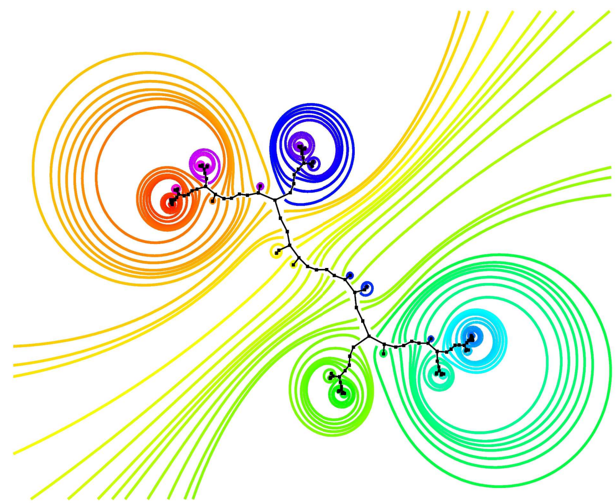
Conformal laminations and trees

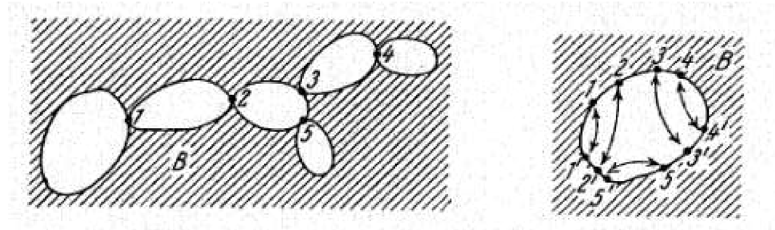
Steffen Rohde, UW

KIAS, Seoul, May 8, 2017



Which laminations are conformal?

$\mathcal{L}$ "small" (Koebe, Leung, Gupta)



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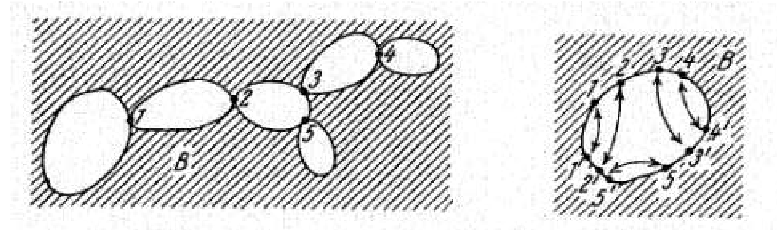
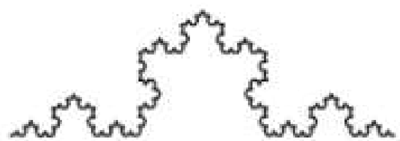


Jordan arcs



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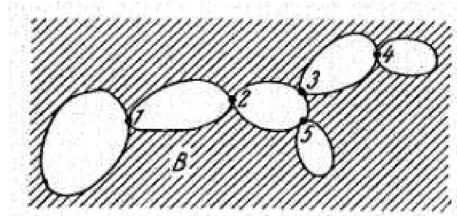

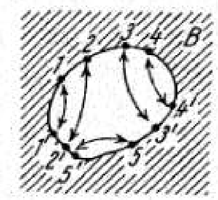

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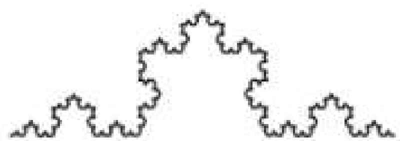


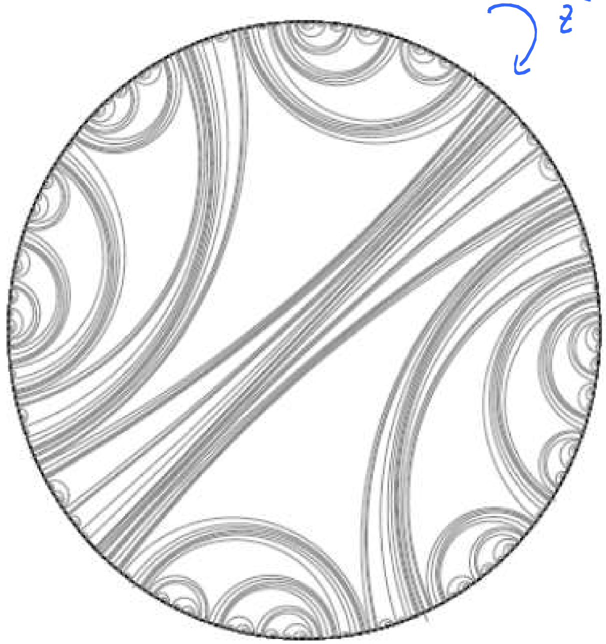
unique solution if quasisymmetric

complex dynamics

Complex dynamics

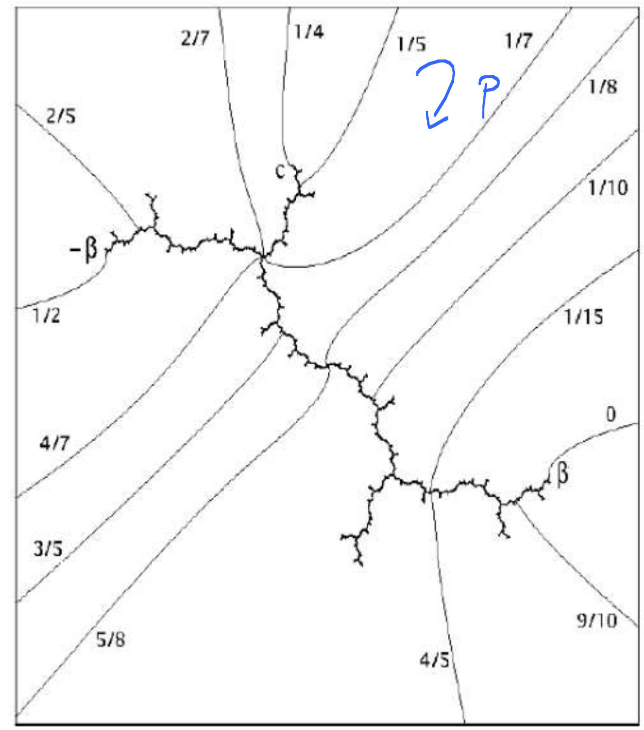
$p(z) = z^2 + c$

$D := \{ z : p_n(z) \longrightarrow \infty \}$ simply connected $\Longleftrightarrow$ $c \in \mathcal{M}$



$\circlearrowright z^2$

$\xrightarrow{f}$



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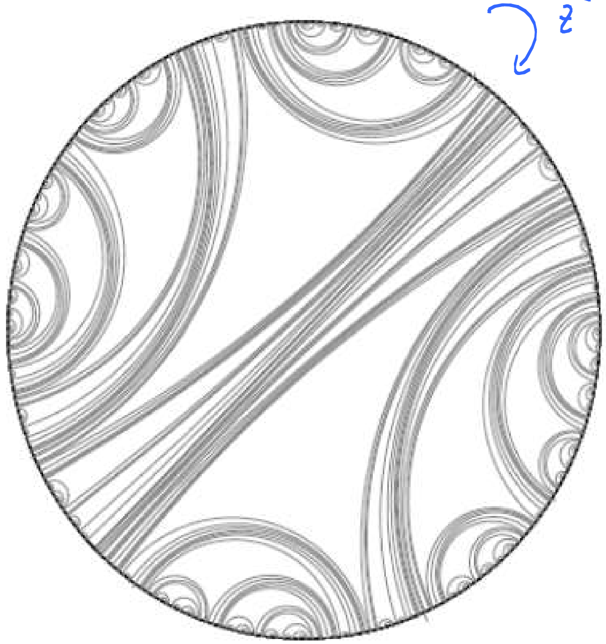

