Clusters, loops and trees in the Ising model

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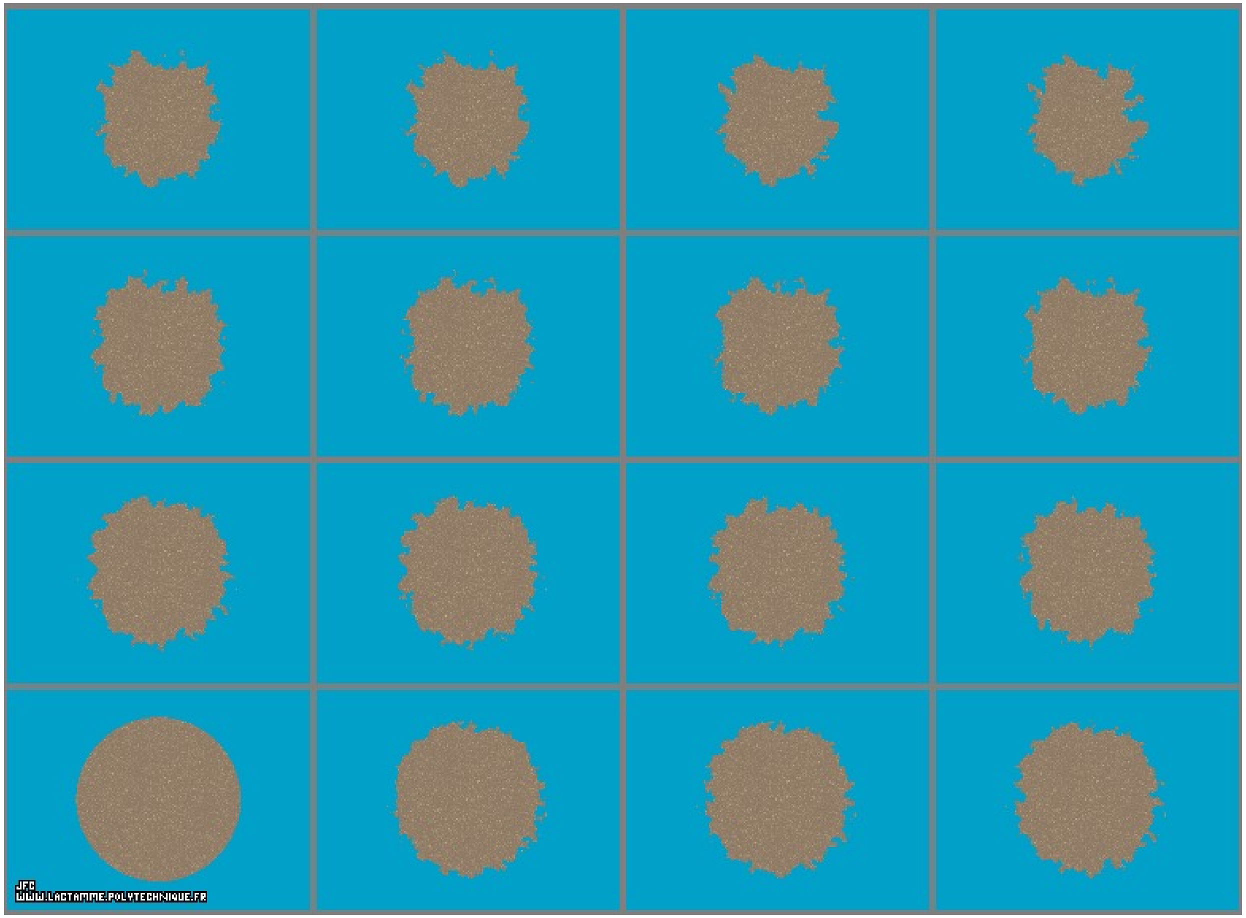
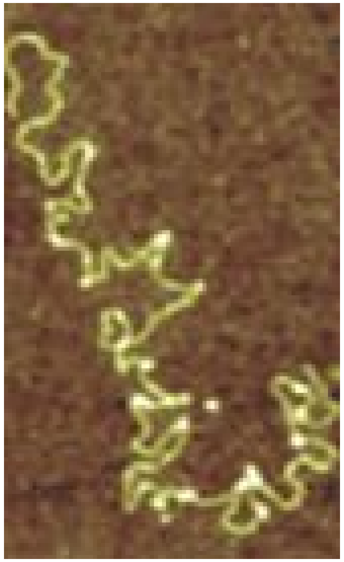
statistical physics:

macroscopic effects of microscopic interactions



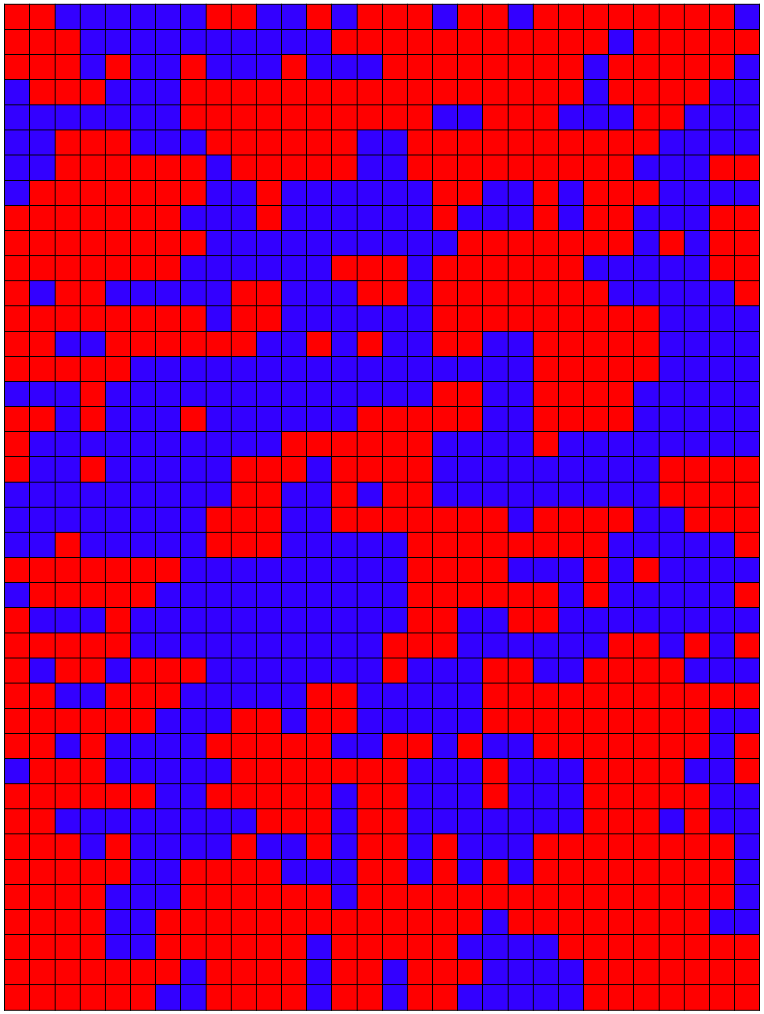
erosion simulation © J.-F. Colonna



DNA by atomic force microscopy © Lawrence Livermore National Laboratory

The Lenz-Ising model

To explain the phase transition in ferromagnetics, **Lenz** with his student **Ising**, suggested a model:



Squares of two colors, representing spins $s=\pm 1$

Nearby spins want to align, temperature parameter x :

$W(\text{config}) = x^{\#\{+-\text{neighbors}\}}$

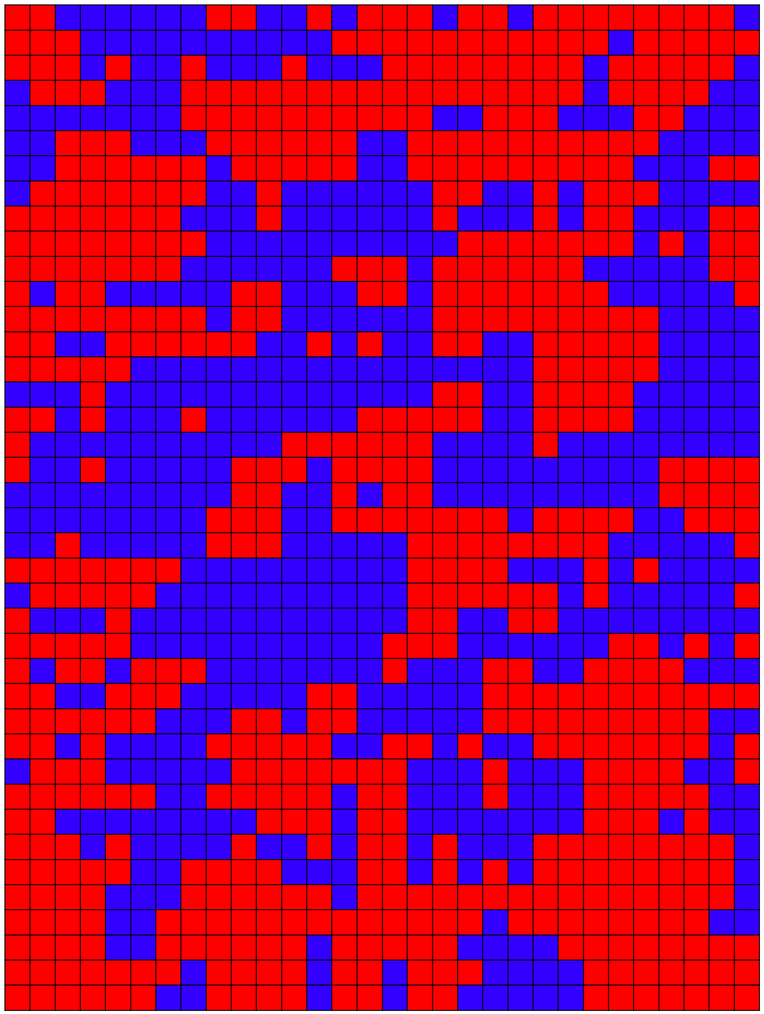
Partition function

$$Z = \sum_{\text{config}} W(\text{config})$$

Probability $P(\text{cfg}) = W(\text{cfg})/Z$

The Lenz-Ising model

To explain the phase transition in ferromagnetics,



Lenz with his student **Ising**, suggested a model:

Often written :

