the cosine angle of spin and ext. field

$$\mathfrak{M} = \frac{1}{N}\sigma^T g$$

We consider the Gibbs distribution of $\mathfrak{M}$ for given disorder variables, i.e. for given (a) external field g, (b) eigenvectors $u_1, \cdots, u_N$, and (c) eigenvalues $\lambda_1, \cdots, \lambda_N$ In particular,

$$n_1 = u_1^T g$$

is given. (This is $O(1)$ with high probability and defined only up to $\pm$)

When $h = 0$, the spin distribution is well-known:

i. When $T = 0$, $\sigma = \pm u_1$

ii. When $T < 1$, σ is uniform on the **double cone** about $\pm u_1$ with half vertex angle $\cos^{-1}(\sqrt{1-T})$ (with high probability). Note that the angle of the cone does not depend on the disorder variables.

iii. When $T > 1$, σ is asymptotically uniform on the sphere.

We focus on the interesting case, $0 \le T < 1$.

For $h = 0$, g is an independent of σ. Thus, with $P(\mathfrak{B} = 1) = P(\mathfrak{B} = -1) = 1/2$,

$$\mathfrak{M} \overset{\mathcal{D}}{\simeq} \frac{|n_1|\sqrt{1-T}\mathfrak{B} + \sqrt{T}\mathfrak{N}(0,1)}{\sqrt{N}} \quad \text{for } h = 0 \text{ and } T < 1$$

where $n_1 = u_1^T g$.

Conjecture [Baik, le Doussal, Wu 2019] For $T < 1$ and $h = HN^{-1/2}$,

$$\mathfrak{M} \overset{\mathcal{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}_H + \sqrt{T}\mathfrak{N}(0,1)}{\sqrt{N}} \quad \text{where } n_1 = u_1^T g$$

with high probability where $\mathfrak{B}_H \in \{-1, 1\}$ and

$$P(\mathfrak{B}_H = 1) = \frac{e^{\frac{H|n_1|\sqrt{1-T}}{T}}}{e^{\frac{H|n_1|\sqrt{1-T}}{T}} + e^{-\frac{H|n_1|\sqrt{1-T}}{T}}}$$

When $H = 0$, we two values ± 1 become equally likely (double cone)

When $H \to \infty$, $\mathfrak{B}_H \to 1$ (single cone)

Conjecture [Baik, le Doussal, Wu 2019] For $T < 1$,

$$\left(\frac{\sigma^T u_1}{\sqrt{N}}\right)^2 \to \begin{cases} 0 & \text{for } h > 0 \\ 1 - T - H^2 \sum_{i=2}^{\infty} \frac{\nu_i^2}{(s+\alpha_1-\alpha_i)^2} & \text{for } h = HN^{-1/6} \\ 1 - T & \text{for } h = 0 \end{cases}$$

with high probability. Here, $s > 0$ is the solution of $\frac{1-T}{H^2} = \sum_{i=1}^{\infty} \frac{\nu_i^2}{(s+\alpha_1-\alpha_i)^2}$

When $h = 0$, the vertex angle of (double) cone does not depend on disorder variables.

When $h = HN^{-1/6}$, the vertex angle depends on disorder variables.

Fluctuations are also obtained. The case of $h = 0$ is related to the work of Sosoe and Vu 2018 and Landon and Sosoe 2019

Summary

1. Spherical spin glass is defined by random Gibbs measure on a sphere
2. Two Hamiltonians were considered: (1) SSK, (2) SSK + external field
3. There is a random integral formula (single-variable!) for the partition function to which the method of steepest-descent is applicable using the rigidity of the eigenvaues
4. The fluctuations of the free energy were obtained. There are interesting transitional behaviors.
5. Overlaps of the spin with the external field and also u_1 are studied.

Thank you for attention!