Product Hardy spaces associated to operators with heat kernel bounds on spaces of homogeneous type

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Theme: Develop Hardy spaces $H^p_{L_1,L_2}(X_1 \times X_2)$.

Joint work with Peng Chen, Xuan Duong, Yongsheng Han, Anna Kairema, Ji Li, Cristina Pereyra, Jill Pipher, and Lixin Yan

References

[KLPW] A. Kairema, J. Li, M.C. Pereyra & W___, *Haar bases on quasi-metric measure spaces, and dyadic structure theorems on product spaces of homogeneous type*, J. Funct. Analysis, 2016.

[HLW] Y. Han, J. Li & W___, *Hardy space theory on spaces of homogeneous type via orthonormal wavelet bases*, Appl. and Comp. Harmonic Analysis, 2017.

[CDLWY1] P. Chen, X.T. Duong, J. Li, W___ & L.X. Yan, *Product Hardy spaces associated to operators with heat kernel bounds on spaces of homogeneous type*, Math. Zeitschrift, 2016.

[CDLWY2] P. Chen, X.T. Duong, J. Li, W___ & L.X. Yan, *Marcinkiewicz-type spectral multipliers on Hardy and Lebesgue spaces on product spaces of homogeneous type*, J. Fourier Analysis & Appl., 2017.

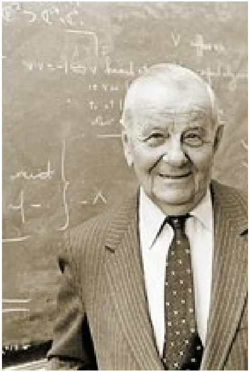
[S] E.M. Stein, *Singular integrals: The roles of Calderón and Zygmund*, Notices Amer. Math. Soc., 1998.

Outline

0. Intro
1. Selected elements of the Calderón–Zygmund theory
2. Spaces of homogeneous type (X, d, μ)
3. The product theory and the multiparameter theory
4. Results I:

   [KLPW]: Haar basis on (X, d, μ)
   [HLW]: $H^p(X_1 \times X_2)$
5. Function spaces H^p_L, BMO_L associated to operators L
6. Results II:

   [CDLWY1]: $H^p_{L_1,L_2}(X_1 \times X_2)$
   [CDLWY2]: boundedness of product spectral multiplier operators



All three have biographical articles in Notices Amer. Math. Soc.

- Prototypical question: Is a singular integral operator (SIO) T bounded from function space X to function space Y?

$$T : X \to Y\,?$$

- Example: Is T bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$? In other words, is there a constant C such that $\|Tf\|_2 \leq C\|f\|_2$ for all $f \in L^2(\mathbb{R})$, meaning

$$\left[\int_{-\infty}^{\infty} |Tf(x)|^2\,dx\right]^{1/2} \leq C\left[\int_{-\infty}^{\infty} |f(x)|^2\,dx\right]^{1/2} \;?$$

- Operator T can be: Riesz transform R_j, Hilbert transform H:

$$Hf(x) := \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy$$

"Calderón–Zygmund operators (CZOs)", multiplier operators.

- Function spaces X, Y can be: $L^p(\mathbb{R})$, the Hardy space H^1, BMO, weighted $L^p(\omega)$ for ω in A_p or RH_p, etc.

1. Selected elements of the Calderón–Zygmund theory

1. Define Hardy spaces H^p, $1 \leq p < \infty$, via square function $S(f)$
2. Atomic decomposition of Hardy space H^1: $f(x) = \sum_i \lambda_i a_i(x)$ for compactly supported atoms $a_i(x)$ with bounds
3. $H^1 = H^1_{\mathrm{at}}$
4. Calderón–Zygmund decomposition $f = g + b$ into good and bad functions, for $f \in H^p$
5. Operator Theory: Interpolation Theorem T a CZO. Then: [Stein]

   (i) $T : L^2 \to L^2$, $T : H^1 \to L^1$ $\quad \Rightarrow T : L^p \to L^p, p \in (1, 2]$.