$W(\text{config}) = x^{\#\{+-\text{neighbors}\}}$

$\asymp \exp(-\beta \sum_{\text{neighbors}} s(u)s(v))$

Here spins $s(u) = \pm 1$

and $x = \exp(2\beta)$

In magnetic field multiply

by $\exp(-\mu \sum_u s(u))$

The 2D Ising model

Ising, 1924 There is **no** phase transition in **1D**

Peierls, 1936 There is a phase transition in 2D

Kramers-Wannier, 1941 Derive $x_{crit} = 1/(1+\sqrt{2})$

Onsager, Kaufman, K-O, 1944-50
Derive the partition function, magnetization,…

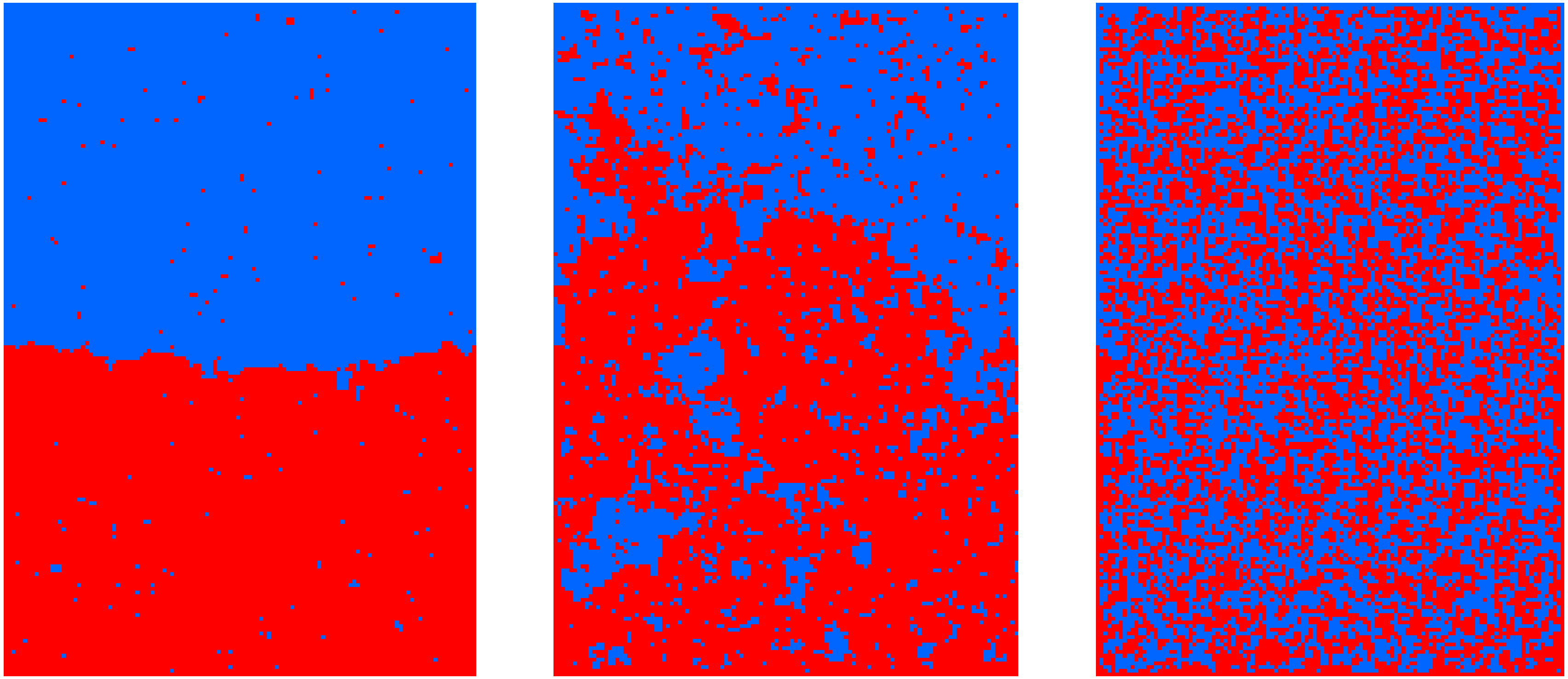
Yang, Kac, Ward, Potts, Montroll, Hurst, Green, Kasteleyn, McCoy, Wu, Tracy, Widom, Vdovichenko, Fisher, Baxter, … many results by different methods.
“Exact solvability” (not always rigorous!)

Petermann-Stueckelberg, Fisher, Kadanoff, Wilson, 1951-66, renormalization group

Belavin-Polyakov-Zamolodchikov, Cardy, 1985,
Conformal Field Theory

The phase transition in 2D

$x < x_{crit}$ $x = x_{crit}$ $x > x_{crit}$



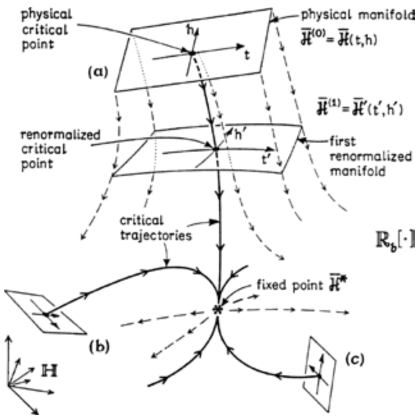
Dobrushin BC: boundary is red below, blue above.

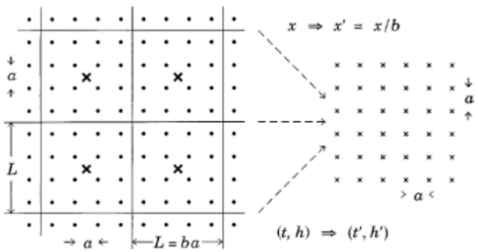
$Prob \asymp x^{\#\{+\text{-}neighbors\}}$

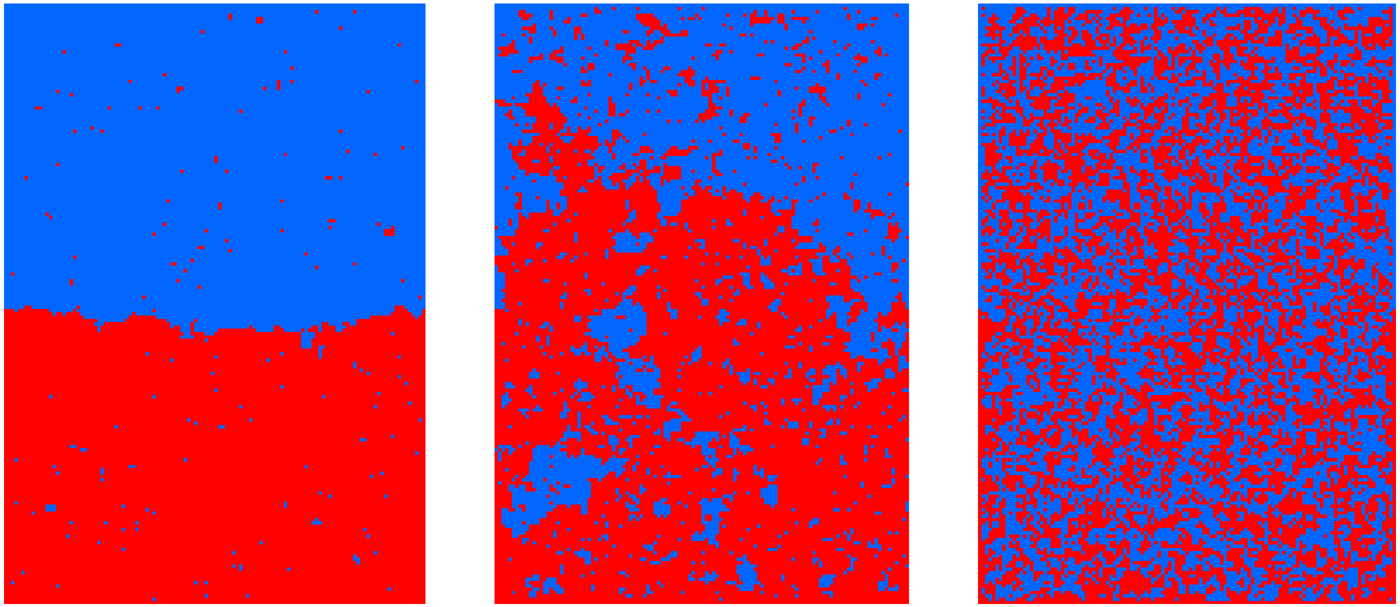
Remark. For square lattice $x_{crit} = 1/(1+\sqrt{2})$

The renormalization picture

[Stueckelberg, Fisher, Kadanoff, Wilson]







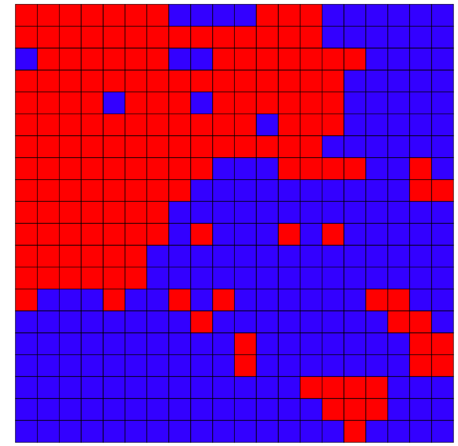
3 fixed points on [0,∞[$\quad 0 \quad < \quad x_{crit} \quad < \quad x^{*}_{crit}$

Well understood: interfaces for $x < x_{crit}$ [Pfister-Velenik], at x_{crit} [Chelkak-Smirnov], for $x > x_{crit}$ should get percolation.

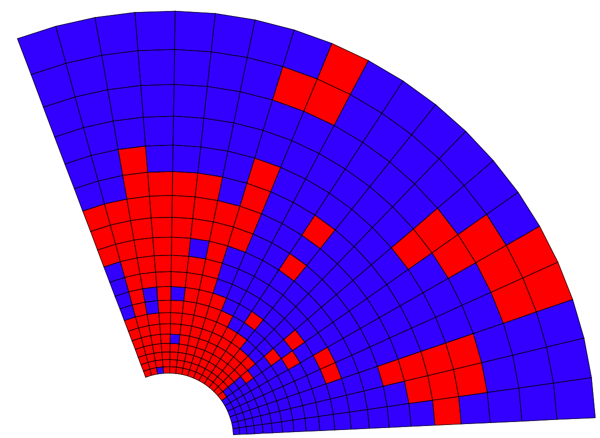
Conformal Field Theory

- a physics approach to critical points

Conformal transformations
= those preserving angles
= analytic maps



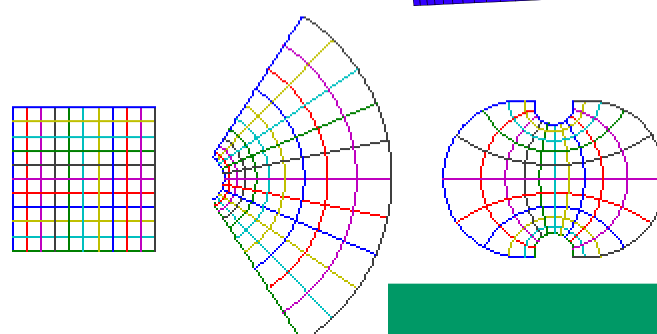
Locally **translation** +
+ **rotation** + **rescaling**



CFT [Belavin, Polyakov, Zamolodchikov 1984]:

In the scaling limit, postulate **conformal invariance**

Infinite symmetries allow to (unrigorously) derive many quantities **[Cardy, …]**



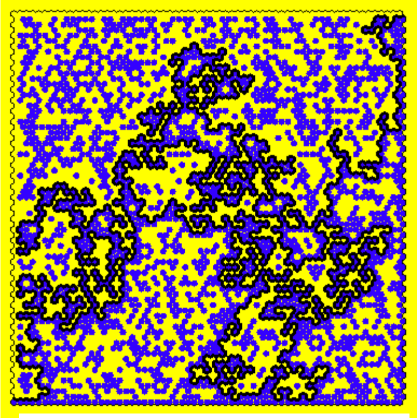
Physics of the Ising model

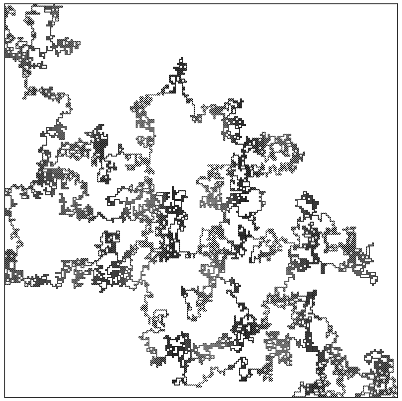
- **Phase transition in D>1**
- Curie point and exponents arise from renormalization
- **Exact solvability in D=2**
  (incl. some math results)
  E.g. magnetization exponent = 1/8
- **Conformal Field Theory in D=2**
- **D=3 expected to be similar**, recent advances of CFT [Rychkov, …]
- **D≥4 easier**

