Sobolev spaces that are not algebras

Luke Rogers

Joint work with Thierry Coulhon, PSL Research University, Paris

Background: The world is not smooth

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(Picture by Michael Frame)

Mathematics for physics on non-smooth spaces?

The world is often non-Euclidean, but it is still structured.

- Address many problems with statistical physics. Structure is conformal invariance. Tools include SLE, CLE, LQV, GFF.
- It is possible to do DE and PDE on spaces with quite limited structure.
  - Analysis on metric measure spaces, first order calculus structure
  - Analysis on fractals, self-similar, self-affine, random self-similar or self-affine structures.

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Diffusion on fractals

- One of the simplest PDE: the heat equation

$$\partial_t u = \Delta u$$

- Solution may be realized using Brownian motion.
- Physicists: random walks on simple fractals like Sierpinski Gasket via renormalization group methods
- Result has “anomolous diffusion”; heat spreads more slowly than in $\mathbb{R}^n$ because must get around “holes”.

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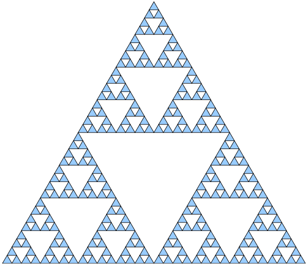
Diffusion on fractals via probability

- (Goldstein ’87, Kusuoka ’87, Barlow-Perkins ’88) Sierpinski Gasket is “nearly disconnected”.



- If there is a random walk and one stops it at scale n disconnection points result is a random walk on graph G_n.
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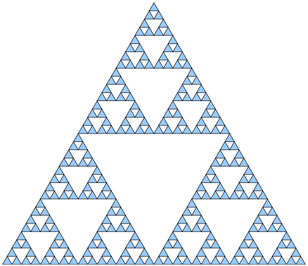

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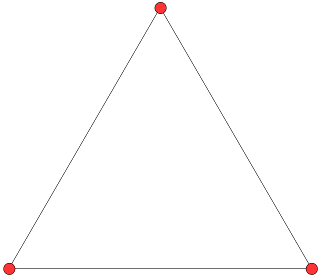

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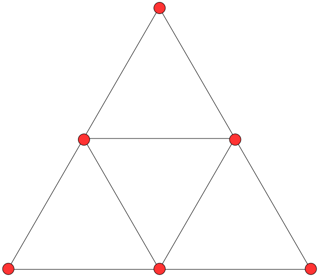

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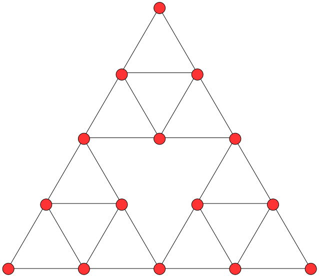

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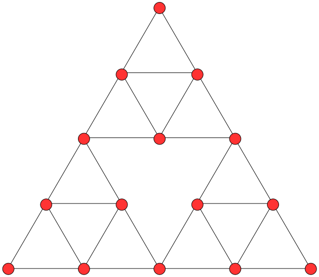

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Diffusion on fractals via probability

- Fractals that are more connected are more difficult. For example, the "pieces" of the Sierpinski carpet connect at edges rather than points.
- Barlow-Bass '89 constructed a Brownian motion on the SC
- Kusuoka-Zhou '92 showed there was a self-similar Brownian motion on SC
- Barlow-Bass-Kumagai-Teplyaev '09 showed uniqueness
- In general it is quite a lot of work to prove existence and uniqueness of Brownian motions on fractals. Most techniques require a lot of symmetry or the solution of a difficult non-linear fixed point problem. (Hambly, Kumagai, Lindstrom, Metz, Sabot, Teplyaev.)

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More PDE

- Classical functional analysis connects Brownian motion to
  - Dirichlet forms (think of $\int |\nabla f|^2$)
  - Self-adjoint Markovian operators (eg Δ)
  - Semigroups like $e^{t\Delta}$
- Can construct a self-similar Dirichlet form and Laplacian directly on certain self-similar fractals (Kigami).
- Each approach has its strengths: probabilistic constructions often come with (very useful) heat kernel estimates, Dirichlet form construction gives more information about functions.
- Having any of these gives access to a basic differential operator of physics, a type of Laplacian, so we get other natural DE and PDE.