   (ii) $T : L^2 \to L^2$, $T : L^\infty \to \mathrm{BMO}$ $\quad \Rightarrow T : L^p \to L^p, p \in [2, \infty)$.
6. Also get $T : H^p \to L^p$ bounded for a range of p
7. H^p and L^p coincide for a range of p

[CDLWY1]: We generalise this from $H^p(\mathbb{R})$ to $H^p_{L_1,L_2}(X_1 \times X_2)$

1. Why we care about dyadic H^1_d, BMO_d

- Interpolation: Can interpolate between L^2 and dyadic H^1, or L^2 and dyadic BMO_d. (Bui & Laugesen, 2013)
- Estimates: The maximal function, which controls size of many SIOs, is itself controlled by dyadic maximal functions: $Mf \leq C(M_d f + M_\delta f)$ pointwise.
- Dyadic version can be model case: John–Nirenberg Theorem was first proved for dyadic case. Product BMO was first defined for dyadic case.
- Exploit easier proofs: Prove dyadic version first, pull across to continuous version via a bridge.
  (Garnett & Jones, BMO *from dyadic* BMO, 1982).
  Corona-type theorem. Peter's distance-to-L^∞-in-BMO theorem. Peter's A_p factorization theorem.
- The "one-third trick": Gives $\mathrm{BMO} = \mathrm{BMO}_d \cap \mathrm{BMO}_d^{1/3}$.

Definition 1.1 (Dyadic H^1)

Dyadic Hardy space $H^1_d(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : S_d(f) \in L^1(\mathbb{R})\}$, where

$$S_d(f)(x) := \left\{ \sum_{I \text{ dyadic}} \left| \langle f, h_I \rangle \frac{\chi_I(x)}{|I|^{1/2}} \right|^2 \right\}^{1/2}$$

is the dyadic square function, $h_I(x)$ is the Haar function of I, and $\langle f, h_I \rangle$ is the Haar coefficient of f for I.

Definition 1.2 (Continuous H^1)

(Continuous) Hardy space $H^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : S(f) \in L^1(\mathbb{R})\}$, where $S(f)$ is the Littlewood–Paley square function

$$S(f)(x) := \left\{ \int_0^\infty |Q_t f(x)|^2 \frac{dt}{t} \right\}^{1/2}, \qquad Q_t f(x,t) := \psi_t * f(x).$$

1. Our project: Extending the CZ theory in several directions

dyadic

product theory

classical CZ
harmonic analysis

$H^p_{L_1,L_2}(X_1 \times X_2)$

doubling measures; non-doubling measures

spaces associated with operators L; singular integrals with non-smooth kernels

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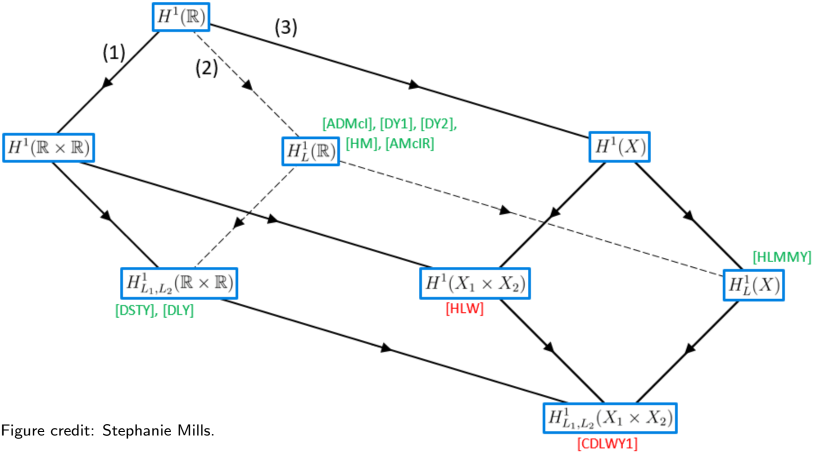


Figure credit: Stephanie Mills.

Similar diagram for H^p, $1 < p < \infty$. [AMcIR], [HM], [KU], [DL].

Underlying space	$(\mathbb{R}^n,$	Euclidean metric,	Lebesgue measure)
becomes	$(X,$	quasimetric d,	doubling measure $\mu)$

Theme: Functions $f : \mathbb{R}^n \to \mathbb{R}$ replaced by $f : X \to \mathbb{R}$.

"One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present.

—Yves Meyer, Abel Prize 2017

Preface to Deng & Han, LNM Vol. 1966, Springer, 2009

Definition: A *space of homogeneous type* is a triple (X, d, μ):

- X a set.
- d a quasimetric: $d(x, y) \leq A_0 d(x, z) + A_0 d(z, y)$.
- μ a doubling measure: $\exists$ C s.t. $\forall$ quasiballs $B(x, r) \subset X$,

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty.$$

Def'n by Coifman & Weiss (1978); their original def'n (1971) was slightly more general. Their insight about the proofs.

Quasiball of centre x and radius r:

$$B(x, r) := \{y \in X : d(y, x) < r\}.$$

We assume μ is defined on a σ-algebra which contains all Borel sets and all quasiballs.