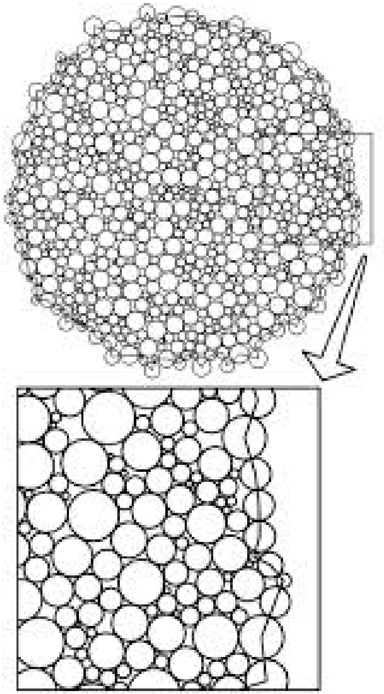
Mathematical approaches

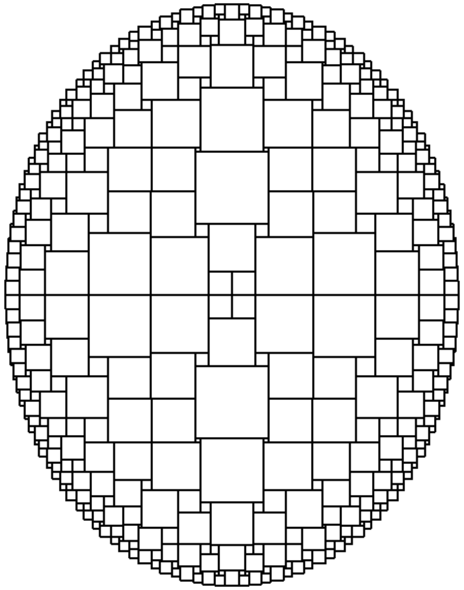
- a recent rigorous alternative

- Schramm-Loewner Evolution: a geometric description of the interfaces scaling limits at criticality – an SLE(κ) random curve
- Discrete complex analysis: a way to rigorously establish existence and conformal invariance of the scaling limit

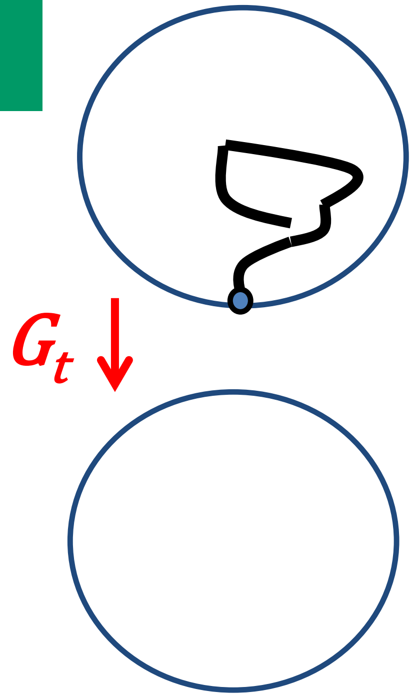








Loewner Evolution



A tool to study variation of conformal maps and domains, introduced to attack Bieberbach's conjecture

K. Löwner (1923), "*Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*. I", *Math. Ann.* **89**

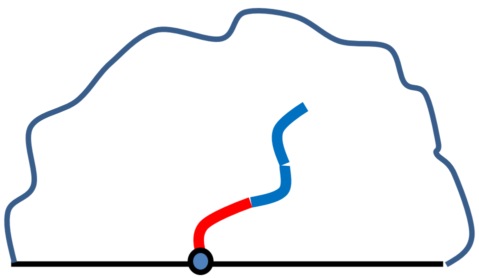
Instrumental in the eventual proof

L. de Branges (1985), "*A proof of the Bieberbach conjecture*", Acta Mathematica **154** (1): 137–152

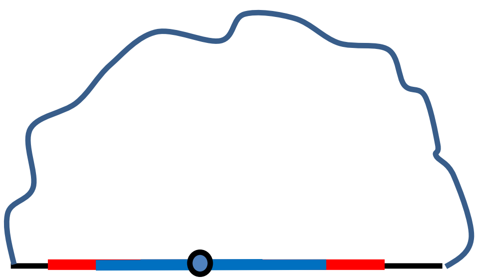
Thm Let $F(z)=\Sigma a_n z^n$ be a map of unit disk into the plane. Then $|a_n|\leq n$, equality for a slit map.

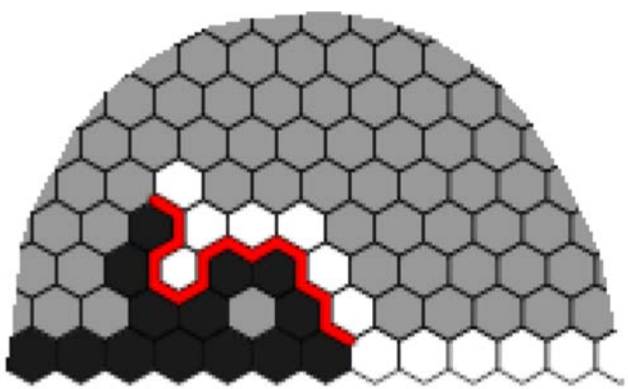
Wide open: find γ s.t. $|a_n|\lesssim n^\gamma$ for bounded maps.

Schramm-Loewner Evolution



conformal $G_t \downarrow$





Loewner Equation

$$\partial_t (G_t + U_t) = \frac{2}{G_t}$$

Schramm's SLE: a fractal curve obtained for random $U_t = \sqrt{\kappa} B_t$

- a Brownian motion

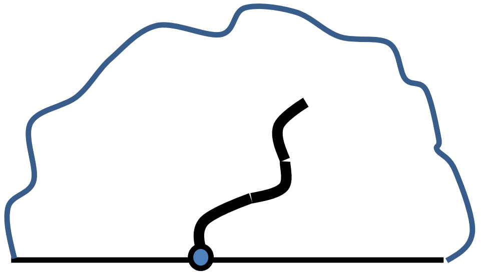
SLE=BM on the moduli space.

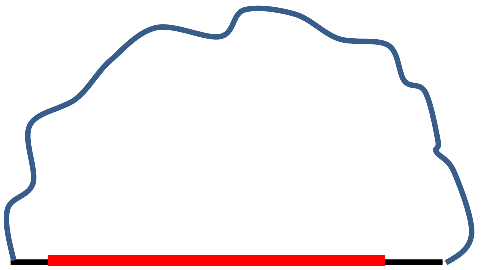
Calculations from Itô calculus, interesting fractal properties

Lemma [Schramm] *If an interface has a conformally invariant scaling limit, it is* ***SLE(κ)***

Schramm-Loewner Evolution

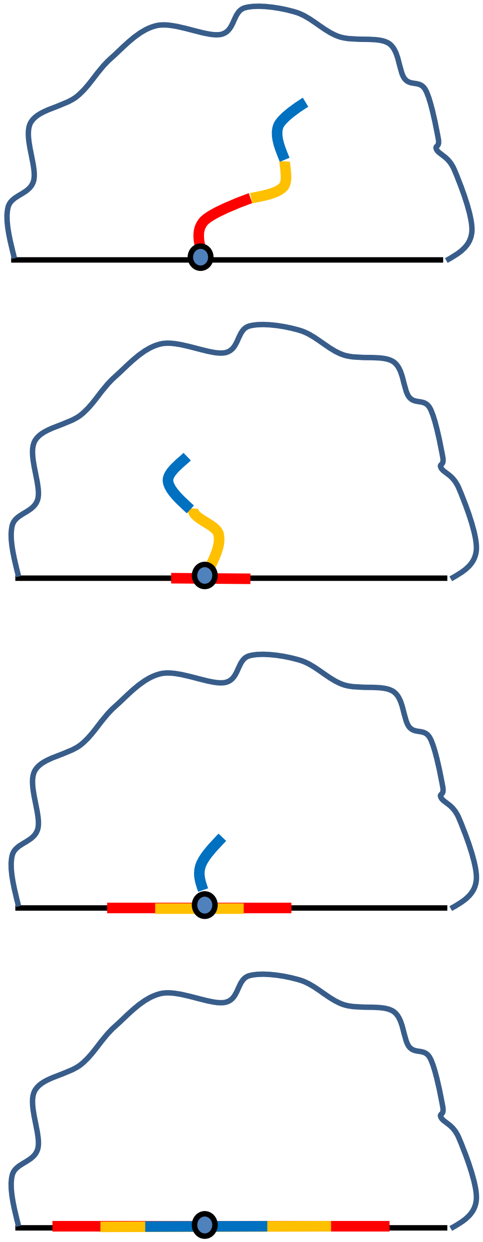
- Draw the slit



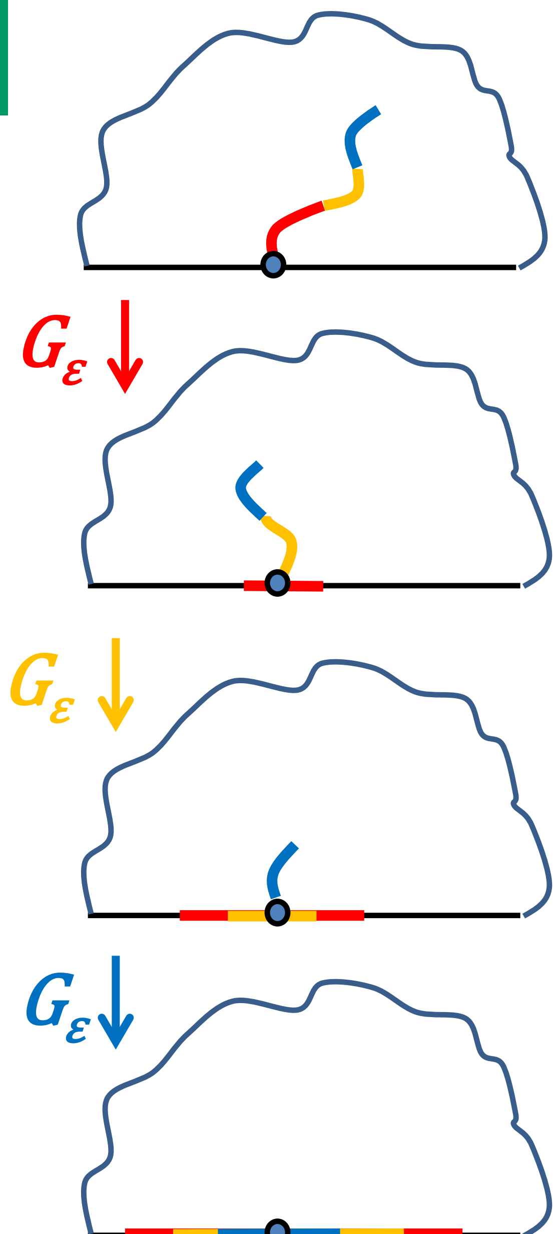


Schramm-Loewner Evolution

- Draw the slit
- Stop at **ε** capacity increments

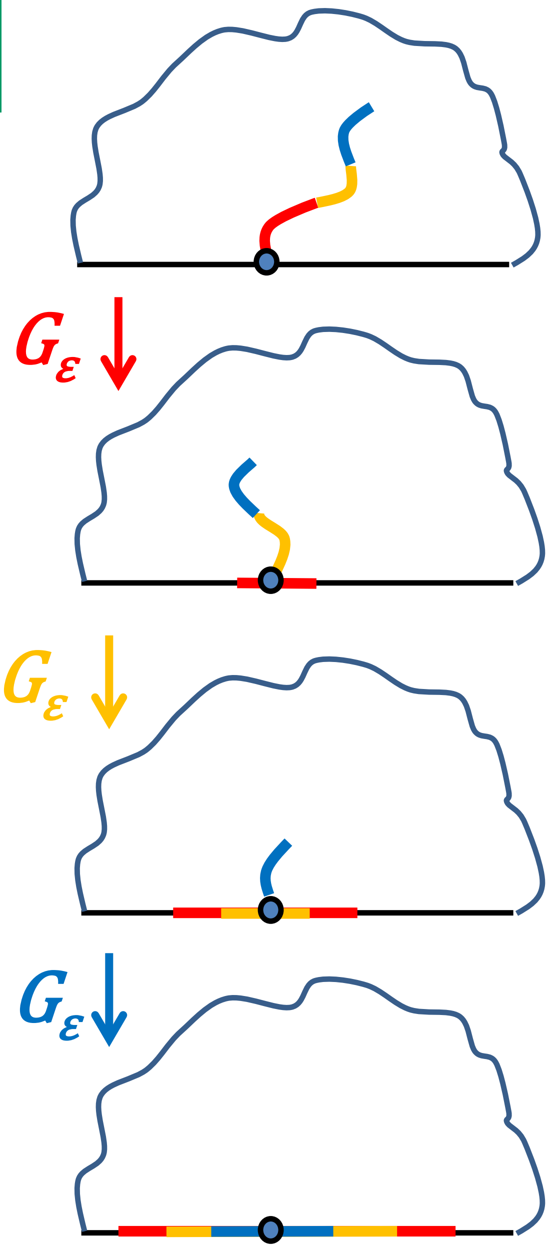


Schramm-Loewner Evolution



- Draw the slit
- Stop at **ε** capacity increments
- Open it up by a conformal map $G_\varepsilon = z - U_\varepsilon + \frac{2\varepsilon}{z} + \dots$