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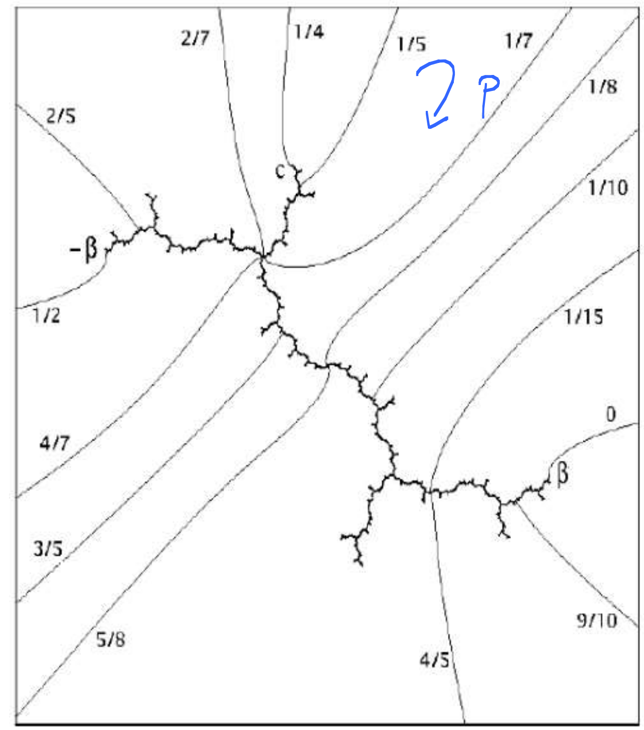

$\circlearrowright p$

Carleson-Jones-Yoccoz ('94): D is John $\iff 0 \notin \overline{\bigcup_n p_n(0)}$

Zdunik ('00), Smirnov ('01), Bruin-Schleicher ('14): $\dim \operatorname{supp} \mathcal{L} < 1 \quad (c \neq -2)$

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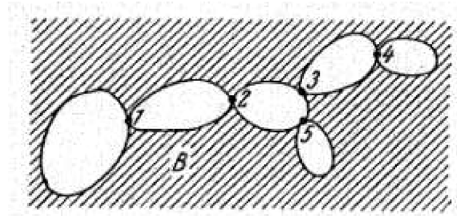

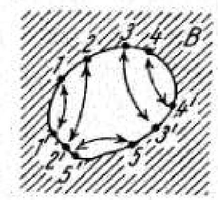

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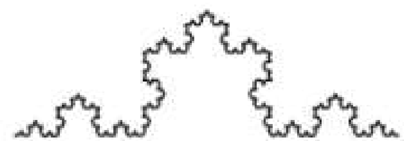

unique solution if quasisymmetric

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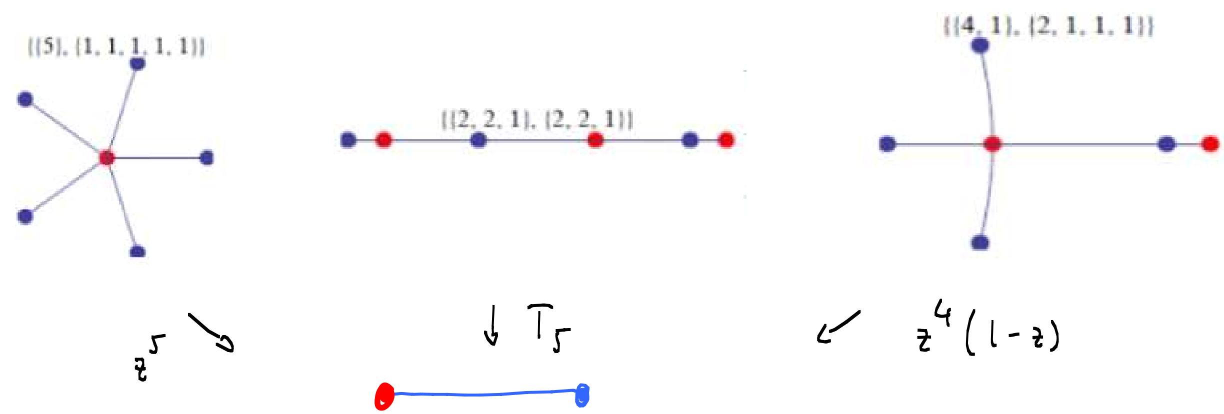
balanced trees / Shabat polynomials

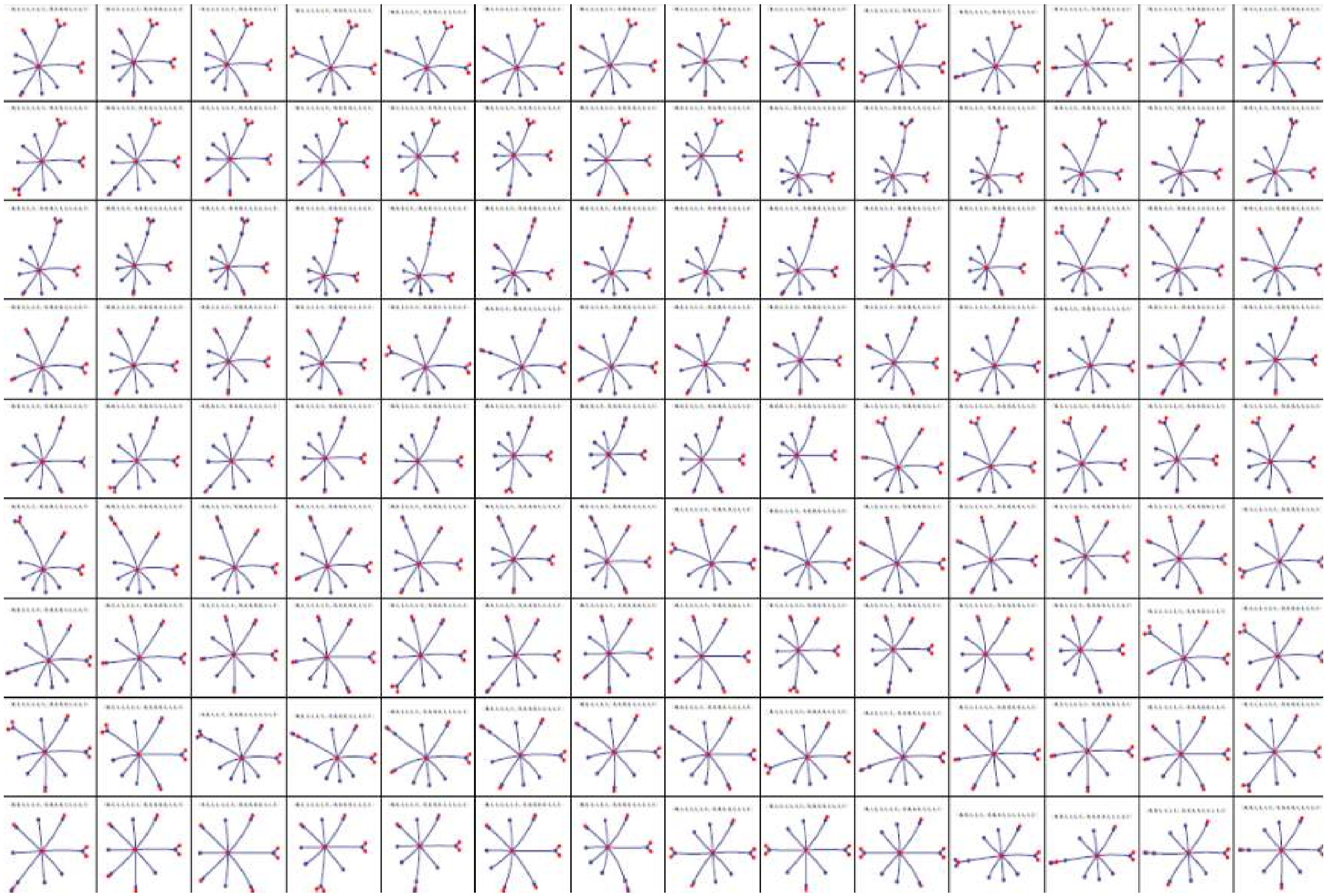
Thm (Shabat)

Every combinatorial tree can be embedded as $T \subset \mathbb{C}$ s.th. there is a polynomial p with $p^{-1}[0,1] = T$ and $p(\mathrm{Crit}) = \{0, 1, \infty\}$



lamination:

How does a large random tree look like?

