On the interface between Hermitian and normal random matrices

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External potential

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$$S \subset \operatorname{Int}\{Q < +\infty\}.$$

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$$d\sigma = \Delta Q \cdot \mathbf{1}_S \cdot dA.$$

Note that $\Delta Q \geq 0$ on S. *We assume that* $\Delta Q > 0$ *on the boundary* ∂S.

Sakai's theorem

The boundary ∂S is regular with possible cusps/double points.

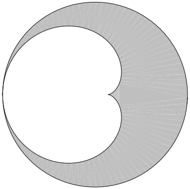

Figure: The figure on the left shows a boundary with two singular points: one double point and one cusp.

Not all cusps are possible: $\frac{3}{2}$, $\frac{7}{2}$,... are excluded.

RNM-model

Particles/eigenvalues $\{\zeta_j\}_1^n$ in external field nQ.
Energy:

$$H_n = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_1^n Q(\zeta_j).$$

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$$dP_n(\zeta) = e^{-H_n(\zeta)} \, dA_n(\zeta) / \int_{\mathbb{C}^n} e^{-H_n} \, dA_n.$$

(dA_n Lebesgue measure).

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Coulomb gas at $\beta = 1/(k_B T)$: replace $H_n \leftrightarrow \beta H_n$.

Ginibre ensemble

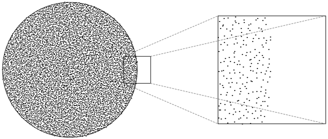


Figure: A sample from the standard Ginibre ensemble.

Correlation kernel

We have a determinantal process, k-point functions

$$\mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k) = \det(\mathbf{K}_n(\zeta_i, \zeta_j))_{k\times k}.$$

Here $\mathbf{R}_n(\zeta) := \mathbf{R}_{n,1}(\zeta)$ is expected number of particles per unit area.
The kernel $\mathbf{K}_n$ is reproducing kernel for the subspace of L^2 of weighted polynomials

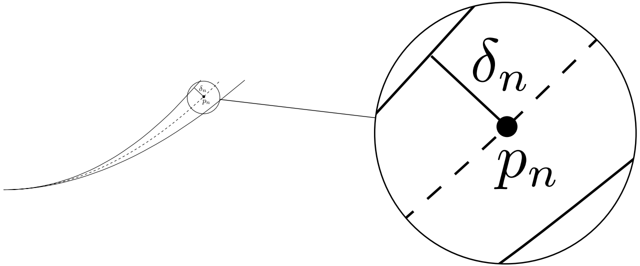
$$p(\zeta)e^{-nQ(\zeta)/2}$$

where degree$(p) \leq n$.

Rescaling

Put $r_n = 1/\sqrt{n\Delta Q(p_n)}$. Rescaled system about $p_n = 0$:

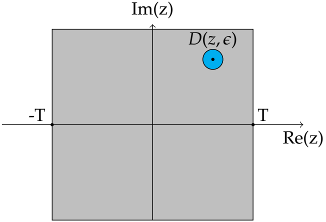
$$\{z_j\}_{j=1}^n, \qquad z_j = r_n^{-1}\zeta_j, \qquad j = 1, \ldots, n.$$



We fix $T > 0$, take $\delta_n = Tr_n$, p_n closest point to the cusp with boundary distance δ_n.

Rescaled density function $R_n(z)$

Rescaled droplet as $n \to \infty$ is *strip* $-T \leq \mathrm{Re}\, z \leq T$.



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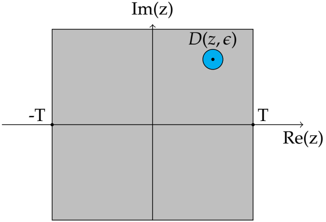

Random sample $\{z_j\}_1^n$.

$$R_n(z) = \frac{\textit{expected number of points in } D(z,\epsilon)}{\epsilon^2},$$

$(\epsilon \to 0)$.

Structure lemma (normal families)

Lemma

Subsequential limits

$$R(z) := \lim_{k\to\infty} R_{n_k}(z)$$

exist. Each such limit determines a unique determinantal point field $\{z_j\}_1^\infty$.