Schramm-Loewner Evolution



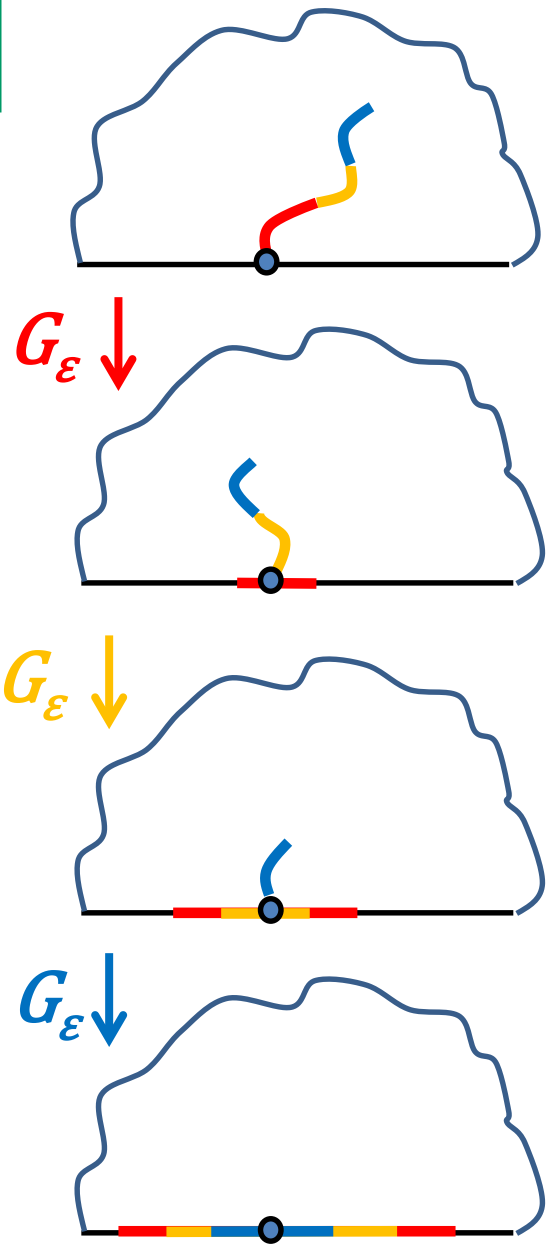
- Draw the slit
- Stop at **ε** capacity increments
- Open it up by a conformal map $G_\varepsilon = z - U_\varepsilon + \frac{2\varepsilon}{z} + \dots$
- Composition of iid maps

$$G_{n\varepsilon} = z - U_{n\varepsilon} + \frac{2n\varepsilon}{z} + \dots =$$

$$= G_\varepsilon(G_\varepsilon(G_\varepsilon(\dots))) =$$

$$= z - (U_\varepsilon + \dots + U_\varepsilon) + \frac{2n\varepsilon}{z} + \dots$$

Schramm-Loewner Evolution



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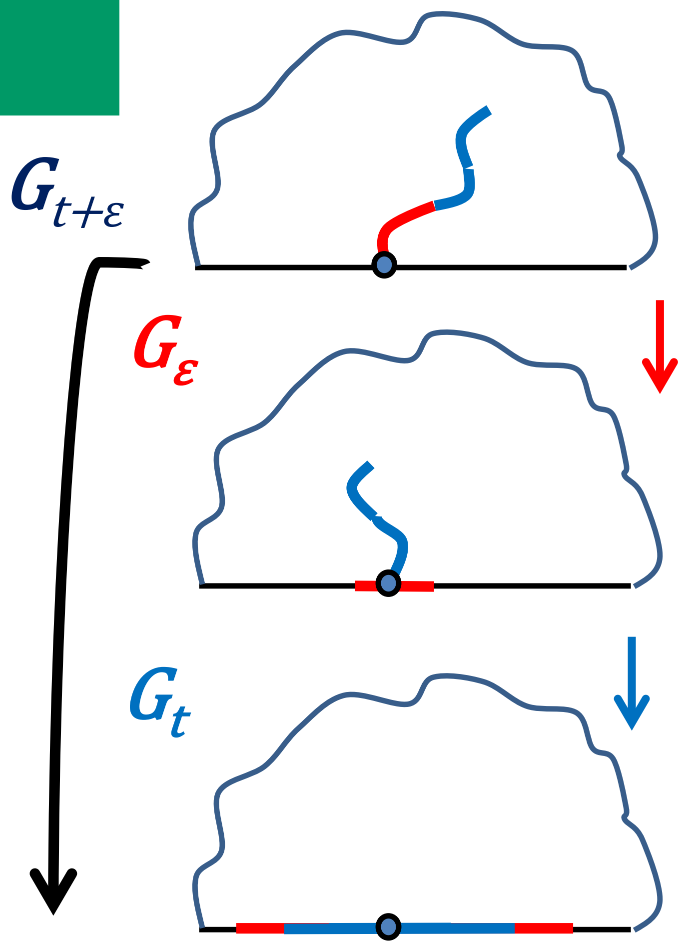
- U_t is a Brownian motion!
- “A random walk on the moduli space”

Schramm-Loewner Evolution



Differentiate the slit map

$$G_t = z - U_t + \frac{2t}{z} + \ldots$$

here $2t$ is the slit capacity

$$\partial_t (G_t + U_t) =$$

$$= \lim \frac{1}{\varepsilon} (G_{t+\varepsilon} + U_{t+\varepsilon} - G_t - U_t)$$

$$= \lim \frac{1}{\varepsilon} \left((G_\varepsilon - \mathrm{Id}) \circ G_t + (U_{t+\varepsilon} - U_t) \right)$$

$$= \lim \frac{1}{\varepsilon} \left(\left(-U_\varepsilon + \frac{2\varepsilon}{z} + \ldots \right) \circ G_t + (U_\varepsilon) \right)$$

$$= \lim \frac{1}{\varepsilon} \left(\frac{2\varepsilon}{z} \circ G_t + \ldots \right) = \frac{2}{G_t}$$

Loewner Equation $\partial_t (G_t + U_t) = \dfrac{2}{G_t}$

Schramm LE: $U_t = \sqrt{\kappa} B_t$, **a Brownian motion**

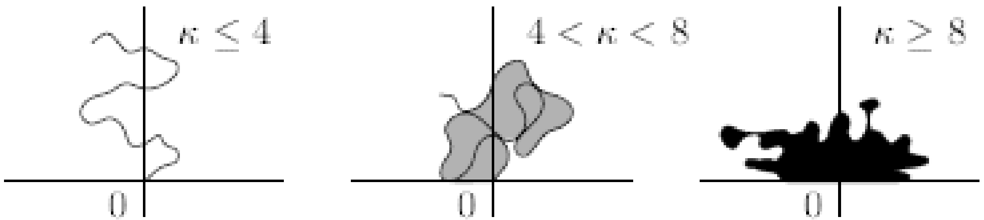
Leads to a random fractal curve

Schramm-Loewner Evolution

SLE=BM on the moduli space. Calculations reduce to Itô calculus, interesting fractal properties

Lemma [Schramm] If an interface has a conformally invariant scaling limit, it is ***SLE(κ)***

Theorem [Schramm-Rohde] SLE phases:



Theorem [Beffara]

$$\mathrm{HDim}\big(SLE(\kappa)\big) = 1 + \frac{\kappa}{8}, \qquad \kappa < 8$$

Theorem [Zhan, Dubedat]

$$SLE(\kappa) = \partial\big(SLE(16/\kappa)\big), \qquad \kappa < 4$$

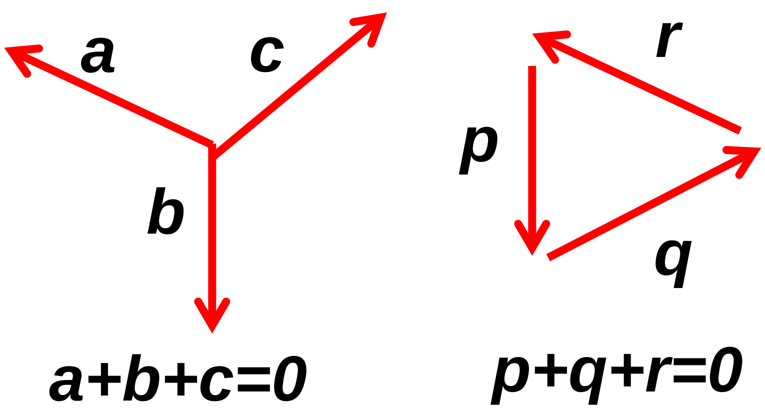
Discrete complex analysis

For 2D models (pioneered by **Kenyon** for dimers)

- Find an observable *F* (edge density, spin correlation, exit probability,…) which is discrete analytic (preholomorphic) or harmonic
- Deduce scaling limit and conformal invariance. Relation to SLE and exponents

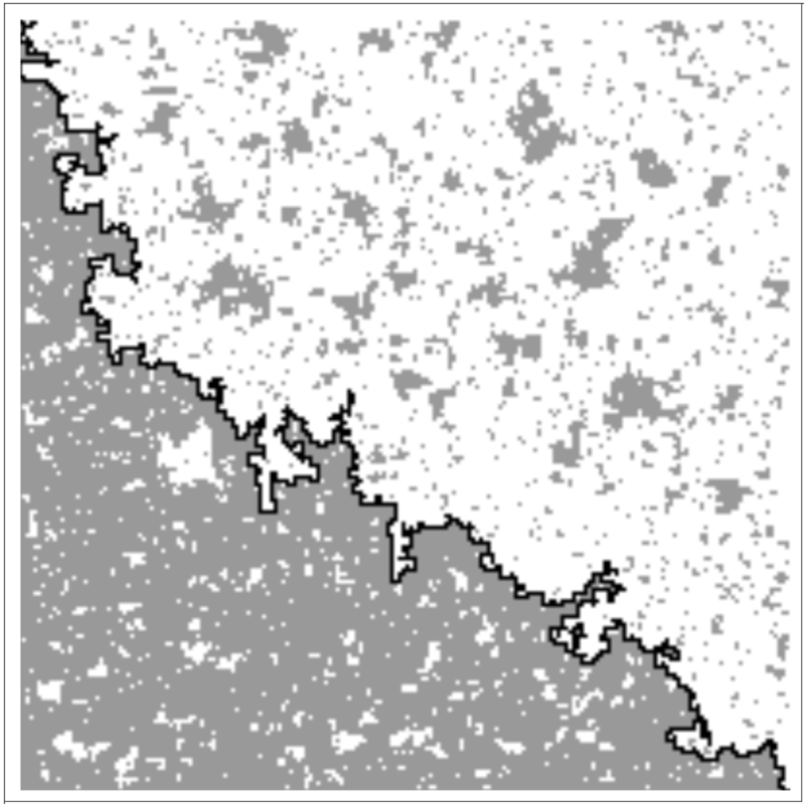
Discrete analytic function

(on a planar graph) = a flow which satisfies two Kirchhoff laws. *Local relations but leads to global info!*