0. $X = \mathbb{R}^n$,
   $d =$ Euclidean metric:
   $$d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$
   $\mu =$ Lebesgue measure: $\mu(E) = n$-dimensional volume of E
1. $X =$ four-corners Cantor set
2. $X =$ graph of Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$
3. $X =$ Heisenberg group $\partial\mathbb{B}^n$ in $\mathbb{C}^n$
4. $X =$ nilpotent Lie group
5. $X = \mathbb{Z}$ with counting measure
6. $X = \{-1\} \cup [0, \infty)$
7. $X =$ compact Riemannian manifold

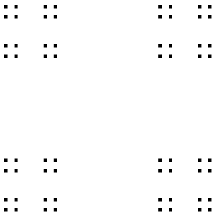
Example 1: Fractals

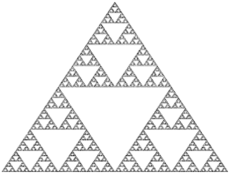
X = Garnett's four-corners Cantor set

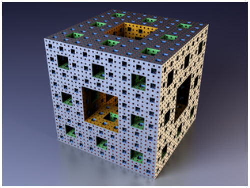
d = Euclidean metric

μ = one-dimensional Hausdorff measure = length

X is totally disconnected, and has finite, positive length.

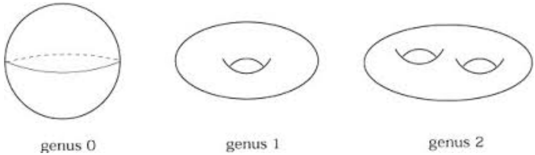






X can be other regular fractals: Sierpiński gasket, Menger sponge.

2. Example 7: X = compact Riemannian manifold



Special case:

X = compact Riemann surface
d = hyperbolic metric
μ = hyperbolic area measure

Doubling ok for small ball: measure almost Lebesgue.
What about large ball? X is compact so ok.

Q: What if X not compact?

What if X not compact?

Hyperbolic crochet, by Dr Daina Taimina, Cornell.

μ-area of ball on this hyperbolic surface X grows exponentially with radius: $\mu(B(x,r)) \sim e^{cr}$.

μ not a doubling measure.

X not a space of homogeneous type.

2. Non-example: X = non-compact Riemannian manifold

TED talk *The Beautiful Math of Coral*, by Margaret Wertheim.

HOW TO CROCHET HYPERBOLIC CORALS

BY THE INSTITUTE FOR FIGURING

Ladies Silurian Atoll. Photo © the IFF.

The Hyperbolic Crochet Coral Reef is a celebration of the intersection of geometry and handicraft and a testimony to the disappearing wonders of the marine world. Launched as a response to the devastation of living reefs from global warming and ocean acidification, the Crochet Reef resides equally in the realms of art, science, mathematics and environmentalism.

HYPERBOLIC CROCHET CORAL REEF

A project by the Institute For Figuring

created and curated by
Margaret and Christine Wertheim

www.crochetcoralreef.org

fig.

HYPERBOLIC WONDERS

In coral reefs we witness an almost endless diversity: wavy strands of kelp, crenellated corals and curlicued sponges. Even those who have never seen a living reef immediately recognize the Crochet Reef's distinctive forms for this woolly wonder takes its cue from nature. In both cases the ruffled shapes are variations on a mathematical structure known as hyperbolic geometry. Nature loves these forms, for this is an ideal way to maximize surface area, allowing filter feeding organisms such as corals to enhance nutrient intake.

Hyperbolic Crochet Sea Anemone

For humans, the best way to make models of hyperbolic geometry is with crochet, a discovery made in 1997 by Dr. Daina Taimina at Cornell University. Nature, however, does not stick to mathematical perfection and just as there is nothing in nature that is perfectly spherical, so there is nothing in nature that is perfectly hyperbolic. Living forms result from deviation and imperfection.

Christine Wertheim installing The People's Reef in Scottsdale AZ. Photo © the IFF.

In 2005, Margaret and Christine Wertheim at the Institute For Figuring, in Los Angeles, began to develop a taxonomy of reef-like forms by building on Dr Taimina's techniques. Instead of adhering to a mathematically pure pattern, they began to use more freeform techniques which give the models a natural and organic look. Tightly bunched mounds of brain coral, towered spires of pillar coral, blooms of carnation coral, and forests of kelp can all be mimicked.

Just as the diversity of living species on earth result from variations in an underlying DNA code, so a huge range of woolen 'species' may be brought into being through modifications in the underlying crochet code. As in nature, organic looking structures are the result of variation and experiementation. Anyone who takes up these techniques may begin to explore what is possible here. There is, as it were, an endlessly diverse, ever-evolving crochet 'tree of life.'