Each limit point field is determined by a Hermitian-entire function $L(z, w)$ via

$$R(z) = L(z, z)e^{-|z|^2}.$$

L is the Bergman kernel of a contractively embedded subspace of Fock space $L_a^2(e^{-|z|^2})$.
Infinite Ginibre ensemble: $L(z, w) = e^{z\bar{w}}$ is Bargmann-Fock kernel.

Ward's identity

Let ψ any test-function and $\{\zeta_j\}_1^n$ random sample. Random variable

$$W_n[\psi] = \frac{1}{2}\sum_{j\neq k}\frac{\psi(\zeta_j)-\psi(\zeta_k)}{\zeta_j-\zeta_k} + n\sum \partial Q(\zeta_j)\psi(\zeta_j) + \sum \partial\psi(\zeta_j).$$

Lemma

$\mathbf{E}_n W_n[\psi] = 0.$

Proofs: Reparametrization invariance of Z_n/integration by parts. Gives an exact relation between 1- and 2-point functions $\mathbf{R}_{n,1}$ and $\mathbf{R}_{n,2}$.

Theorem

(AKMW)

1. *Zero-one law: either* $R = 0$ *identically or* $R > 0$ *everywhere.*
2. *If the width* T *is large enough then* $R > 0$ *everywhere.*
3. *If* $R > 0$ *then*
$$\bar{\partial} C = R - 1 - \Delta \log R$$
where $B(z,w) = |K(z,w)|^2 / K(z,z)$ *and* $C(z) = \int \frac{B(z,w)}{z-w}\, dA(w)$.
4. *If* $z = x + iy$ *then*
$$R(z) \leq Ce^{-2(|x|-T)^2}.$$

Limiting correlation kernel

With

$$K(z,w) = L(z,w)e^{-|z|^2/2-|w|^2/2}$$

the k-point intensities are given by

$$R_k(z_1,\ldots,z_k) = \det(K(z_i,z_j))_{i,j=1}^k.$$

Often easier to write

$$L(z,w) = e^{z\bar{w}}\Psi(z,w)$$

and look for Ψ.
Then $R(z) = \Psi(z,z)$.

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Then $R(z) = \Psi(z,z)$.
Infinite Ginibre: $\Psi = 1$.
Regular boundary point: $\Psi(z,w) = \frac{1}{2}\operatorname{erfc}(\frac{z+\bar{w}}{\sqrt{2}})$.

Translation invariant solutions

Theorem

Suppose

1. T *is large enough that* $R > 0$,
2. *Translation invariance:* $R(z)$ *depends only on* $x = \mathcal{R}e(z)$.

Then there is an interval $[A, B] \subset [-2T, 2T]$ *such that*

$$R(z) = \Psi(z + \bar{z}) := \frac{1}{\sqrt{2\pi}} \int_A^B e^{-(z+\bar{z}-t)^2/2}\, dt,$$

Can be written

$$\Psi(z) = \gamma * \mathbf{1}_{[A,B]}(z), \qquad \gamma(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Conjecture of AKMW

We believe that R is nontrivial and translation invariant for every $T > 0$ and that $[A, B] = [-2T, 2T]$.

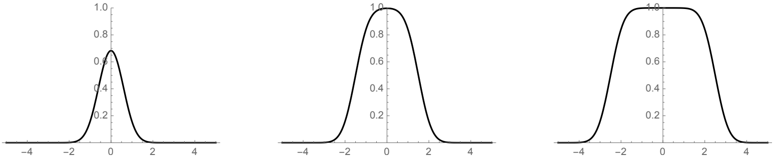


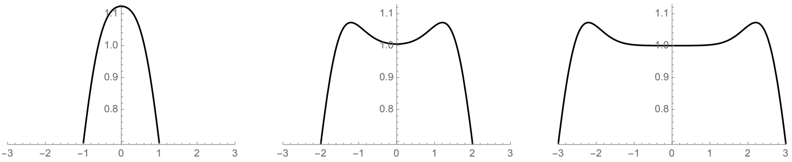
Figure: The graph of $R(x) := \gamma * \mathbf{1}_{[-2T,2T]}(2x)$ for $T = 1/2$, $T = 3/2$, and $T = 5/2$.