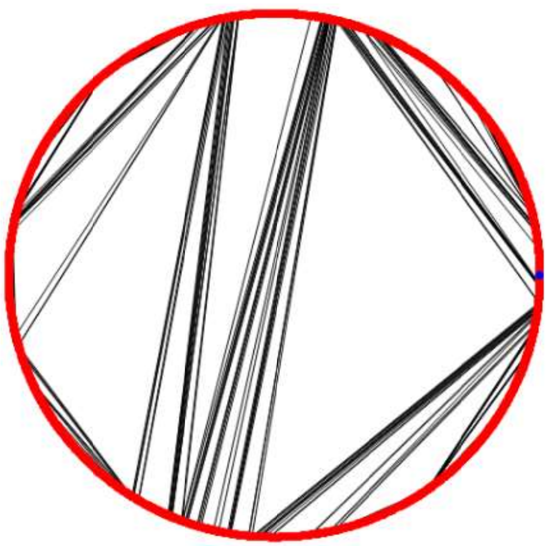


How does a large random tree look like?

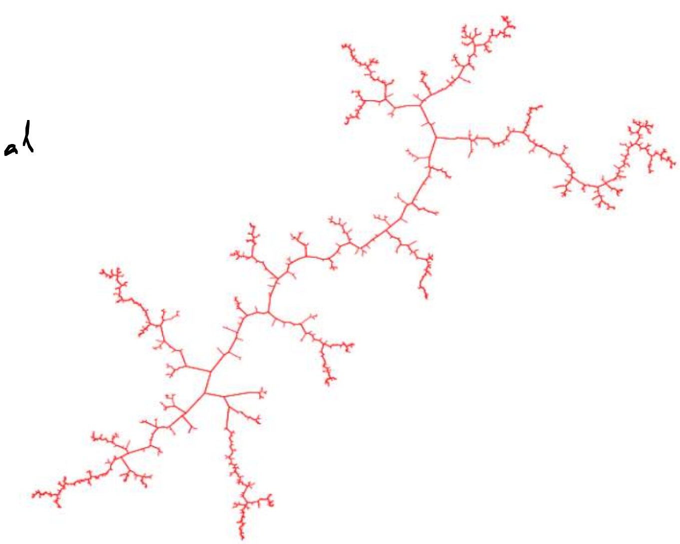
$\exists$ conformal maps $f_n : \Delta \to S^2 \setminus T_n$

<u>Q</u> Does $f_n \to f$ $(n \to \infty)$?

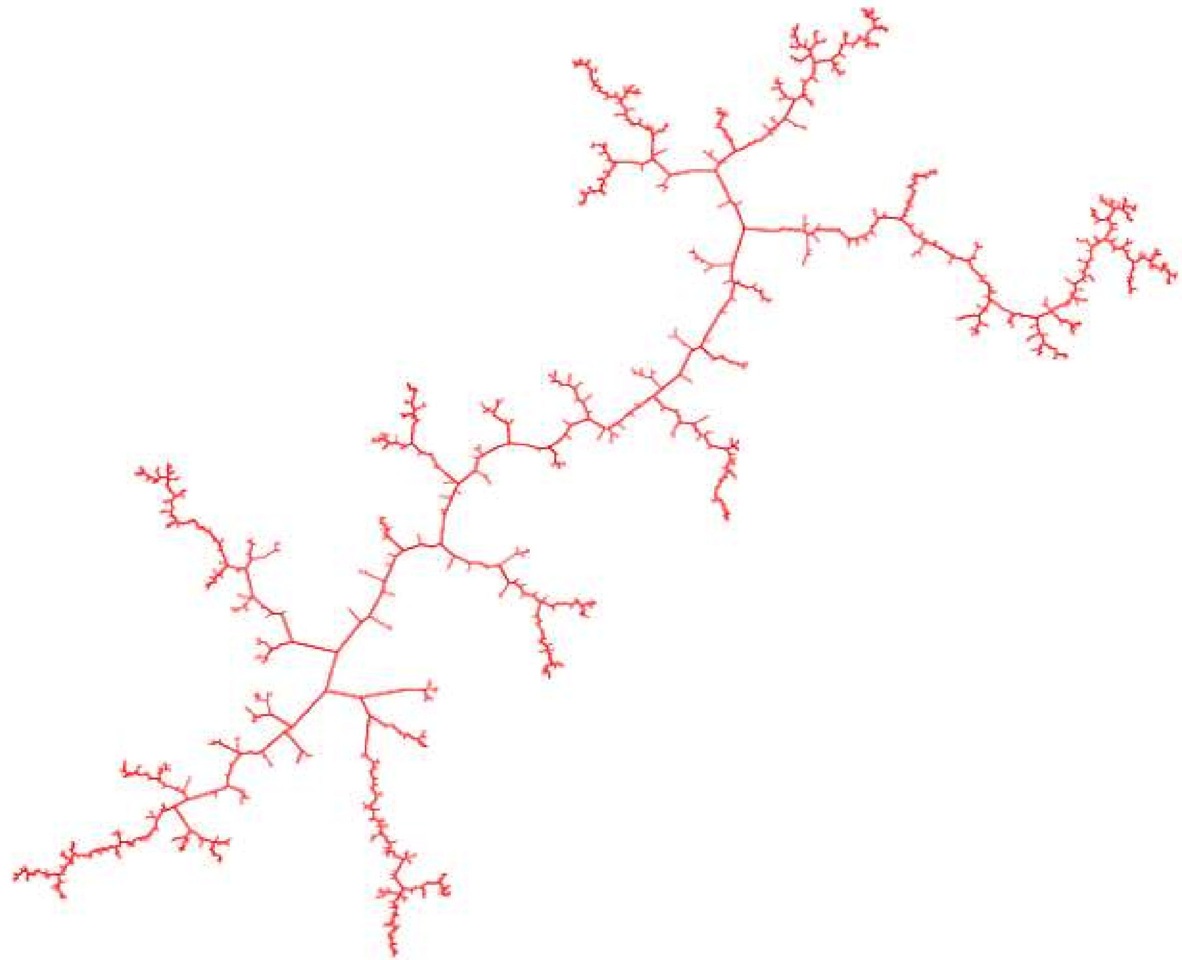
Is the Brownian lamination $\mathcal{L}_B$ conformal, and $\mathcal{L}_B = \mathcal{L}_f$?