In addition to the "Core Collection" of Crochet Reef's created by the Institute, since 2006 the IFF has been working with cities and communities around the globe to create local "Satellite Reefs". As of 2010, Satellite Reefs have been made across the USA, and in the UK, Australia, Latvia, Ireland, and South Africa.

For more information about the Hyperbolic Crochet Coral Reef project visit: **www.crochetcoralreef.org.**

To learn more about hyperbolic crochet see the book that shows you how:

A Field Guide to Hyperbolic Space
By Margaret Wertheim (Institute For Figuring Press)

Books may be purchased online:
www.theiff.org/publications

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2. Features of spaces of homogeneous type X

- (X, d, μ) is geometrically doubling: $\exists\ N$ s.t. every ball $B(x, r)$ can be covered by at most N balls of radius $r/2$.
- Quasi-metric d need not be Hölder regular.
  (Can pass to regular Macías & Segovia quasimetric d'.)
- Ball $B(x, r)$ need not be open.
- No 0 element. No coordinate directions.
- No addition $+$.
  (Can replace by random dyadic lattices $\{\mathcal{D}(\omega)\}_{\omega\in\Omega}$; [NTV], [HK], ...).
- No translation in X.
- No $1/3$ trick.
  (Can replace by collection of adjacent systems of dyadic cubes $\{\mathcal{D}^1, \ldots, \mathcal{D}^T\}$; [HK]).

2. Replace underlying space $\mathbb{R}^n$ by X

The Calderón–Zygmund theory deals with collections of functions $f : \mathbb{R}^n \to \mathbb{R}$.

Theme: Replace $\mathbb{R}^n$ by any space of homogeneous type X: $f : X \to \mathbb{R}$.

Includes functions

f : four-corners Cantor set $\to \mathbb{R}$.

f : graph of Lipschitz function $F \to \mathbb{R}$.

f : Heisenberg group $\to \mathbb{R}$.

f : nilpotent Lie group $\to \mathbb{R}$.

$f : \mathbb{Z} \to \mathbb{R}$.

$f : \{-1\} \cup [0, \infty) \to \mathbb{R}$.

f : compact Riemannian manifold $\to \mathbb{R}$.

Goal: Develop Calderón–Zygmund theory on all X; covers all this. Many contributors.

- Product space of homogeneous type: $(\widetilde{X}, d, \mu)$

$$\widetilde{X} := X_1 \times \cdots \times X_k,$$
$$d := d_1 \times \cdots \times d_k,$$
$$\mu := \mu_1 \times \cdots \times \mu_k.$$

- Example 1: $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$.
- Example 2: Nagel–Stein: Carnot–Carathéodory spaces $\widetilde{M} = M_1 \times \cdots \times M_k$ formed by vector fields satisfying Hörmander's finite-rank condition.
  They study non-isotropic smooth SIOs.

"To oversimplify, ... "product theory" is that part of harmonic analysis in $\mathbb{R}^n$ which is invariant with respect to the n-fold dilations

$$x = (x_1, \ldots, x_n) \mapsto (\delta_1 x_1, \ldots, \delta_n x_n), \qquad \delta_j > 0.$$

Its initial concern is with operators that are essentially products of operators acting on each variable separately, and then more generally with operators (and associated function spaces) that retain some of these characteristics." —E.M. Stein [S]

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- Compare one-parameter case (identical δs):

$$x = (x_1, \dots, x_n) \mapsto (\delta x_1, \dots, \delta x_n), \qquad \delta > 0.$$

Operators (such as Δ, Riesz transforms) compatible with these uniform dilations.

- Compare multiparameter case: intermediate.

Operators compatible with specified subgroups of the group of n-parameter dilations.

Not only higher-dimensional, but also:

- Allow different dilations in each direction.
  Multiparameter example:

$$(x_1, x_2, x_3) \mapsto (\delta_1 x_1, \delta_2 x_2, \delta_1 \delta_2 x_3) \qquad \text{Zygmund dilation}$$

- Deal with rectangles, not just cubes
- No canonical decomposition of open set into rectangles
- No Vitali-type covering lemmas
- Substitute: Journé's Lemma; complicated even for $n = 2$
- No stopping-time arguments
- (Non-trivial) Define product function spaces $H^1(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k})$, $\mathrm{BMO}(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k})$
- Definitions of product BMO no longer coincide:

$$\mathrm{bmo}(\mathbb{R}\times\mathbb{R}) \quad \subsetneq \quad \mathrm{BMO}_{\mathrm{prod}}(\mathbb{R}\times\mathbb{R}) \quad \subsetneq \quad \mathrm{BMO}_{\mathrm{rec},k}(\mathbb{R}\times\mathbb{R})$$

3. A note on notation

"Any product theory tends to be burdened with notational complexities."