The hard edge strip: natural candidates

The limit $R = \lim R_n$ for the strip satisfies Ward's equation

$$\bar{\partial} C = R - 1 - \Delta \log R.$$

Natural solutions (AKMW):



Fyodorov-Khoruzhenko-Sommers setting

Potential

$$Q(\zeta) = \frac{1}{1-\tau^2}(|\zeta|^2 - \tau \mathrm{Im}\,(\zeta^2)) = \frac{x^2}{1-\tau} + \frac{y^2}{1+\tau}.$$

FKS introduced *weakly skew-Hermitian regime*

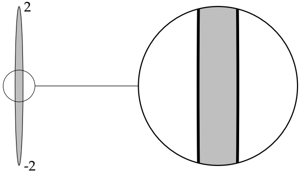
$$\tau = \tau_n = 1 - \frac{(\pi\alpha)^2}{2n}.$$

The droplet is a narrow ellipse about the y-axis:

$$C\frac{x^2}{(\alpha^2/n)^2} + \frac{y^2}{2^2} \leq 1 + o(1).$$

Rescaling

The droplet has width $\sim \alpha^2/n$ and area $\sim \alpha^2/n$.



Let $\{\zeta_j\}_1^n$ random sample. Particle density is $\sim n^2$ so it is natural to rescale by

$$z = cn\zeta, \qquad \textit{i.e.} \quad R_n(z) = \frac{1}{c^2 n^2}\mathbf{R}_n(\zeta).$$

FKS choice is $c = 1/\pi$.

Fyodorov-Khoruzhenko-Sommers theorem

Theorem

(FKS) $R_n \to R$ *where*

$$R(z) = \frac{1}{\sqrt{2\pi}\alpha} \int_{-\pi}^{\pi} e^{-\frac{2}{\alpha^2}\left(y+\frac{\alpha^2 t}{2}\right)^2} dt.$$

We make the *transformation*

$$R_{n,\alpha}(z) := \alpha^2 R_n(i\alpha z).$$

Corollary

As $n \to \infty$, $R_{n,\alpha}$ *converges to*

$$R_\alpha(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} e^{-(z+\bar{z}-t)^2/2} dt, \quad T = \frac{\alpha\pi}{2}.$$

Note: Corollary is equivalent to theorem and is well-suited for our approach.

FKS densities

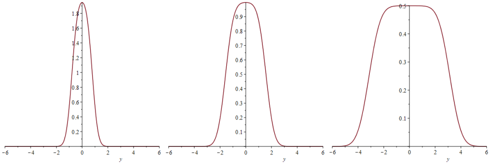


Figure: Density profiles $R^{\alpha}(y)$, $\alpha = 1/\sqrt{2}, 1, \sqrt{2}$.

FKS noted that

$$\lim_{\alpha \to 0} K^{\alpha}(x,y) = K^{\sin}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}, \quad (x,y) \in \mathbb{R}^2$$

and

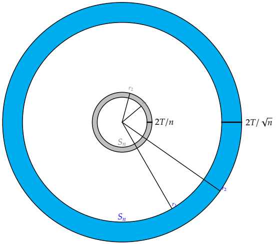
$$\lim_{\alpha \to \infty} K_{\alpha}(z,w) = K^{\mathrm{Ginibre}}(z,w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}.$$

Model case: thin annular ensembles

Put

$$Q_n(\zeta) = \frac{1}{a_n}(|\zeta|^2 - 2c_n \log|\zeta|).$$

For suitable choices of a_n, c_n the droplet is a thin annulus S_n.



Fix $T > 0$ and choose:

1. $\int_{S_n} \Delta Q_n \, dA = 1$, i.e. $a_n = \text{Area}(S_n) = 1/\Delta Q_n$,
2. $r_2 - r_1 \sim 2T/\sqrt{n \Delta Q_n}$.

Point fields in a strip

Let $r_* = \sqrt{na_n}/4T$, $a_n = 1/\Delta Q_n$.
Then

$$r_1 \sim r_* - \frac{T}{\sqrt{n\Delta Q}}, \quad r_2 \sim r_* - \frac{T}{\sqrt{n\Delta Q}}.$$

Note: If $\Delta Q = 1$ then $r_* \sim \sqrt{n}$; if $\Delta Q = n$ then $r_* \sim 1$.