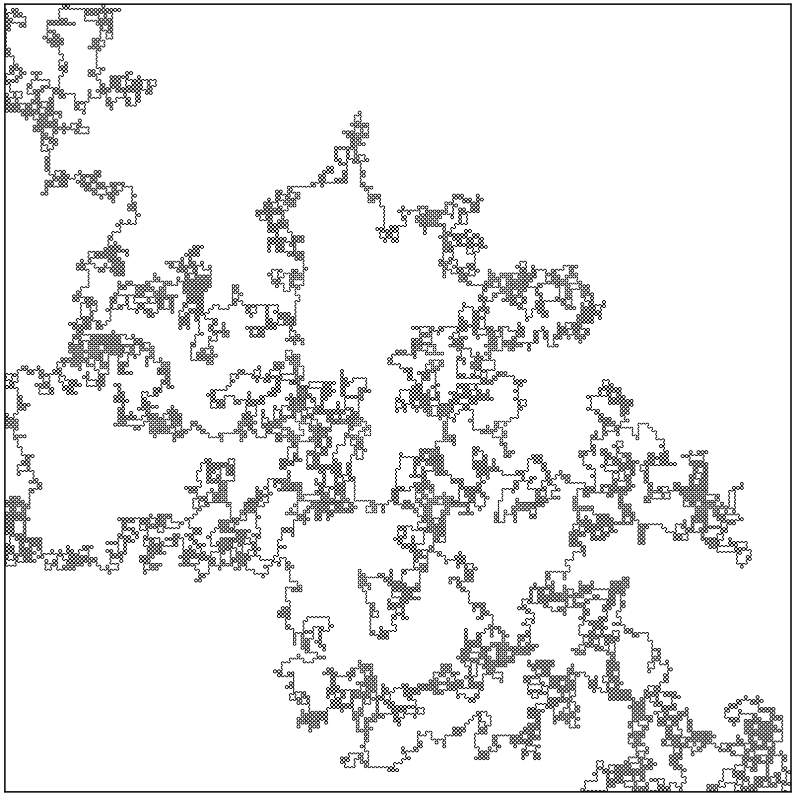


Ising and SLE

[Chelkak, S] Critical Ising and FK-Ising models have preholomorphic observables. Interfaces converge weakly to SLE(3) & SLE(16/3). *Strong convergence – more work [Chelkak, Duminil-Copin, Hongler, Kemppainen, S]*



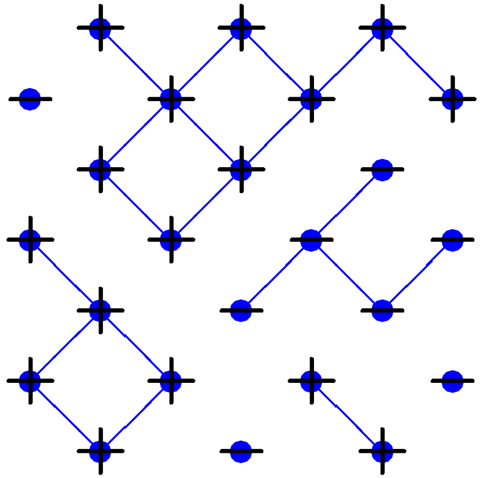
Hdim = 11/8



Hdim = 5/3

Random cluster (FK) model

Fortuin-Kasteleyn mapping (to the random cluster model)



Rewrite Ising probability:

$$\mathbf{Prob} \asymp \prod_{<ij>} \left((1-p) + p\, \delta_{s(i)-s(j)} \right)$$

Expand, to each term prescribe an edge configuration:

p $\rightarrow$ edge is open, $(1-p)$ $\rightarrow$ edge is closed

Edges only connect neighbors of same spin, but not all of them

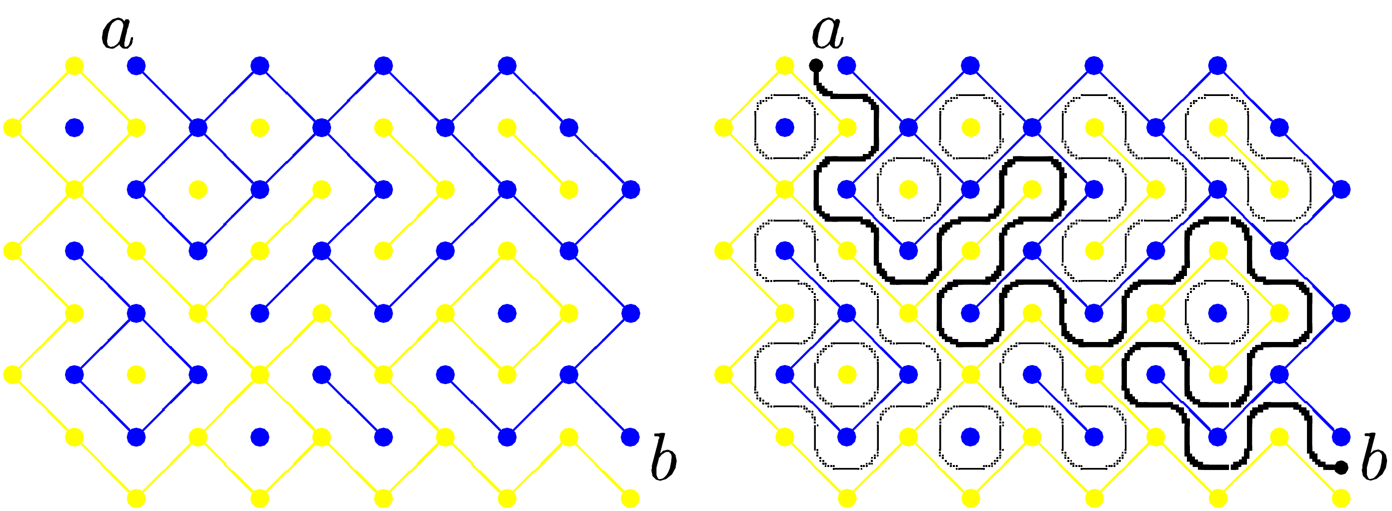
Erase spins, probability of edge configuration is

$$\mathbf{Prob} \asymp p^{\#\ \text{open edges}} (1-p)^{\#\ \text{closed edges}}\, q^{\#\ \text{open clusters}}$$

All sites in a cluster are of the same spin, q ways to chose it.

$$\mathbf{Prob}_{\text{Potts}}\,(s(i) = s(j)) = \mathbf{Prob}_{\text{FK}}\,(i \leftrightarrow j)\, \frac{q-1}{q} + \frac{1}{q}$$

Loop representation of the FK model



Configurations are dense loop collections on the medial lattice

Loops separate clusters from dual clusters

Dobrushin b.c.: besides loops an interface $\gamma : a \leftrightarrow b$

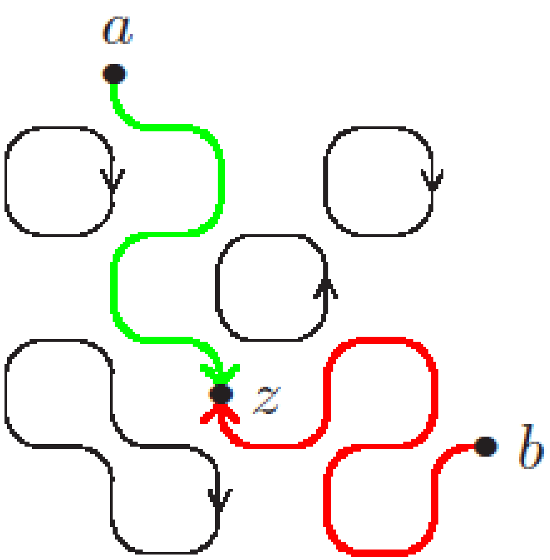
For $p = \sqrt{q}/(1+\sqrt{q})$ the probability $\mathbf{Prob} \asymp (\sqrt{q})^{\#\text{ loops}}$

A preholomorphic observable

Which observable is discrete analytic for the FK Ising?

$$F(z) := \mathbb{E}\, \chi_{z\in\gamma} \cdot \exp\left(-i\,\frac{1}{2}\,\text{winding}(\gamma, b \to z)\right)$$

- **A fermion**, spin $\sigma = 1/2$
- For general q-FK model take spin $\sigma = 1 - 2\arccos(\sqrt{q}/2)$



Motivation: orient loops randomly $\Leftrightarrow$ height function changing by ± 1 whenever crossing a loop

(geographic map with contour lines)

Orient interface $b \to z$ and $a \to z$ $\Leftrightarrow$ $+2$ monodromy at z

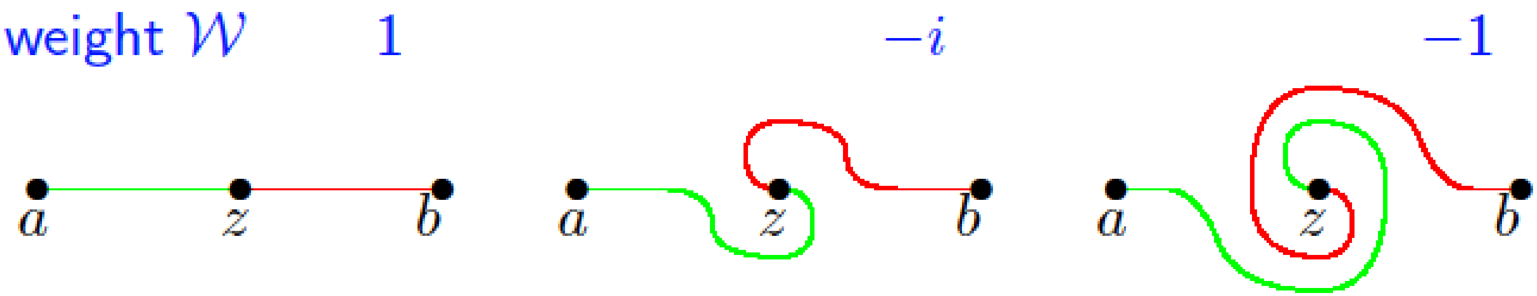
$$F(z) = Z_{+2 \text{ monodromy at z}}$$

Complex weights per loop and interface make Z local (cf. [Baxter])

A preholomorphic observable

FK Ising preholomorphic observable: $F(z) := \mathbb{E}\, \chi_{z\in\gamma} \cdot \mathcal{W}$

- Fermionic weight $\mathcal{W} \;:=\; \exp\left(-i\,\frac{1}{2}\,\mathbf{winding}(\gamma, b \to z)/2\right)$



Note: through a given edge interface always goes in the same direction, so complex weight is uniquely defined up to sign.