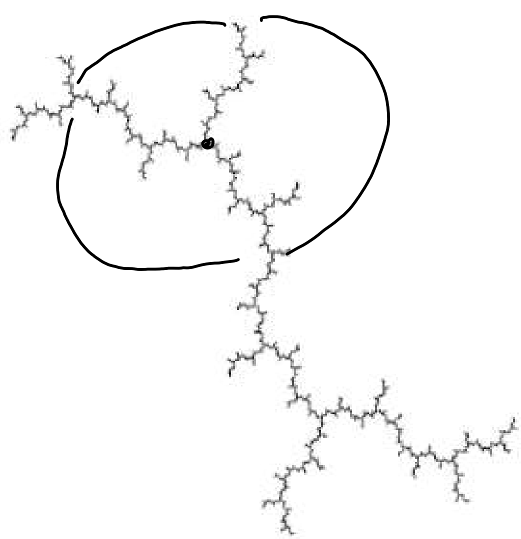
conformal



Conformal welding of Aldous' Continuum Random Tree



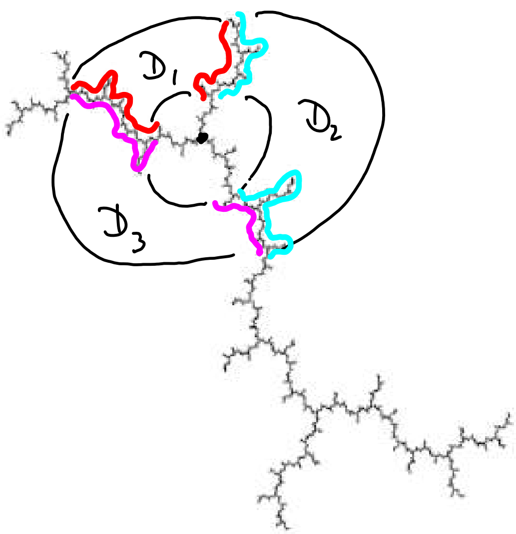
Prop.: T Gehring tree $\Rightarrow$ $\forall x \in T$, $r < \operatorname{diam} T$



Prop.: T Gehring tree $\Longrightarrow$ $\forall x \in T$, $r < \operatorname{diam} T$

$\exists\, D_1, \dots, D_n \subset A\left(\frac{r}{C}, r\right) \setminus T$ s.th.

- $n \le N(T)$, $\bigcup D_i$ separates



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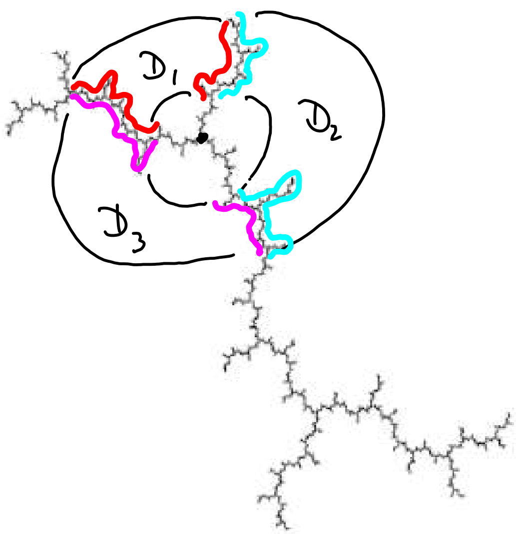

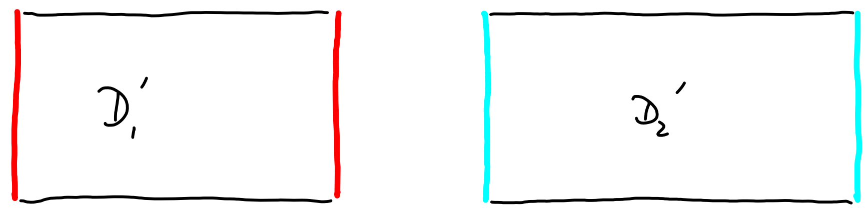

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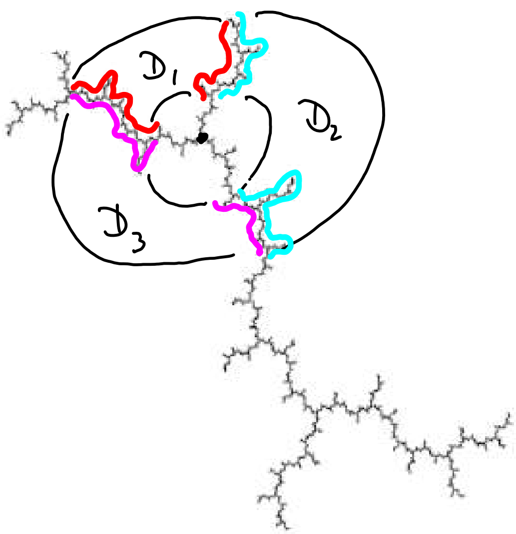
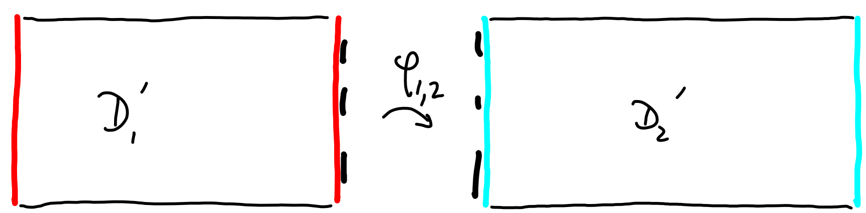
- D_i and D_{i+1} are "nicely" glued along "large" set





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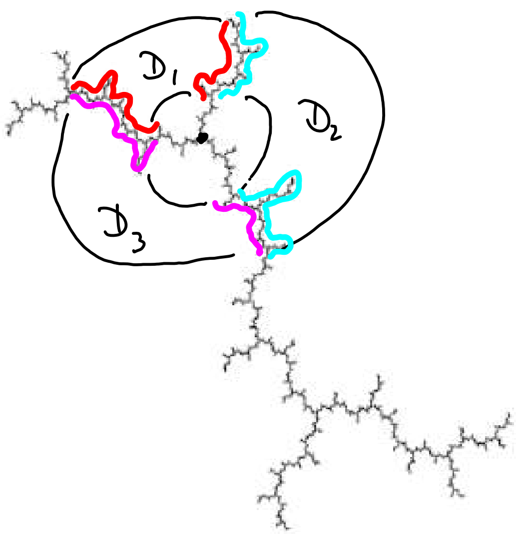

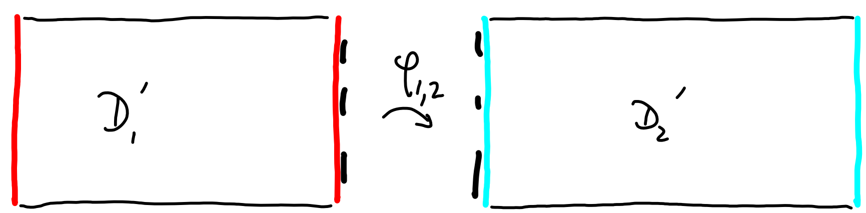

"nice" : $K(T)$ - quasisymmetric

"large" : $c(T)$ - uniformly perfect

$(\operatorname{cap} A \cap B(x,\rho) \ge c\rho \quad \forall x \in A,\ \rho < |I|)$

(Caution: the obvious choice is not even continuous)

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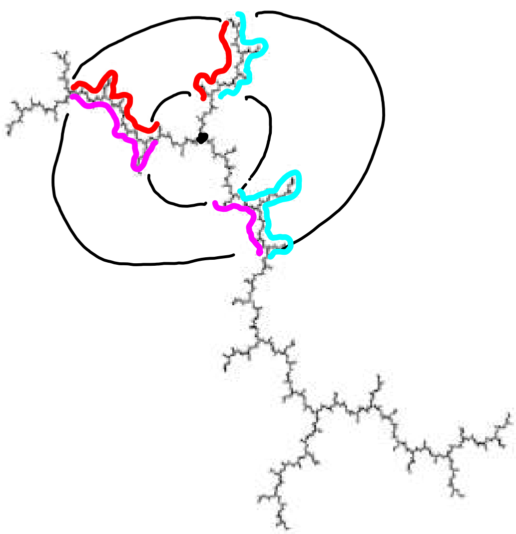