Theorem

(ABS) If we rescale about r_* on the scale $1/\sqrt{n\Delta Q}$, we obtain the rescaled droplet $[-T, T]$ and the limiting 1-point function

$$R^T(z) = \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} e^{-(2x-t)^2/2}\, dt.$$

Question of AKMW

Taking $a_n \sim 1$ and rescaling on scale $1/\sqrt{n}$ we answer in the affirmative the question of AKMW for the case of weakly circular ensembles.

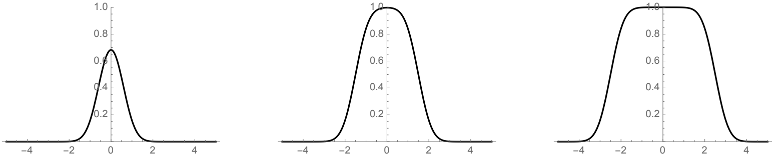


Figure: The graph of $R^T(x)$ for $T = 1/2$, $T = 3/2$, and $T = 5/2$.

Weakly circular case

Taking $a_n \sim 1/n$ and rescaling on scale $1/n$ we again get

$$R^T(z) = \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} e^{-(2x-t)^2/2}\, dt.$$

Changing to $R_\alpha(z) = \alpha^{-2} R^T(iz/\alpha)$, $\alpha = 2T/\pi$ we recover the densities of Fyodorov, Khoruzhenko and Sommers.

FKS type fields with a hard edge

Our strategy works also for hard edge confinement. Corresponding FKS density profiles are given here:

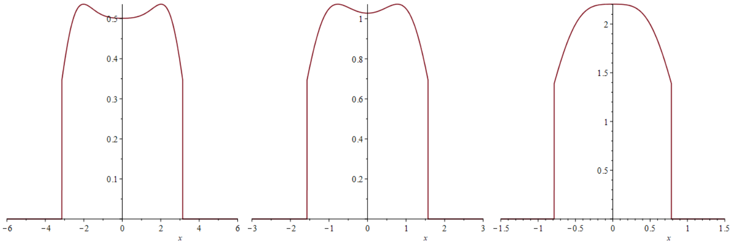
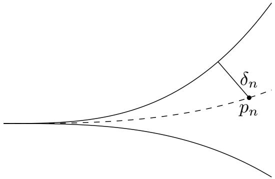


Figure: Density profiles $R^{\alpha}_{\text{hard}}(y)$, $\alpha = 1/\sqrt{2}, 1, \sqrt{2}$.

AKMW comparison



What if $\delta_n \sim 1/n$ and we rescale at scale $1/n$: $z = n(\zeta - p_n)$?

$$R_n(z) = \frac{1}{n^2}\mathbf{R}_n(\zeta).$$

Since $\mathbf{R}_n \leq Cn$ it follows that $R_n \to 0$.
In thin annular case $r_1 = 1$, $r_2 = 1 + c/n$ we have $\mathbf{R}_n \sim n^2$ and so $R_n \sim 1$ which is why we can get nontrivial limits.

Weakly Hermitian ellipse ensembles: bulk universality

Let $V(x) = x^2/4$. Wigner's semi-circle law:

$$\sigma_V(x) = \frac{1}{2\pi}\sqrt{4 - x^2} \cdot \mathbf{1}_{[-2,2]}(x).$$

Now consider a *bulk point* x: $-2 < x < 2$,

$$Q_n(x + iy) = \frac{x^2}{1+\tau} + \frac{y^2}{1-\tau}, \quad \tau = 1 - \alpha^2/2n\sigma_V(x)^2,$$

so

$$Q_n(x + iy) \sim \frac{1}{4}x^2 + \frac{n\sigma_V(x)^2}{\alpha^2}y^2.$$

The secret behind the rescaling

The scaling is chosen so that *cross-section equation* holds:

Lemma

As $n \to \infty$

$$\frac{1}{\pi n} \int_{-\infty}^{-\infty} \mathbf{R}_n(x + iy)\, dy \to \sigma_V(x).$$

This corresponds to the *mass-one equation* of AKM.