A. Nagel and E.M. Stein, 2006

On the product theory of singular integrals

Today $\widetilde{X} := X_1 \times X_2$ has only two factors.

Definition 1.3 (KLPW: dyadic H^1, biparameter version)

$$H^1_d(\widetilde{X}) := \{f \in L^1(X_1 \times X_2) : S_d(f) \in L^1(X_1 \times X_2)\},$$

with $S_d(f)$ Littlewood–Paley g-function for Haar basis $\{h^Q_u\}$:

$$S_d(f)(x_1, x_2) :=$$

$$\left\{ \sum_{Q_1 \in \mathcal{D}_1} \sum_{Q_2 \in \mathcal{D}_2} \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} \left| \langle f, h^{Q_1}_{u_1} h^{Q_2}_{u_2} \rangle \frac{\chi_{Q_1}(x_1)}{\mu_1(Q_1)^{1/2}} \frac{\chi_{Q_2}(x_2)}{\mu_2(Q_2)^{1/2}} \right|^2 \right\}^{1/2}.$$

Definition 1.4 (HLW: continuous H^1, biparameter version)

To define continuous $H^1(\widetilde{X})$: use Auscher–Hytönen [AH] orthonormal spline wavelet coefficients $\langle f, \psi^{k_1}_{\alpha_1} \psi^{k_2}_{\alpha_2} \rangle$ instead.

(A) Construct Haar basis on space of homogeneous type (X, d, μ).

(B) On X and $\widetilde{X} = X_1 \times X_2$, define BMO, H^1 and dyadic versions.

(C) Kairema–Li–Pereyra–W___:

Intersections

$\mathrm{BMO}(\widetilde{X}) = \bigcap_{t=1}^{T} \mathrm{BMO}_{d,t}(\widetilde{X})$

Sums

$H^1(\widetilde{X}) = \sum_{t=1}^{T} H^1_{d,t}(\widetilde{X})$

(D) Chen–Li–W___:

Translation-averaging

$\mathrm{BMO}_{d,\omega}(\widetilde{X}) \to \mathrm{BMO}(\widetilde{X})$

Translates

$H^1(\widetilde{X})$ Davis' Theorem

(C) extends Mei, Hytönen–Kairema, Li–Pipher–W___.

(D) extends Garnett–Jones, Pipher–W___, Treil.

(E) On $\widetilde{X} = X_1 \times X_2$, define $H^p_{L_1,L_2}$ and atomic and dyadic versions. Prove boundedness of spectral multiplier operators.

4. (A) Construction of Haar basis on (X, d, μ) [KLPW]

- [KLPW]: Explicit construction of Haar wavelet basis $\{h_u^Q\}$ on space of homogeneous type X.

$$h_u^Q := \frac{\mu(E_{u+1})^{1/2}}{\mu(Q_u)^{1/2}\mu(E_u)^{1/2}}\chi_{Q_u} - \frac{\mu(Q_u)^{1/2}}{\mu(E_u)^{1/2}\mu(E_{u+1})^{1/2}}\chi_{E_{u+1}}$$

- Built on M. Christ et al. constructions of "dyadic cubes" on X. Number of basis functions per cube depends on number of children.
- Compare earlier work of Giraldi–Sweldens, Nazarov–Treil–Volberg, Aimar et al., Hytönen.
- On $\widetilde{X} := X_1 \times \cdots \times X_n$ use product Haar wavelet basis.
- [KLPW]: This Haar basis is key ingredient in definition of dyadic product H_d^1, BMO_d and VMO_d on product spaces of homogeneous type $\widetilde{X}$.

Definition 1.5 (Dyadic Bounded Mean Oscillation)

$f \in L^1_{\text{loc}}(\mathbb{R})$ belongs to dyadic $\mathrm{BMO}_d(\mathbb{R})$ if

$$\|f\|_d := \sup_{\text{dyadic intervals } I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty.$$

Equivalent to Carleson packing condition:

$$\sup_J \frac{1}{|J|} \sum_{I \subset J, I \in \mathcal{D}} |\langle f, h_I \rangle|^2 < \infty.$$

Continuous $\mathrm{BMO}(\mathbb{R})$: similar but with all intervals (not just dyadic), and with an integral (not sum) using different, smoother wavelets.