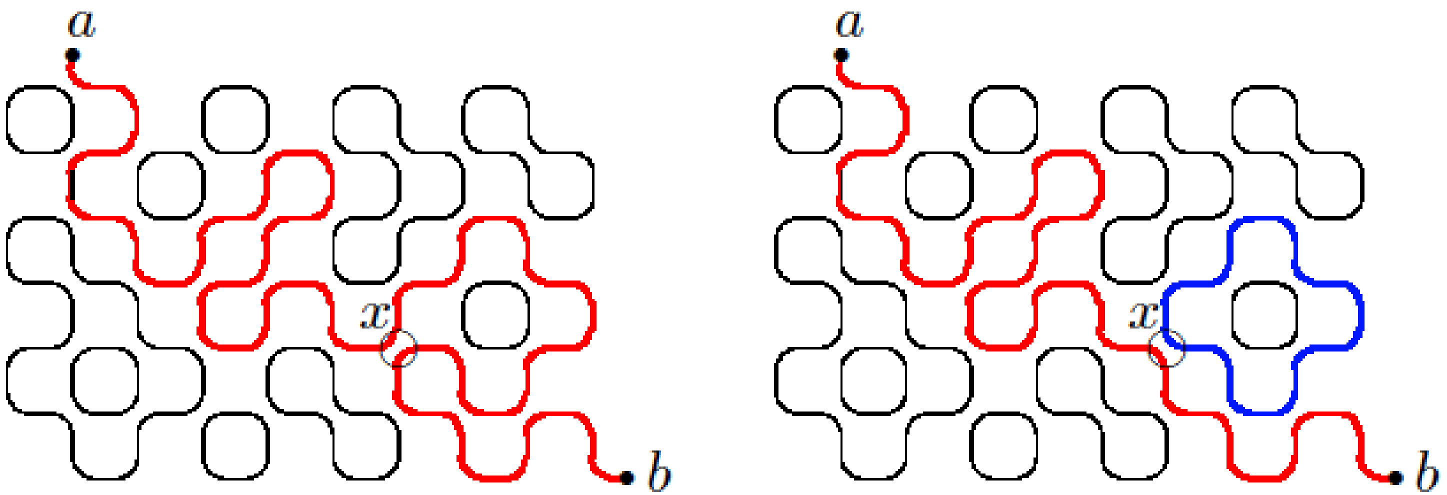
Theorem [S, Chelkak & S]. *For FK Ising when lattice mesh $\epsilon \to 0$*

$$F(z)\,/\,\sqrt{\epsilon} \rightrightarrows \sqrt{\Phi'(z)} \;\; \text{inside } \Omega,$$

where Φ maps conformally Ω to a horizontal strip, $a, b \mapsto$ ends.

A preholomorphic observable

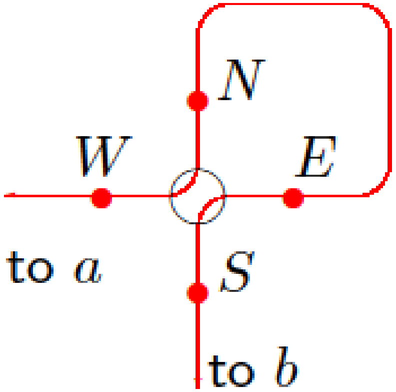
Proof: discrete analyticity by local rearrangement



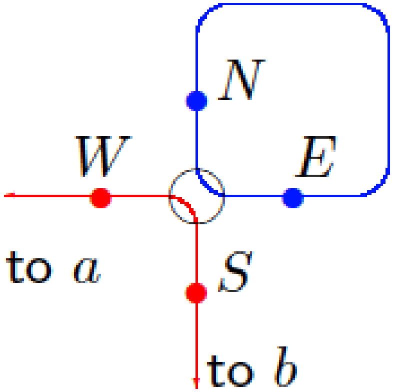
If a-b interface passes through x,
changing connections at x creates two configurations.
Additional loop on the right $\Rightarrow$ weights differ by a factor of $\sqrt{q} = \sqrt{2}$

A preholomorphic observable

Proof: discrete CR relation $F(N)+F(S)=F(E)+F(W)$



$X\lambda^2$	$F(N)$	0
X	$F(S)$	$X\sqrt{2}$
$X\lambda$	$F(W)$	$X\lambda\sqrt{2}$
$X\bar{\lambda}$	$F(E)$	0



$\lambda=\exp(-i\pi/4)$ is the weight per $\pi/2$ turn. Two configurations together contribute equally to both sides of the relation:

$$X\lambda^2+X+X\sqrt{2}=X\bar{\lambda}+X\lambda+X\lambda\sqrt{2}$$

$$i+1+\sqrt{2}=\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\sqrt{2} \qquad \square$$

A preholomorphic observable

Proof: Riemann-Hilbert boundary value problem

When z is on the boundary, winding of the interface $b \to z$ is uniquely determined, same as for $\partial\Omega \Rightarrow$ determine $\mathrm{Arg}(F)$ on $\partial\Omega$.

F solves the discrete version of the covariant Riemann BVP
$\mathrm{Im}\left(\boldsymbol{F}(\boldsymbol{z}) \cdot (\text{tangent to } \boldsymbol{\partial\Omega})^{\boldsymbol{\sigma}}\right) = \mathbf{0}$ with $\sigma = 1/2$.

Continuum case: $F = (\Phi')^\sigma$, where $\Phi : \Omega \to \text{horizontal strip}$.

Proof: convergence Consider $\int_{z_0}^{z} F^2(u)du$ – solves Dirichlet BVP.

Big problem: in the discrete case F^2 is no longer analytic!!!

A preholomorphic observable

Proof of convergence: set $H := \frac{1}{2\epsilon}\mathrm{Im}\int^z F(z)^2 dz$

- well-defined
- approximately discrete harmonic: $\Delta H = \pm\left|\partial F\right|^2$
- $H = 0$ on the arc ab, $H = 1$ on the arc ba

$\Rightarrow H \rightrightarrows \mathrm{Im}\Phi$ where Φ is a conformal map to a strip

$\Rightarrow \nabla H \rightrightarrows \Phi' \Rightarrow \frac{1}{\sqrt{\epsilon}} F \rightrightarrows \sqrt{\Phi'}$ □

Problems: we must do all sorts of estimates (Harnack inequality, normal familes, harmonic measure estimates, . . .) for *approximately* discrete harmonic or holomorphic functions in the absence of the usual tools. For more general graphs even worth.

Question: what is the most general discrete setup when one can get the usual complex analysis estimates? (while being unable to multiply functions)

Interfaces and SLEs

Proof: convergence of interfaces. Assume $\exists$ observable with a conformally invariant limit $\Rightarrow$ [Kemppainen-Smirnov] $\Rightarrow$ a priori estimates $\Rightarrow \{\gamma\}_{\text{mesh}}$ is precompact in a nice space. Enough to show: limit of any converging subsequence $=$ SLE.

Pick a subsequential limit, map to $\mathbb{C}_+$, describe by Loewner Evolution with unknown random driving force $w(t)$.

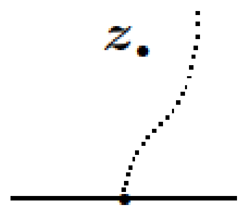
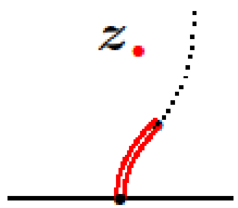
From the martingale property $\boldsymbol{F(z,\Omega) = \mathbb{E}_{\gamma'} F(z, \Omega \setminus \gamma')}$ of the observable extract expectation of increments of $w(t)$ and $w(t)^2$, conclude that $w(t)$ and $w(t)^2 - \frac{8t}{\sigma+1}$ are martingales.

By Lévy characterization theorem $\mathbf{w(t)} = \sqrt{\frac{\mathbf{8}}{\sigma+\mathbf{1}}}\mathbf{B_t}$.

So interface converges to $\text{SLE}\left(\frac{8}{\sigma+1}\right)$, i.e. $\text{SLE}\left(\frac{16}{3}\right)$ when $\sigma = \frac{1}{2}$. $\square$

Interfaces and SLEs

Proof: convergence of interfaces. Recall normalization $G_t(z) = z - w(t) + 2t/z + \mathcal{O}\left(1/z^2\right)$ at ∞.

$F\Big($  $\Big) = \mathbb{E}_{\gamma[0,t]}\, F\Big($  $\Big)$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(\log(z)')^{\sigma} \qquad\qquad \mathbb{E}_{G_t}\,(\log(G_t)')^{\sigma}$$

$$\| \qquad\qquad\qquad\qquad \|$$

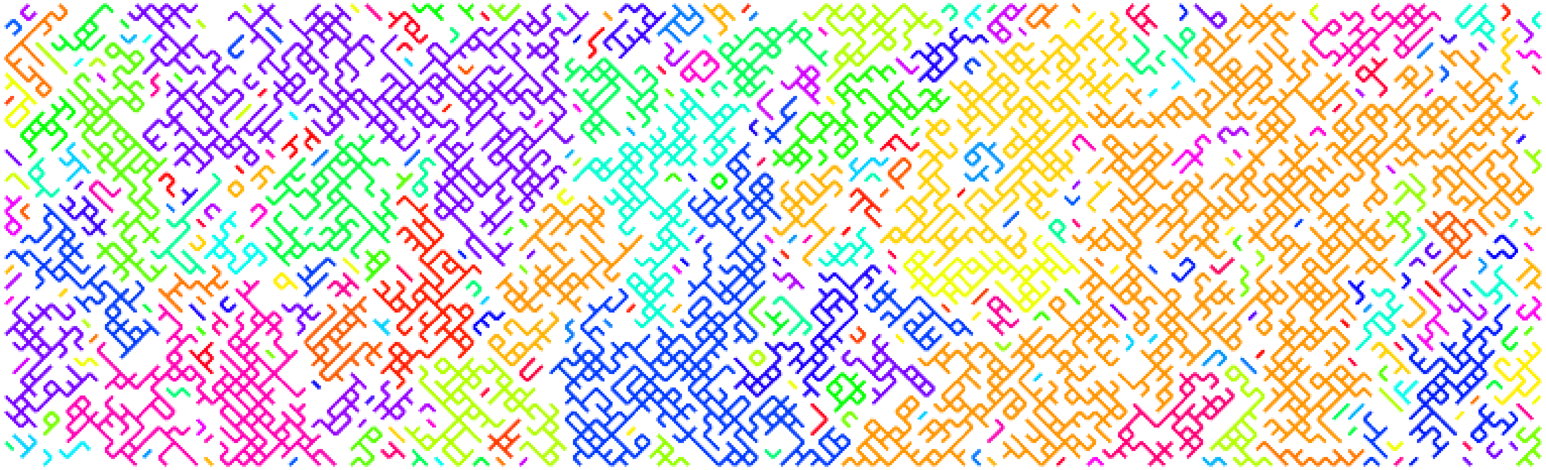
$\frac{1}{z^{\sigma}}$ $\qquad$ use expansion of G_t at ∞

$$\|$$

$$\mathbb{E}_{G_t}\frac{1}{z^{\sigma}}\left(1 + \frac{\sigma}{z}w(t) + \frac{\sigma(\sigma+1)}{2z^2}\left(w(t)^2 - \frac{8t}{\sigma+1}\right) + \mathcal{O}\left(\frac{1}{z^3}\right)\right)^{\sigma}$$

Rem A posteriori the method calculates all martingale observables for SLE!

How to describe the full scaling limit?