Limit of weak Hermiticity

Fix p, $-2 < p < 2$ and rescale on the scale $n\sigma_V(p)$:

$$z = n\sigma_V(p)(\zeta - p).$$

Theorem

(FKS, ACV) The rescaled density function converges to

$$R^\alpha(z) = \frac{1}{\sqrt{2\pi}\alpha} \int_{-\pi}^{\pi} e^{-\frac{2}{\alpha^2}\left(x+\frac{\alpha^2 t}{2}\right)^2} dt.$$

Ward's equations

The Ward's equation for $R = R^\alpha$ reads

$$\bar{\partial} C = R - \frac{1}{\alpha^2} - \Delta \log R.$$

However if we transform to $\hat{R}(z) = \alpha^2 R(\alpha z)$ then Ward's equation for $R = \hat{R}$ becomes just

$$\bar{\partial} C = R - 1 - \Delta \log R.$$

This is *standard Ward equation*.
Furthermore the cross-section equation reduces to

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} R(x + iy)\, dy \equiv 1.$$

Translation invariance and AKM theorem

A 1-point function R is called *horizontal translation invariant* if $R(x + iy) = R(iy)$ for all x, *symmetric* if also $R(-iy) = R(iy)$.

Theorem

(AKM) If R is translation invariant, symmetric and satisfies cross-section equation then we obtain precisely the FKS limits, after transforming back via $R^\alpha(z) = \alpha^{-2} R_\alpha(z/\alpha)$.

Proof.

Uniqueness of solution to Ward's equation under these conditions is proven in AKM, using Fourier analysis. □

So we need merely prove translation invariance in order to obtain a new proof.

New proof of FKS theorem

We must show translation invariance $\partial_x R(x+iy) = 0$ i.e.

$$\partial_x R_n(x+iy) \to 0, \qquad (n \to \infty).$$

However $\mathbf{R}_n$ is expressible by *Hermite polynomials* H_j,

$$q_j(\zeta) = \frac{\sqrt{n}}{(1-\tau^2)^{1/4}} \frac{1}{\sqrt{j!}} \sqrt{\left(\frac{\tau}{2}\right)^j} H_j\left(\sqrt{\frac{n}{2\tau}}\zeta\right),$$

$$\mathbf{R}_n(\zeta) = e^{-nQ_n(\zeta)} \sum_{j=0}^{n-1} |q_j(\zeta)|^2.$$

Continuation

Rescaling we find that translation invariance is equivalent to the convergence

$$\left(\frac{\tau}{2}\right)^n \frac{\mathrm{Re}\,[H_{n-1}(c_n z)H_n(c_n \bar{z})]}{\sqrt{n}(n-1)!} \to 0, \quad (n \to \infty).$$

which follows from standard properties of Hermite polynomials.
□

Advantages of this proof:

- It is a good deal easier to show translation invariance rather than full convergence.
- Possibly generalizes to potentials of the form

$$Q(z+iy) = V(x) + cny^2.$$

  Role of Hermite polynomials is then believed to be o.p.'s with respect to $V(x)$ continued analytically to $\mathbb{C}$.

Bender's theorem

Put

$$Q(x+iy) = \frac{x^2}{1+\tau} + \frac{y^2}{1-\tau}$$

where $\tau = 1 - \alpha^2/n^{1/3}$.
Rescale at the right edge $p = 1+\tau$ by

$$z = (\zeta - p)n^{2/3}.$$

Theorem

$R_n \to R^\alpha$ where

$$R^\alpha(z) = \frac{\sqrt{\pi}}{\alpha} e^{-\frac{(Im\, z)^2}{\alpha^2} + \alpha^2\left(Re\, z + \frac{1}{6}\right)} \int_0^\infty e^{\alpha^2 t} \left|\mathrm{Ai}\left(z + t + \frac{\alpha^2}{4}\right)\right|^2 dt.$$

NOT translation invariant!

Twisted convolution formulation

If we change to $R_\alpha(z) = 2\alpha^2 R^\alpha(\sqrt{2}\alpha z)$ we obtain a solution to Ward's equation

$$\bar{\partial} C = R - 1 - \Delta \log R.$$

Two-dimensional Fourier transform

$$\hat{f}(w) = \int_{\mathbb{C}} f(z) e^{-2i\mathrm{Re}\,(z\bar{w})}\, dA(z).$$

We represent the Fourier transform of R_α in the form

$$\hat{R}^\alpha(w) = \hat{r}^\alpha(w) \cdot e^{-|w|^2/2}$$

for some function $r^\alpha(z)$.