Recall

$$\sup_J \frac{1}{|J|} \sum_{I \subset J, I \in \mathcal{D}} |\langle f, h_I \rangle|^2 < \infty.$$

Definition 1.6 (KLPW: dyadic BMO, biparameter version)

$$\mathrm{BMO}_d(\widetilde{X}) := \{ f \in L^1_{\mathrm{loc}}(X_1 \times X_2) : \mathcal{C}_1^d(f) < \infty \},$$

with $\mathcal{C}_1^d(f)$ defined via Haar basis $\{h_u^Q\}$ on each factor:

$$\mathcal{C}_1^d(f) :=$$

$$\sup_\Omega \left\{ \frac{1}{\mu(\Omega)} \sum_{R \subset \Omega, R = Q_1 \times Q_2 \in \mathcal{D}_1 \times \mathcal{D}_2} \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} \left| \langle f, h_{u_1}^{Q_1} h_{u_2}^{Q_2} \rangle \right|^2 \right\}^{1/2}.$$

[HLW]: For continuous $\mathrm{BMO}(\widetilde{X})$: use Auscher–Hytönen o.n. spline wavelet coefficients $\langle f, \psi_{\alpha_1}^{k_1} \psi_{\alpha_2}^{k_2} \rangle$ instead.

With no additional assumptions on $\widetilde{X}$:

- Prove Auscher–Hytönen o.n. spline wavelet expansions also converge in suitable spaces of test functions and distributions
- Define square function via AH coefficients
- Define H^p and its dual CMO^p ($p = 1$ gives $\mathrm{BMO} = \mathrm{CMO}^1$)
- Define VMO, prove $\mathrm{VMO}^* = H^1$
- Calderón–Zygmund decomposition
- Interpolation theorem

[HLW] comes after a long history: previously with additional assumptions on the space of homogeneous type.

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4. Results I:

   [KLPW]: Haar basis on (X, d, μ)
   [HLW]: $H^p(X_1 \times X_2)$

5. Function spaces H^p_L, BMO_L associated to operators L
6. Results II:

   [CDLWY1]: $H^p_{L_1,L_2}(X_1 \times X_2)$
   [CDLWY2]: boundedness of product spectral multiplier operators

- Classical function spaces $H^p(\mathbb{R}^n)$ are entwined with Laplacian $\Delta = \nabla \cdot \nabla$
- Generalise Δ to wider class of operators L
- Define function spaces associated to operators L:

$$H^1_L(\mathbb{R}^n), \qquad \mathrm{BMO}_L(\mathbb{R}^n)$$

  Duong & Yan (L as below), Hofmann & Mayboroda ($L = \nabla \cdot A\nabla$), Auscher, Russ & McIntosh, ...
- Nonnegative self-adjoint operators L acting on $L^2(\mathbb{R}^n)$ with conditions (GE), (GGE_p), (DG), (FS) on kernel $p_t(x, y)$ of heat semigroup e^{-tL}
- Hence prove boundedness of associated Riesz transforms $T = \dfrac{\partial}{\partial x_i} L^{-1/2}$, $T = \dfrac{\partial^2}{\partial x_i \partial x_j} L^{-1}$.

  Riesz transforms: the components of $\nabla L^{-1/2}$, ΔL^{-1}

1. $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$, Schrödinger operator with nonnegative $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

2. $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$, with inverse square potential $V = c/|x|^2$, $c > -(n-2)^2/4$.

3. L second-order Maxwell operator with measurable coefficient matrices

4. L Stokes operator with Hodge boundary conditions on bounded Lipschitz domains in $\mathbb{R}^3$

5. L time-dependent Lamé system with homogeneous Dirichlet boundary conditions

5. Typical conditions on heat kernel $p_t(x,y)$: stronger to weaker

(GE) Gaussian estimates (Gaussian upper bounds): $\exists\ C,\ c > 0$ s.t. $\forall\ x,\ y \in X,\ \forall\ t > 0$,

$$|p_t(x,y)| \leq \frac{C}{V(x,t^{1/2})} \exp\left(-\frac{d(x,y)^2}{ct}\right).$$

(GGE_p) $p \in [1,2]$. Generalised Gaussian estimates: $(1/p + 1/p' = 1)$

$$\|P_{B(x,t^{1/2})} e^{-tL} P_{B(y,t^{1/2})}\|_{L^p(X) \to L^{p'}(X)} \leq CV(x,t^{1/2})^{-(1/p-1/p')} \exp\left(-\frac{d(x,y)^2}{ct}\right).$$

(DG) Davies–Gaffney estimates: $\forall$ open $U_i \supset \operatorname{supp} f_i$, $f_i \in L^2(X)$,

$$|\langle e^{-tL} f_1, f_2\rangle| \leq C \exp\left(-\frac{\operatorname{dist}(U_1,U_2)^2}{ct}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)}.$$