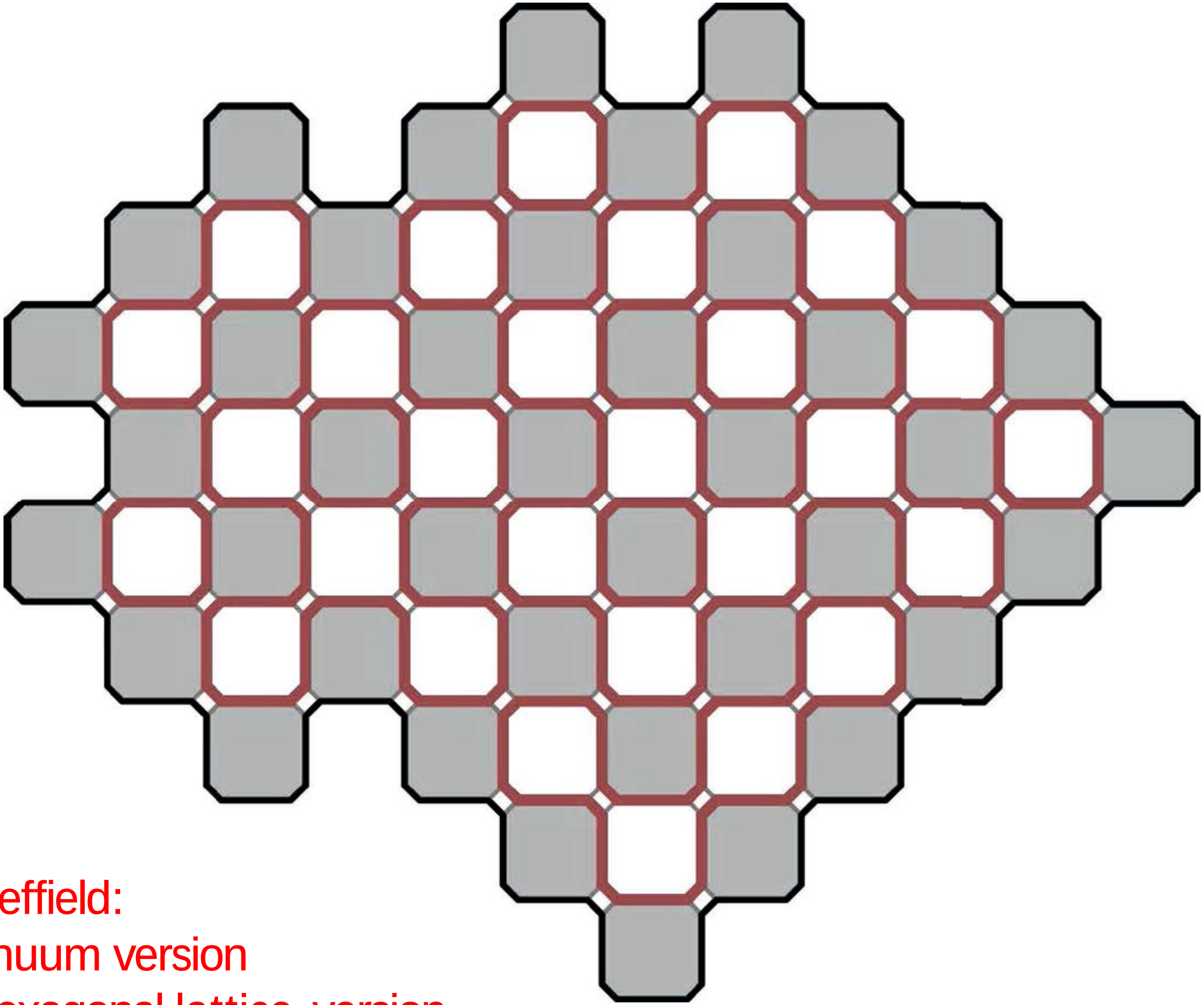
Spin correlations.
(Chelkak, Hongler, Izyurov, Smirnov, . . .)

A collection of clusters or crossings.
(Schramm, Smirnov)

A collection of loops.
(Lawler, Werner, Sheffield, Miller, Wu, . . .)

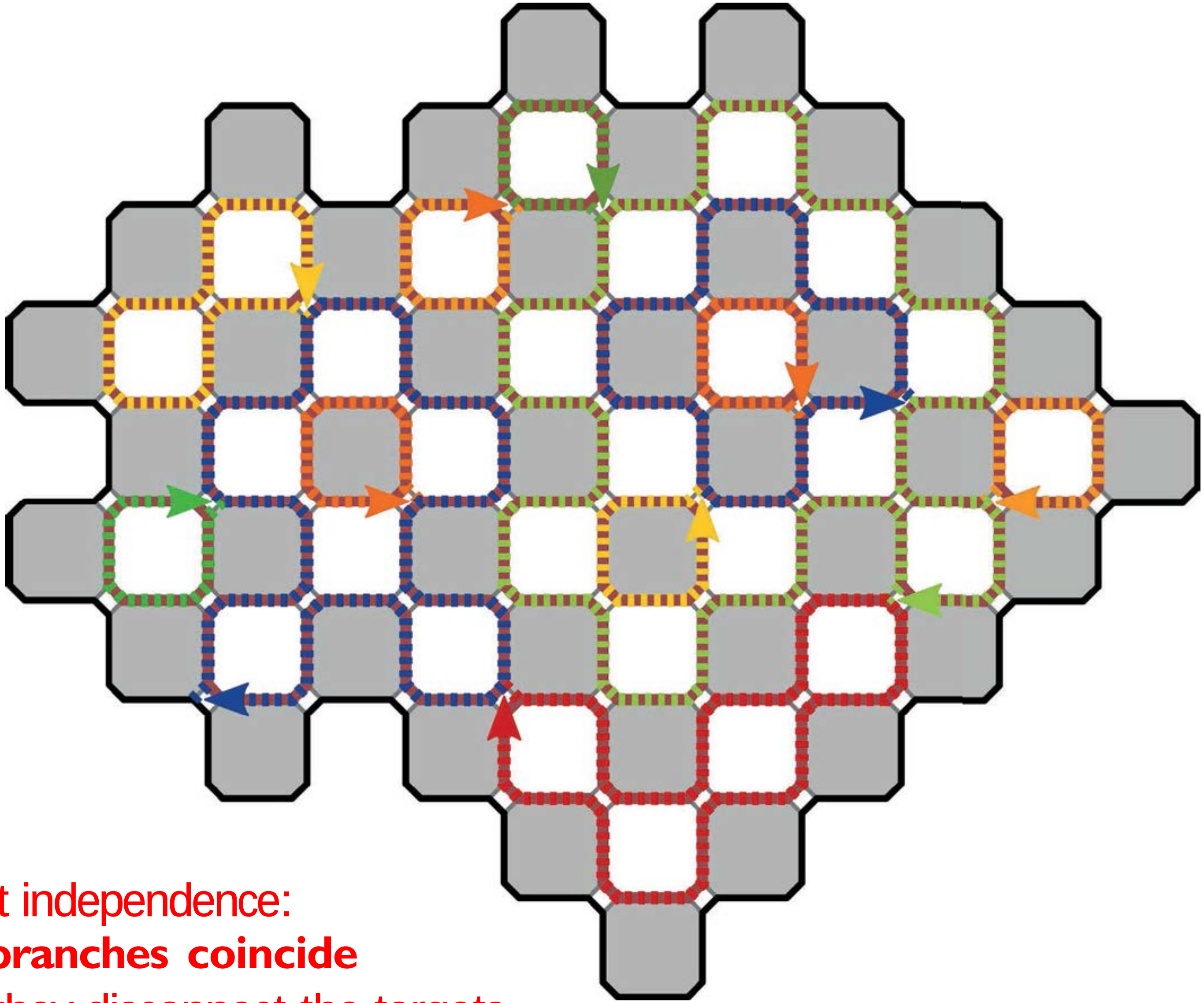
A branching tree of SLEs.
(Camia-Newman, Sheffield, Kemppainen-Smirnov, . . .)

Exploration tree



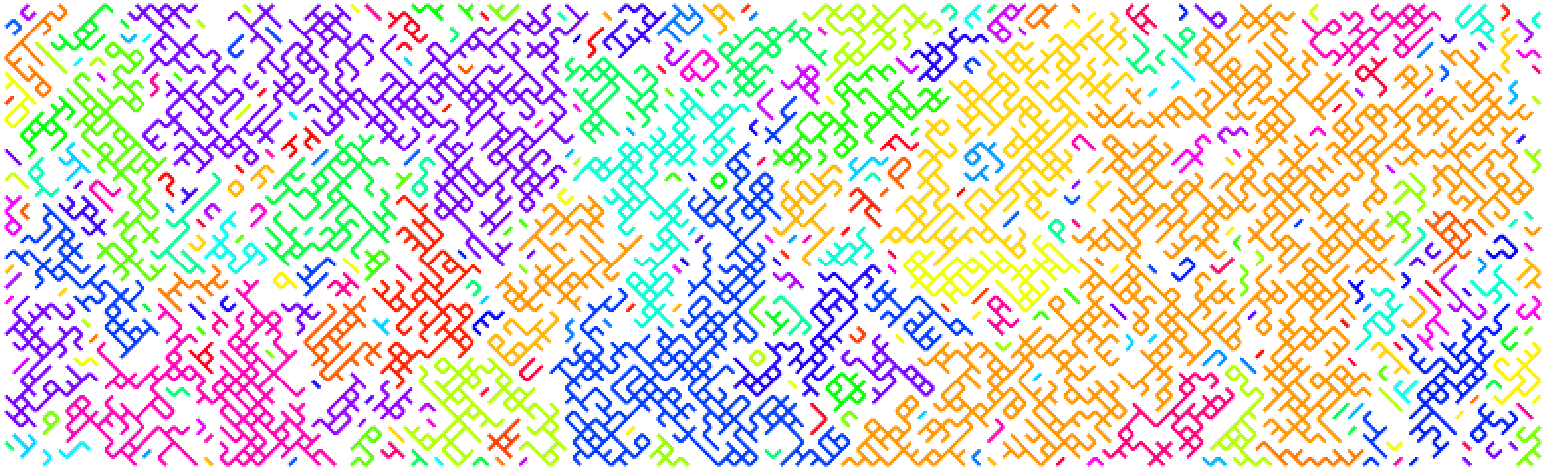
cf. Sheffield:
Continuum version
and hexagonal lattice version
(definition less starlight-forward and non-canonical).

Exploration tree

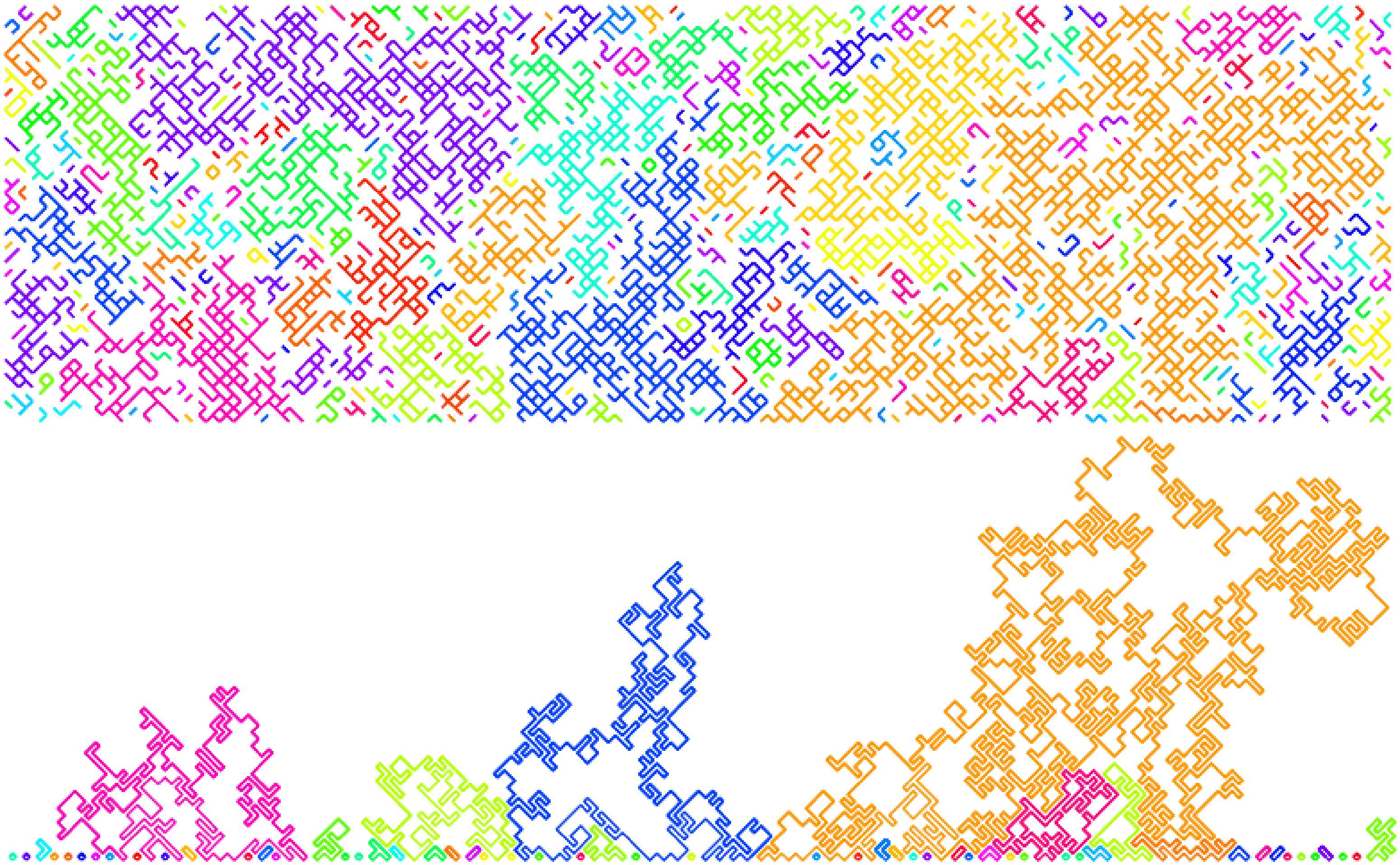


Target independence:
Two branches coincide
until they disconnect the targets,
then they are independent.