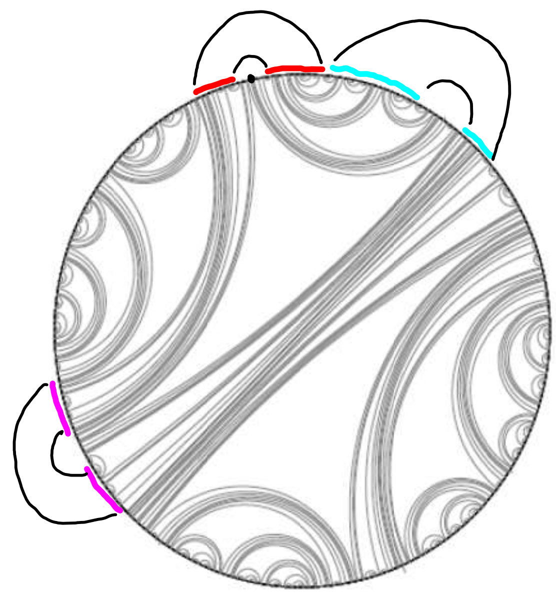

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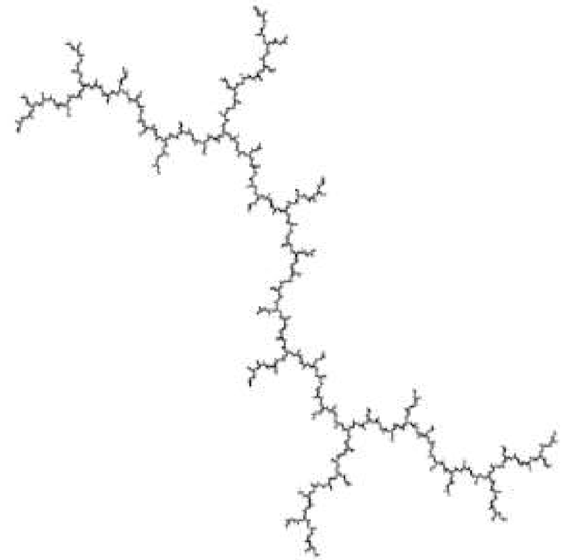

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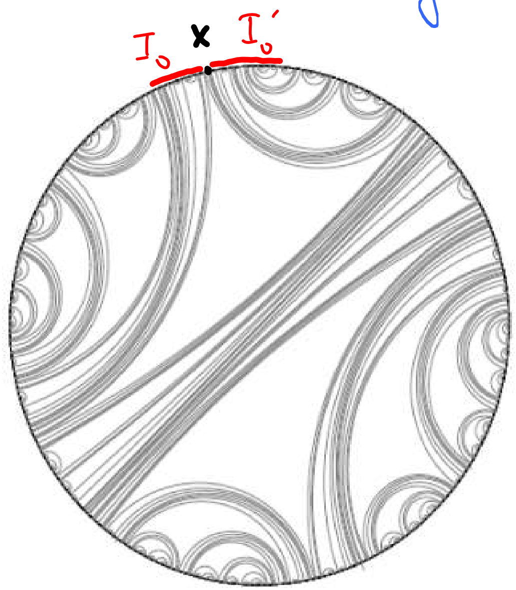

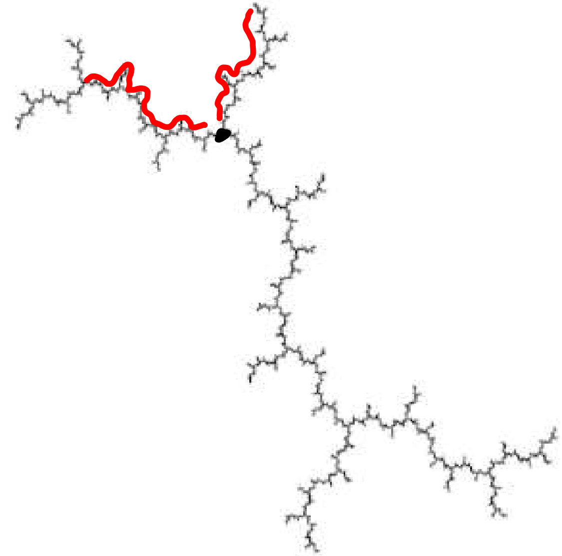

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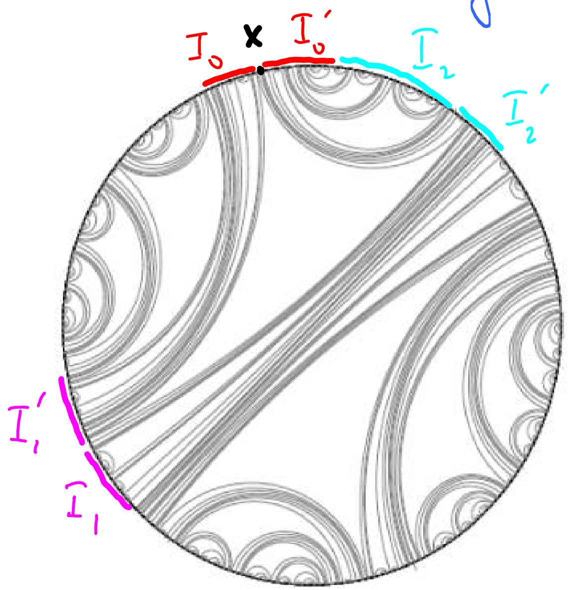

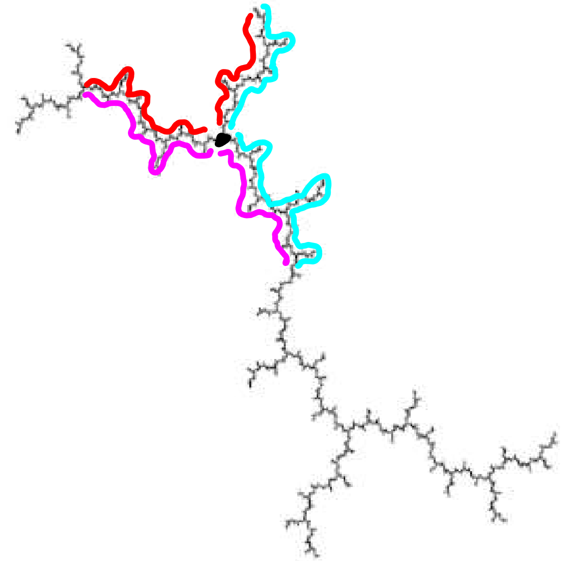

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$\exists\, I_1, I_1', \ldots, I_n, I_n'$ (I_j, I_j' adjacent, $|I_j| \asymp |I_j'|$; $n \le N(\alpha)$)

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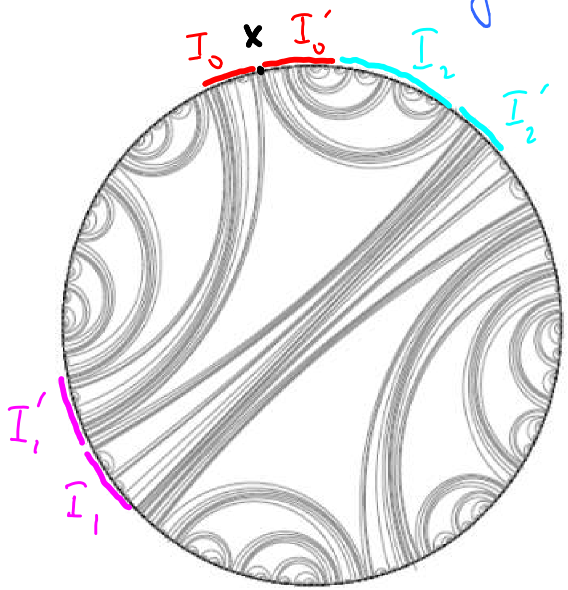