(FS) Finite propagation speed property: $\forall f_i \in L^2(U_i)$, $\forall t \in (0, \operatorname{dist}(U_1,U_2))$,

$$\langle \cos(t\sqrt{L}) f_1, f_2\rangle = 0.$$

$$\text{“(GE)} \Longrightarrow (\mathrm{GGE}_p) \Longrightarrow \text{(DG)} \Longleftrightarrow \text{(FS)”}$$

Legalese:

$(\mathrm{GE}) \Longleftrightarrow (\mathrm{GGE}_1) \Longrightarrow (\mathrm{GGE}_p)$ for all $p \in (1, 2]$.

(GGE_p) for some $p \in [1, 2) \Longrightarrow (\mathrm{GGE}_2) \Longleftrightarrow (\mathrm{DG}) \Longleftrightarrow (\mathrm{FS})$

(GGE_p) for some $p \in [1, 2) \Longrightarrow L$ is one-to-one on $L^2(X)$ (Theorem 5.1)

1. $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$, Schrödinger operator with nonnegative $V \in L^1_{\text{loc}}(\mathbb{R}^n)$.
   Has (GE), hence (DG).
2. $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$, with inverse square potential $V = c/|x|^2$, $c > -(n-2)^2/4$.
   Has (GGE_p) for $p \in ((p_c^*)', 2n/(n+2)]$ where $p_c^* = p_c^*(n, c)$, hence (DG).
3. L second-order Maxwell operator with measurable coefficient matrices
   Has (GGE_p) for some $p \in [1, 2)$, hence (DG).
4. L Stokes operator with Hodge boundary conditions on bounded Lipschitz domains in $\mathbb{R}^3$
   Has (GGE_p) for some $p \in [1, 2)$, hence (DG).
5. L time-dependent Lamé system with homogeneous Dirichlet boundary conditions
   Has (GGE_p) for some $p \in [1, 2)$, hence (DG).

6. [CDLWY1] Our main results (α): $p = 1$

Assume L_1, L_2 have (DG).

1. Define $H^1_{L_1,L_2}(X_1 \times X_2)$ via square function $Sf = S_{L_1,L_2}f$.

$$Sf(x) := \left\{ \iint_{\Gamma(x)} \left|\left(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2}\right) f(y)\right|^2 \times \frac{d\mu_1(y_1)dt_1 d\mu_2(y_2)dt_2}{t_1 V(x_1,t_1) t_2 V(x_2,t_2)} \right\}^{1/2}.$$

   where $\Gamma(x)$ is a suitable product cone.

2. $H^1_{L_1,L_2}(X_1 \times X_2) := \overline{\{f \in H^2(X_1 \times X_2) : \|Sf\|_1 < \infty\}}$.
3. Then $C_1\|f\|_2 \leq \|Sf\|_2 \leq C_2\|f\|_2$.

Assume L_1, L_2 have (DG).

1. Define $H^1_{L_1,L_2}(X_1 \times X_2)$ via square function $Sf = S_{L_1,L_2}f$.
2. Define atomic $H^1_{L_1,L_2,at,N}(X_1 \times X_2)$ via $(H^1_{L_1,L_2}, 2, N)$-atoms $a(x_1, x_2)$.
3. Theorem (CDLWY1): The square function and atomic definitions of H^1 coincide:

$$H^1_{L_1,L_2}(X_1 \times X_2) = H^1_{L_1,L_2,at,N}(X_1 \times X_2)$$

for all N sufficiently large.

Assume L_1, L_2 have (DG).

1. Define $H^p_{L_1,L_2}(X_1 \times X_2)$ via square function $S_{K_0} f = S_{K_0,L_1,L_2} f$.
2. Theorem (CDLWY1): Establish Calderón–Zygmund decomposition $f = g + b$ of $f \in H^p_{L_1,L_2}(X_1 \times X_2)$.
3. Theorem (CDLWY1: Interpolation theorem #1): For sublinear T with T bounded on L^2 and bounded from H^1 to L^1 as follows:

$$T : L^2(X_1 \times X_2) \to L^2(X_1 \times X_2),$$
$$T : H^1_{L_1,L_2}(X_1 \times X_2) \to L^1(X_1 \times X_2),$$

get T bounded from H^p to L^p for p between 1 and 2:

$$T : H^p_{L_1,L_2}(X_1 \times X_2) \to L^p(X_1 \times X_2) \quad \text{for all } p \in (1, 2).$$

Assume L_1, L_2 have (GGE_{p_0}) for some $p_0 \in [1, 2)$.