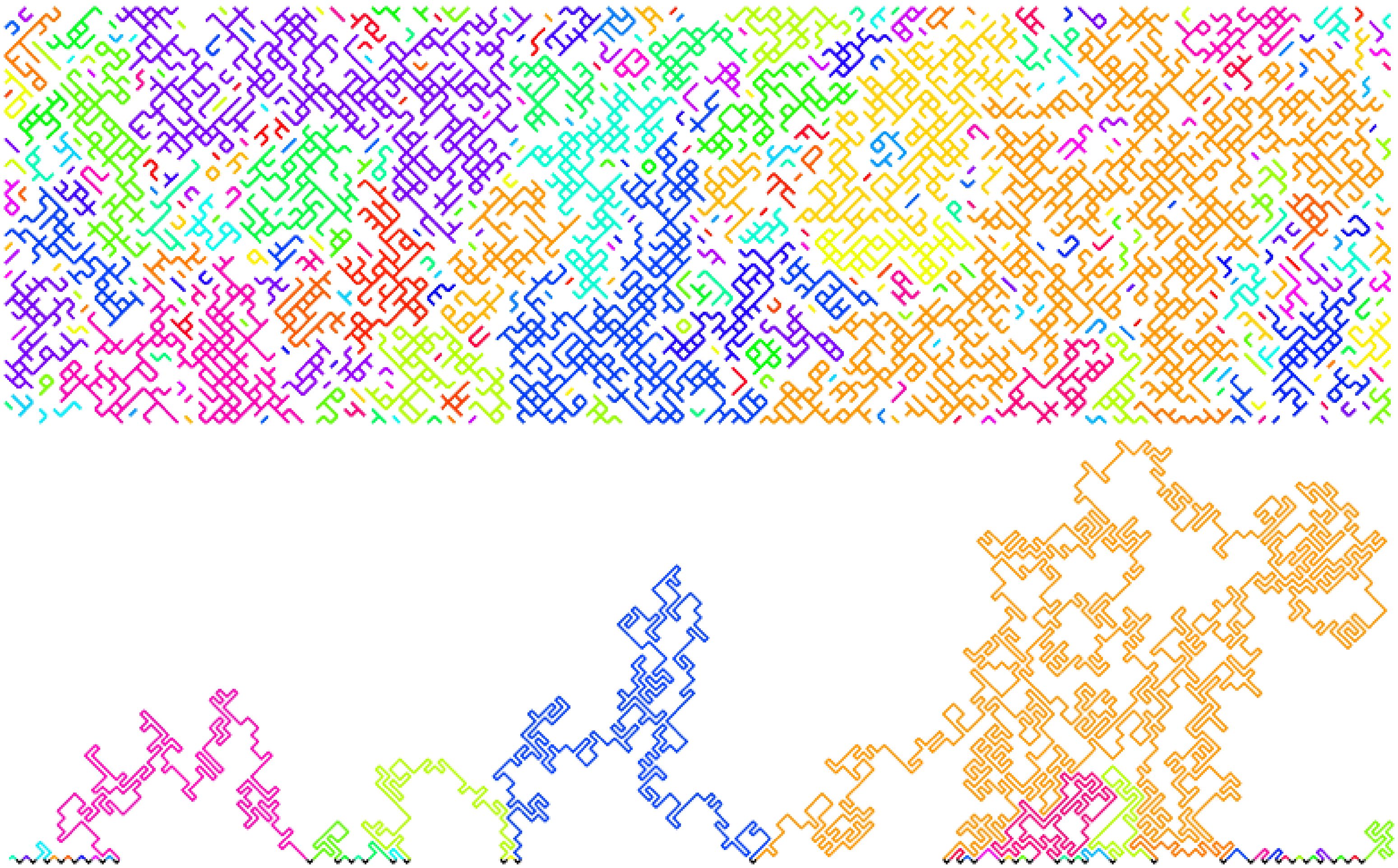
Boundary touching loops and exploration tree in FK Ising



Boundary touching loops and exploration tree in FK Ising



Boundary touching loops and exploration tree in FK Ising



The loop soup is canonically converted into a tree.

There is also an inverse map from the tree to the loops.

Main theorem

Theorem (Antti Kemppainen – Stanislav Smirnov)

Suppose a sequence of discrete domains $(z_0 \in) \quad \Omega_{\delta_n}$ *converges to* Ω *in Carathéodory metric. Then*

- *Branches of discrete exploration trees converge to* $SLE(16/3, 16/3 - 6)$.
- *Exploration tree converges to a tree of canonically coupled SLEs.*
- *Soup of loops converges to* $CLE(16/3)$ *– a soup of SLE loops.*

- **Chordal case – arXiv:1509.08858**
- **Radial case – arXiv:1609.08527**
- **4-point case – arXiv:1704.02823**

SLE generalizations

- When exploring a branch, besides **the tip** and **the target** we have **an additional marked point** on the boundary: the rightmost point in the explored hull (= the other endpoint of the current loop)
- (Chordal case) The Loewner map $g_t(z) := G_t(z) + U_t$ uniformizes the tip to U_t, and the new point to V_t, which flows with the Loewner flow:

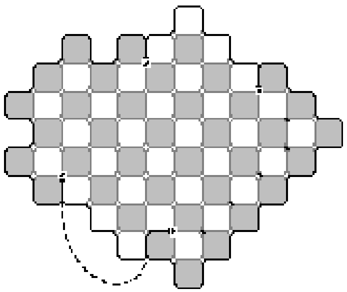
$$\mathrm{d}(G_t(z) + U_t) = \frac{2}{G_t(z)}\mathrm{d}t \quad \text{hence} \quad \mathrm{d}(V_t) = \frac{2}{(V_t - U_t)}\mathrm{d}t.$$

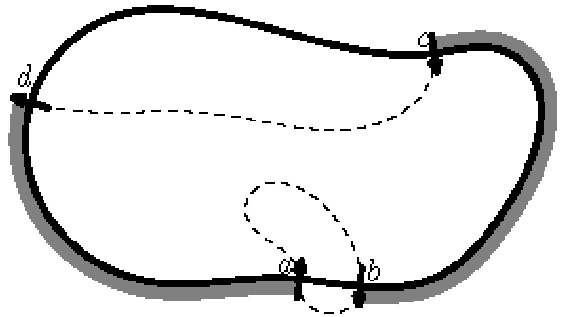
  But now the driving force U_t is a **BM + a drift depending on** V_t.
- A special role is played by SLE(κ, ρ) processes. In this case $V_t - U_t$ is a Bessel process of dimension $D = 1 + 2(\rho + 2)/\kappa$ times $\sqrt{\kappa}$:

$$\mathrm{d}(V_t - U_t) = -\sqrt{\kappa}\mathrm{d}B_t + \kappa\frac{D-1}{2}\frac{\mathrm{d}t}{V_t - U_t}.$$

  Note: $\mathrm{d}U_t = \sqrt{\kappa}\mathrm{d}B_t + \ldots$
- SLE$(\kappa, \kappa - 6)$ enjoys locality property: it does not see where it is going until it hits it (a theorem by [**Lawler, Schramm & Werner**]. *E.g., SLE*$(6) =$ *SLE*$(6, 0)$ *is local and so describes percolation.*
- We see how SLE$(\kappa, \kappa - 6)$ appear naturally as tree branches.

New observables (from adding a long edge)





γ_1 starts at a

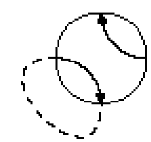
γ_2 starts at c

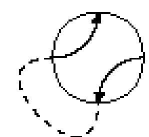
α external connection

$b \to a$

- The branch γ follows γ_1 until the next loop-to-loop jump where it starts to follow new γ_1.
- Set

$$\hat{\gamma} = \begin{cases} \gamma_2 & \text{if } \gamma_2 \text{ exits through } d \\ \gamma_2 \sqcup \alpha \sqcup \gamma_1 & \text{if } \gamma_2 \text{ exits through } b \end{cases}$$





- Observable is defined as before, and is discrete holomorphic for the same reasons.
- The boundary conditions are similar with jumps at 4 marked points.
- The tree observable is obtained by fusing two marked points.

Simple martingales from the observable

- The tree branch observable can be calculated: e.g., in the upper half-plane $\mathbb{H}$ with $a = u, b = v$ and $c = \infty$ with $u < v$, it is

$$f^{\mathbb{H},u,v}(z) = \sqrt{1 + \beta \left(-\frac{1}{z-u} + \frac{1}{z-v} \right)}$$

$$f^{\mathbb{H},U_t,V_t}(z) = \sqrt{g_t'(z)} \sqrt{1 + \beta_t \left(-\frac{1}{g_t(z)-U_t} + \frac{1}{g_t(z)-V_t} \right)}$$

- $U_t = g_t(\gamma(t))$, $V_t = \max g_t(K_t)$ " $= g_t(v)$".
- $t \mapsto V_t$ is increasing and satisfies LE when t satisfies $U_t \neq V_t$.
- Then

$$\beta_t = \frac{1}{4}(V_t - U_t).$$

- Take the leading non-trivial coefficient of $f^{\mathbb{H},u,v,w}$ in the expansions as $z \to \infty$. Then

$$M_t := \pm\sqrt{4\beta_t}$$

$$N_t := 4\big(\beta_t(V_t - U_t) - 2t\big) = M_t^4 - 8t$$

are martingales.
$\pm$ *independent random sign for each excursion.*

Determination of the driving process

- Note that $\{t : U_t = V_t\} = \{t : M_t = 0\}$.
- **What is the law of (M_t) if we know that**

$$M_t \quad \textbf{and} \quad N_t = M_t^4 - 8t$$

  are martingales?
  Cf. Lévy chracterization of BM and Stroock-Varadhan.
- Then $\mathbb{E}[\int_0^t \chi_{M_s=0} \mathrm{d}s] = 0$ and (M_t) defines a standard Brownian motion (B_t) so that $M_t = M_0 + \int_0^t \sigma_s \mathrm{d}B_s$.
- N_t is martingale only when

$$\sigma_t = \sqrt{\frac{4}{3} \frac{1}{|M_t|}}.$$

- If we define $X_t = \sqrt{3/16}(V_t - U_t) = \sqrt{3/16} M_t^2$, then

$$\mathrm{d}X_t = \mathrm{d}\tilde{B}_t + \frac{1}{4} \frac{\mathrm{d}t}{X_t}$$

- **Bessel process of dimension** $D = 3/2$.
  SLE(κ, ρ) with $\kappa = 16/3$ and $\rho = -2/3$. $\implies \rho = \kappa - 6$.

Clusters, loops and trees

- There is a canonical way to explore FK clusters, leading to a branching tree
- Tree are equivalent to clusters and loops
- Branches are nicely coupled (coincide up to separation, then independent)
- Branch independence suggests locality, so expect a continuum tree of branching SLE(κ,κ-6)
- There are observables related to the tree
- Characterization of a diffusion by two moments is possible, gives us a Bessel process, and hence SLE(16/3,-2/3)
- Similar approach to spin Ising?

Happy birthday!