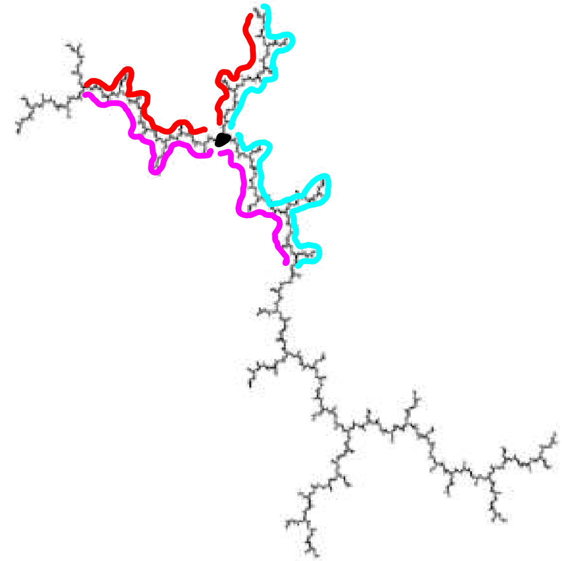

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$\exists\, A_j \subset I_j$, $A_j' \subset I_j'$ "fat" sets

"nice" homeomorphisms $\varphi_j : A_j \to A_{j+1}'$: $(x, \varphi(x)) \in \mathcal{L}$ $\forall\, x \in A_j$

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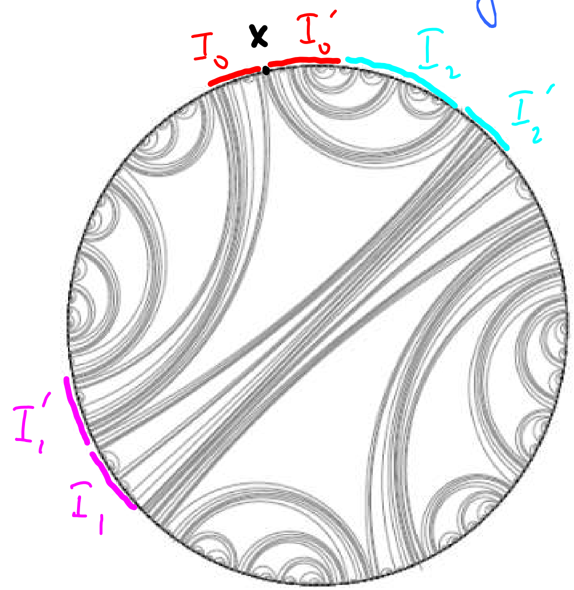

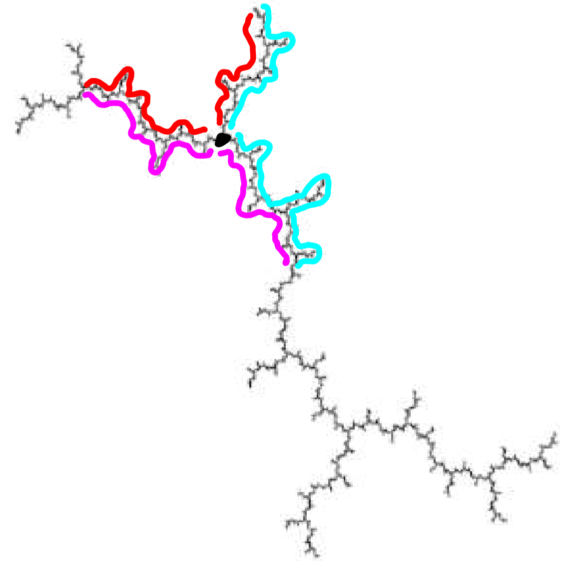

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"nice" homeomorphisms $\varphi_j : A_j \to A_{j+1}'$: $(x, \varphi(x)) \in \mathcal{L}$ $\forall x \in A_j$

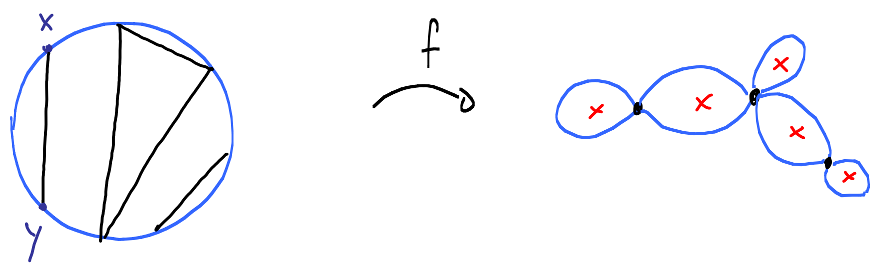
"fat": c-uniformly perfect (cap $A \cap B(x,\rho) \geq c\rho$ $\forall x \in A$, $\rho < |I|$)

"nice": φ_j c-quasi-symmetric

Ingredients of proof: Approximate $\mathcal{L}_n \uparrow \mathcal{L}$.

<u>Lemma</u>: $\forall\, \mathcal{L}$ finite $\exists!\, f$ s.th.

$$\frac{d\omega_i}{d\omega_\infty} \equiv c_i$$



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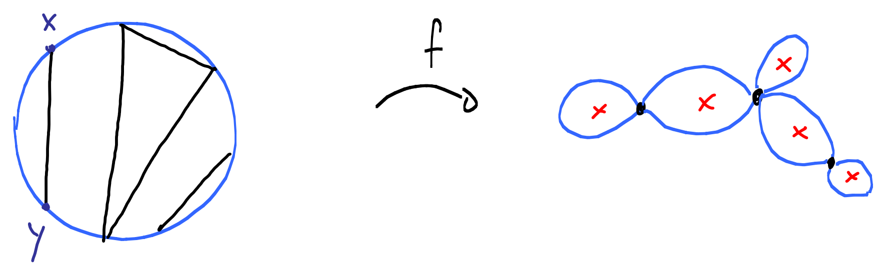
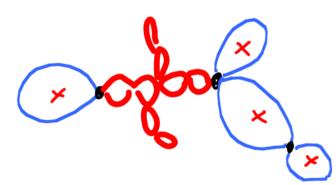


Goal: $f_n \longrightarrow f_\infty$ via equicontinuity

<u>Lemma</u>: If $\displaystyle\max_{P \in \mathcal{P}_n} \sup_{k \geq n} \operatorname{diam} f_k(\overline{P} \cap \mathbb{T}) \to 0$,

and if $f_{n_k} \to f$, then $f \in C(\overline{\mathbb{D}})$ and realizes $\mathcal{L}$.