1. Theorem (CDLWY1): H^p and L^p coincide for suitable p:

$$H^p_{L_1,L_2}(X_1 \times X_2) = L^p(X_1 \times X_2)$$

   for $p_0 < p < p_0'$, where $1/p_0 + 1/p_0' = 1$.

2. Theorem (CDLWY1: Interpolation theorem #2): For sublinear T with T bounded on L^2 and bounded from H^1 to L^1 as follows:

$$T : L^2(X_1 \times X_2) \to L^2(X_1 \times X_2),$$
$$T : H^1_{L_1,L_2}(X_1 \times X_2) \to L^1(X_1 \times X_2),$$

   we get T bounded on L^p for suitable p:

$$T : L^p(X_1 \times X_2) \to L^p(X_1 \times X_2) \qquad \text{for } p_0 < p < p_0'.$$

Theorem (CDLWY2): Assume L_1, L_2 have (DG) and satisfy Stein-Tomas restriction-type estimates $(\mathrm{ST}^2_{p_i,2})$ for some $p_i \in [1,2)$, for $i = 1, 2$. Suppose $s_i > n_i/2$ where $V(x_i, \lambda r) \leq C\lambda^{n_i} V(x_i, r)$ for $i = 1, 2$. Suppose F is a bounded Borel function satisfying these Sobolev conditions:

$$\sup_{t_1,t_2>0} \left\|\eta_{(1,2)}\delta_{(t_1,t_2)}F\right\|_{W^{(s_1,s_2),2}(\mathbb{R}\times\mathbb{R})} < \infty,$$

$$\sup_{t_1>0} \left\|\eta_1\delta_{(t_1,1)}F(\cdot,0)\right\|_{W^{s_1,2}(\mathbb{R})} < \infty,$$

$$\sup_{t_2>0} \left\|\eta_2\delta_{(1,t_2)}F(0,\cdot)\right\|_{W^{s_2,2}(\mathbb{R})} < \infty.$$

Then

(i) $F(L_1, L_2)$ extends to a bounded operator from $H^1_{L_1,L_2}(X_1 \times X_2)$ to $L^p(X_1 \times X_2)$, and

(ii) $F(L_1, L_2)$ is bounded on $L^p(X_1 \times X_2)$ for $p_{\max} < p < p'_{\max}$, where $p_{\max} := \max\{p_1, p_2\}$.

1. Calderón–Zygmund theory is robust: can extend most of it to cover functions defined on spaces of homogeneous type (X, d, μ), and on $\widetilde{X}$.
2. A key ingredient: Can explicitly construct Haar basis $\{h_u^Q\}$ on a space of homogeneous type (X, d, μ).
3. Can develop definitions and theory of function spaces H^p, BMO, VMO, on (X, d, μ) and on product $(\widetilde{X}, d, \mu)$. Can dispense with earlier "additional assumptions" on X, $\widetilde{X}$.
4. Can develop definitions and theory of function spaces $H^p_{L_1,L_2}$, BMO_{L_1,L_2} associated to operators, on product $(\widetilde{X}, d, \mu)$.

Garnett conference, UCLA, 2001

HAPPY BIRTHDAY, PETER,

AND THANK YOU!

Averaging theorem on $\widetilde{X}$.

Theorem 1.7 (Chen–Li–W___)

Take X_1, X_2, $\{\mathcal{D}(\omega_1)\}_{\omega_1 \in \Omega_1}$ and $\{\mathcal{D}(\omega_2)\}_{\omega_2 \in \Omega_2}$ as above. Given family of dyadic BMO functions $\{f^\omega\}_{\omega \in \Omega_1 \times \Omega_2}$ such that

(i) *$\omega \mapsto f^\omega$ is measurable, and*

(ii) *$\|f^\omega\|_{\mathrm{BMO}_{\mathcal{D}(\omega_1) \times D(\omega_2)}(X_1 \times X_2)} \leq C_d$ for all $\omega \in \Omega_1 \times \Omega_2$.*

Then their expectation

$$f(x) = \mathbb{E}_\omega f^\omega(x) := \int_{\Omega_1 \times \Omega_2} f^\omega(x)\, d\mathbb{P}(\omega)$$

belongs to $\mathrm{BMO}(X_1 \times X_2)$, and $\|f\|_{\mathrm{BMO}(X_1 \times X_2)} \lesssim C_d$.

Generalises one-parameter: Garnett–Jones 1982;
two-param: Pipher–W___ 2008; n-param, $n \geq 3$: Treil 2009.