Continuation

The twisted convolution form of Ward's equation means that there shall exist a smooth function $P^\alpha(z)$ whose Fourier transform takes the form $\hat{P}^\alpha(w) = \hat{p}^\alpha(w) \cdot e^{-|w|^2/2}$ where $p^\alpha(z)$ solves the twisted convolution problem

$$\hat{r}^\alpha \star \hat{p}^\alpha = 0.$$

Here *twisted convolution* is

$$f \star g(z) = \int_{\mathbb{C}} f(z-w)g(w)e^{i\mathrm{Im}\,(\bar{z}w)}\,dA(w).$$

d-interactions

Let d a positive integer and define the energy of a configuration $\{\zeta_1, \ldots, \zeta_n\}$

$$H_n = \sum_{j \neq k} \log \frac{1}{|\zeta_j^d + \zeta_k^d|} + n \sum Q(\zeta_j).$$

If there is a particle at ζ then there are particles at $\zeta e^{2\pi i k/d}$, $k = 1, .., d$.

• Case $d = 2$ occurs in connection with QCD: Akemann, Osborn and Katori's recent work.

• Jellium of HW and others seems somewhat different.

Lemma

For each d, the density behaves as $\mathbf{R}_n(\zeta) \sim \frac{n}{d} \Delta Q(\zeta) \mathbf{1}_{S_d}(\zeta)$ where S_d is solution to a modified obstacle problem.

QCD setting

$$Q_n(\zeta) = -\frac{1}{n} \log \left(e^{b\mathrm{Re}\,(\zeta^2)} K_\nu(a|\zeta|^2)|\zeta|^{2\nu+2} \right)$$

$$a \sim n^2/4\kappa^2$$

$$b \sim n^2/4\kappa^2$$

$$\frac{a^2 - b^2}{2b} \sim n/4.$$

Let $d = 2$ and $\{\zeta_j\}_1^n$ random sample and rescale about 0 on $1/n$-scale.

Akemann-Osborn theorem

If $Q_0(z) = \lim nQ_n(z/n)$ then

$$Q_0(z) = -\mathrm{Re}\,(z^2)/4\kappa^2 - \log\left[K_\nu(|z|^2/4\kappa^2)|z|^{2\nu+2}\right].$$

Note! $\int e^{-Q_0} = +\infty$ so it is *NOT* clear that limiting distribution should be rotationally symmetric.

Theorem

$$R(z) = \frac{1}{8\kappa^2} e^{-Q_0(z)} |z|^{-2\nu} \int_0^1 e^{-2\kappa^2 t} \left| J_\nu(\sqrt{t}z) \right|^2 dt.$$

Remark: equilibrium measure with d-interaction

Energy functional

$$I_Q^d[\mu] = \int \log \frac{1}{|\zeta^d - \eta^d|} \, d\mu(\zeta) d\mu(\eta).$$

Minimizer σ_d takes the form

$$d\sigma_d(\zeta) = d^{-1} \Delta Q(\zeta) \mathbf{1}_{S_d}(\zeta) \, dA(\zeta).$$

Euler-Lagrange eq's

$$-\int \log |\zeta^d - \eta^d| \, d\sigma_d(\eta) + Q(\zeta)/2 = F_Q^d, \quad \zeta \in S_d,$$
$$-\int \log |\zeta^d - \eta^d| \, d\sigma_d(\eta) + Q(\zeta)/2 > F_Q^d, \quad \zeta \notin S_d.$$

Thin ellipse ensembles: under investigation

For fixed τ, $0 < \tau < 1$ let

$$\tilde{Q}_\tau(x+iy) = (1-\tau)^{-1}x^2 + (1+\tau)^{-1}y^2.$$

Droplet is the elliptic disk

$$\frac{x^2}{(1-\tau)^2} + \frac{y^2}{(1+\tau)^2} \leq 1.$$

Now set

$$Q_n(\zeta) = a_n \tilde{Q}_\tau(\zeta) + c_n \log(1/\tilde{Q}_\tau(\zeta)).$$

For suitable values of a_n, c_n the droplet is a thin ellipse

$$r_1 \leq \frac{x^2}{(1-\tau)^2} + \frac{y^2}{(1+\tau)^2} \leq r_2$$

where $r_2 - r_1 \sim 1/\sqrt{n\Delta Q_n}$.

THANK YOU!