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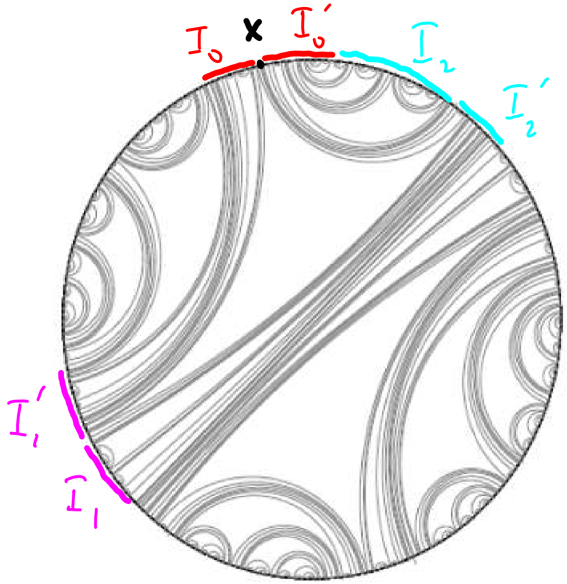

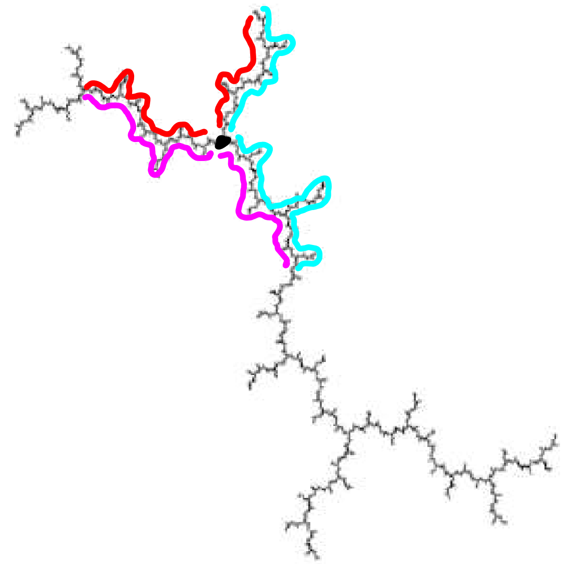

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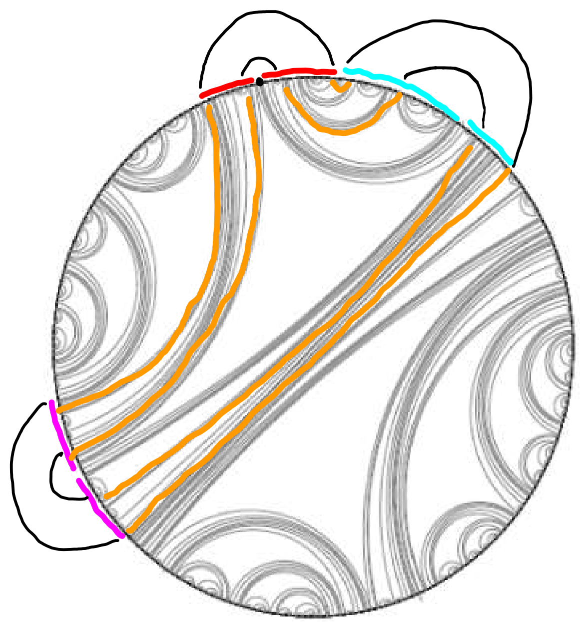

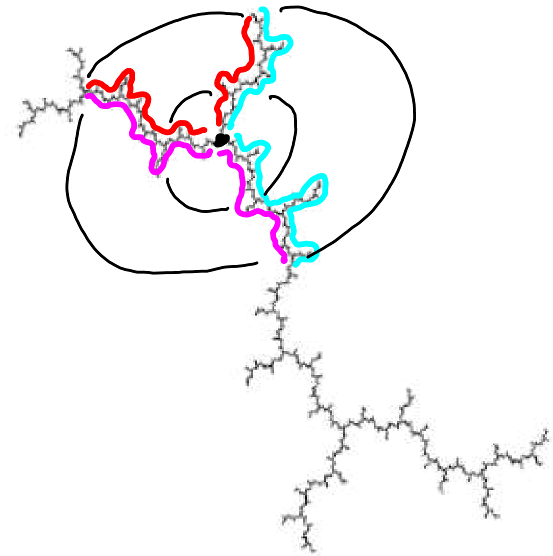

$B_r(x)$ contained in nested sequence of annuli

<u>Lemma</u>: $M(A) \geq c(\mathcal{L}) > 0$

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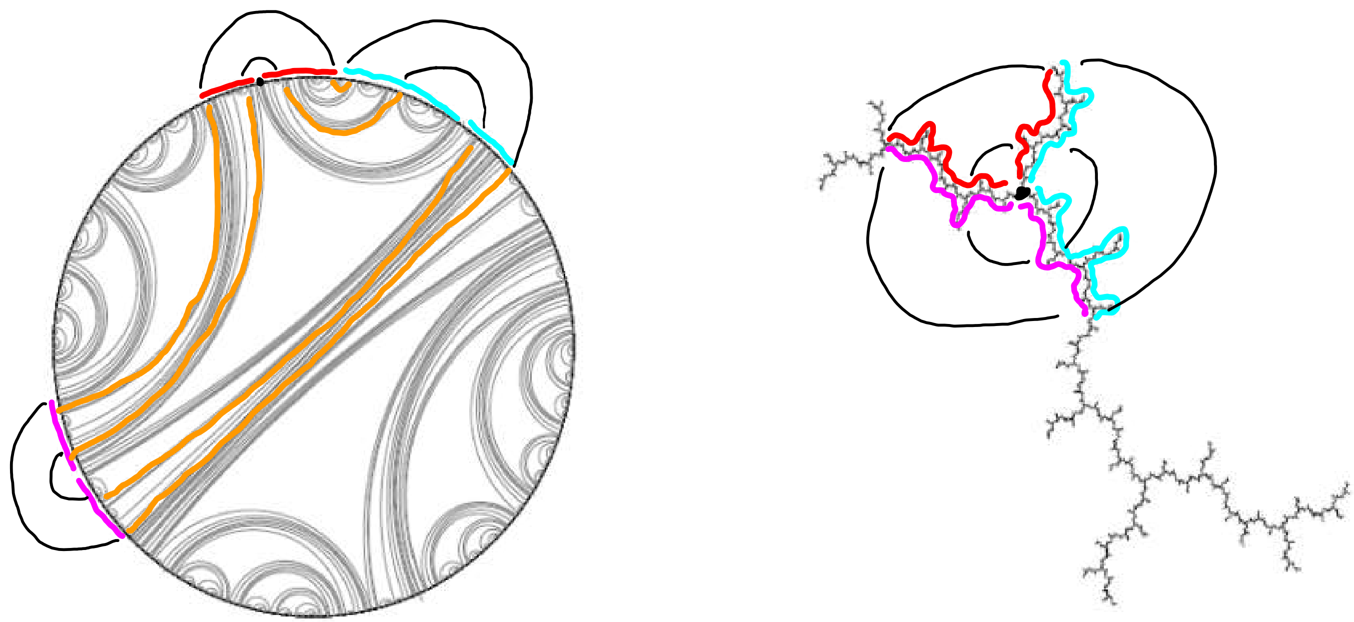

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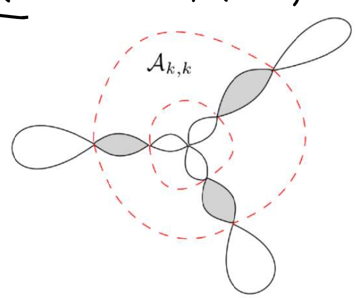

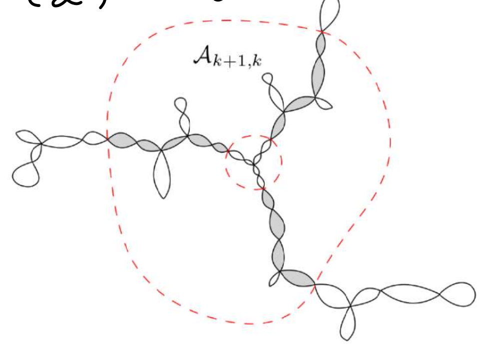

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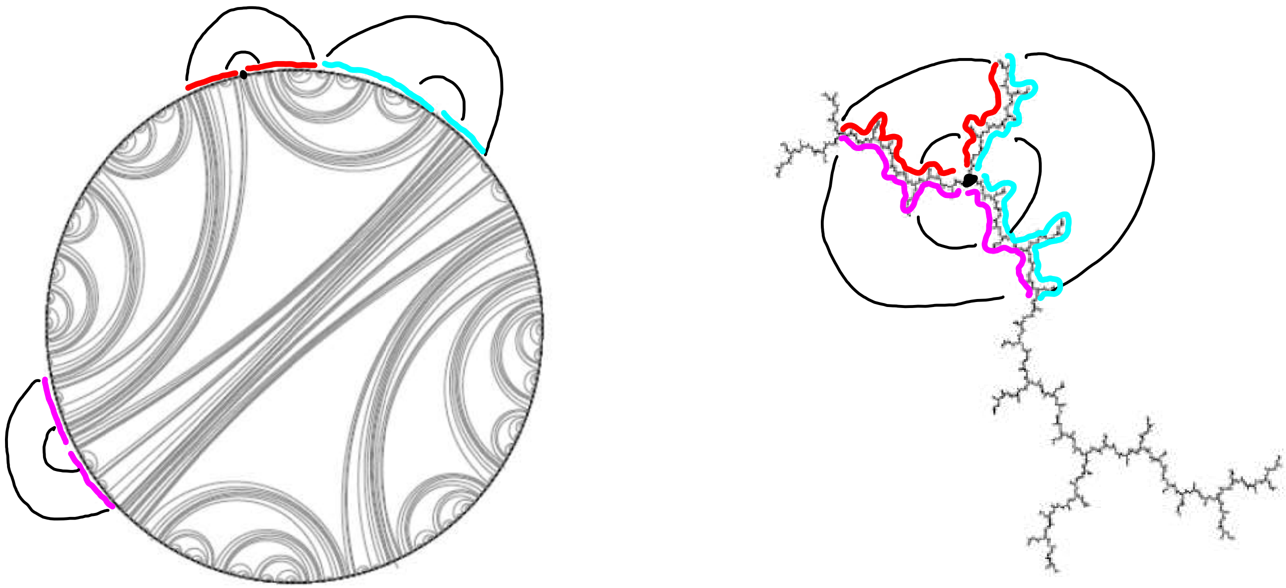
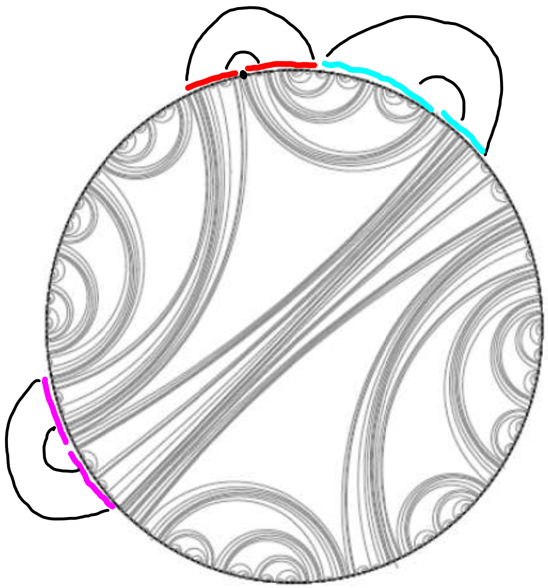
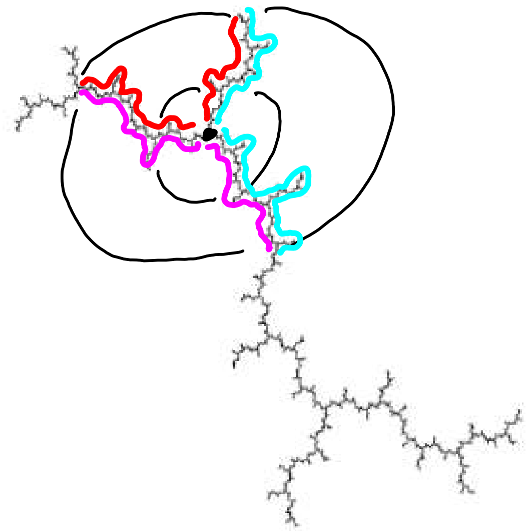
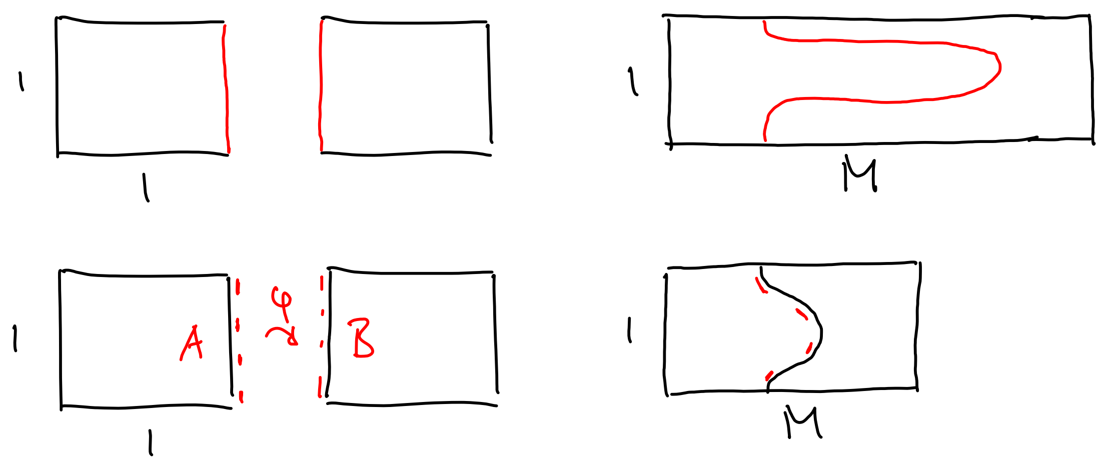


Lemma: $M(A) \geq c(\mathcal{L}) > 0$





Pf uses Lemma:



If A, B uniformly perfect, φ quasisymmetric $\Rightarrow M \leq C$.

Goal: $f_n \to f_\infty$ via equicontinuity:

- If A separated by nested A_k,

$$M(A) \geq \sum M(A_k)$$

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- $$M(A) = \log \frac{R(A)}{r(A)} + O(1)$$

$$\Rightarrow \quad \max_{\mathcal{P}_n} \cdots \leq C e^{-nc}, \quad f_\infty \text{ Hölder}$$

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- John $\Leftrightarrow$ ω doubling: $\exists \beta, c > 0$ s.th.

if $A \subset I \subset \mathbb{I}$, $|A| < \beta |I| \Rightarrow \operatorname{diam} f(A) \leq \frac{1}{2} \operatorname{diam} f(I)$

Goal: $f_n \to f_\infty$ via equicontinuity:

- If A separated by nested A_k,

$$M(A) \geq \sum M(A_k)$$

- $$M(A) = \log \frac{R(A)}{r(A)} + O(1)$$

$$\Rightarrow \max_{P_n} \dots \leq C e^{-nc}, \quad f_\infty \text{ Hölder}$$

- John $\Leftrightarrow$ ω doubling: $\exists \beta, c > 0$ s.th.

if $A \subset I \subset \mathbb{I}$, $|A| < \beta |I| \Rightarrow \operatorname{diam} f(A) \leq \frac{1}{2} \operatorname{diam} f(I)$

Cor.: If $\mathcal{L}$ has "good scales" with positive density, then $\mathcal{L}$ is conformal, D Hölder

Thank you!