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A setting for PDE in “rough spaces”

- Metric measure space (X, d, μ).
- No assumption of Euclidean structure
- Instead assume there is a Dirichlet form $\mathcal{E}$, meaning:
  - non-negative definite symmetric quadratic form
  - dense domain $\mathrm{dom}(\mathcal{E}) \subset L^2(\mu)$.
  - Markov property: if $u \in \mathrm{dom}(\mathcal{E})$ then so is $\tilde{u} = \min\{1, \max\{0, u\}\}$, and $\mathcal{E}(\tilde{u}) \leq \mathcal{E}(u)$.
- Associated “Laplacian” $\mathcal{L}$, with $\mathcal{E}(u, v) = \int(-\Delta u)v\, d\mu$.
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Sobolev Algebra problem

- A natural domain for PDE based on $\mathcal{L}$ is a Sobolev space defined using a Riesz potential

$$W^{\alpha,p}_{\mathcal{L}} = \left\{ f \in L^p : (-\mathcal{L})^{\alpha/2} f \in L^p \right\} \text{ with } \|f\|_{\alpha,p} = \|f\|_p + \|(-\mathcal{L})^{\alpha/2} f\|_p$$

- If we want to consider non-linear PDE then it is important to know (at least) whether products and powers of functions in $W^{\alpha,p}$ are again in $W^{\alpha,p}$.
- **Sobolev Algebra Problem:** given $\mathcal{L}$, find conditions on α and p that imply $W^{\alpha,p}$ is an algebra (or $W^{\alpha,p} \cap L^\infty$ or $\dot{W}^{\alpha,p} \cap L^\infty$ is an algebra).

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History

In the Euclidean setting, so for now $X = \mathbb{R}^d$, $\mathcal{L} = \Delta$.

- Strichartz, '67: $W^{\alpha,p}$ is an algebra if $1 < p < \infty$ and $\alpha p > d$.
- The proof uses a characterization

$$\|f\|_{\alpha,p} \equiv \|f\|_p + \|S_\alpha f\|_p \qquad \text{when } \alpha \in (0,1)$$

$$S_\alpha(f)^2 = \int_0^\infty \left(⨍_{B(x,r)} \big| f(y) - f(x) \big| dy \right)^2 \frac{dr}{r^{1+2\alpha}}$$

as well as a Leibniz calculation for $S_\alpha f$ and Sobolev embedding.

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- Kato-Ponce, '88: $W^{\alpha,p} \cap L^\infty$ is an algebra if $1 < p < \infty$ and $\alpha > 0$
- Proof by paraproducts: some algebra reduces Fourier side to Coifman-Meyer multiplier. Use David-Journe for L^p bounds.
- Gulisashvili-Kon, '96 $\dot{W}^{\alpha,p} \cap L^\infty$ is an algebra if $1 < p < \infty$ and $\alpha > 0$

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- Mazya and Shaposhnikova have an enormous book on this problem for Sobolev spaces, pairs of Sobolev spaces, Besov pairs, domains in $\mathbb{R}^d$, etc.
- Runst and Sickel also have many results.

Further Generalizations

- Coulhon-Russ-Tardivel, ’00: Lie groups with polynomial volume growth and Riemannian manifolds under certain conditions, using estimates like those of Strichartz. Method requires volume doubling and pointwise Gaussian bounds for the gradient of the heat kernel.
- Badr-Bernicot-Russ ’11: Riemannian manifolds with Poincare inequality, boundedness of Riesz transforms and some other conditions weaker than previously known, using heat operator paraproducts.
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Obstructions and counterexamples

- Ben-Bassat–Strichartz–Teplyaev '99. On Sierpinski gasket with self-similar Laplacian $\mathcal{L}$ and self-similar measure μ, if u is a nonconstant function with $\mathcal{L}u$ continous then $\mathcal{L}(u^2)$ is a measure singular to μ.
- Coulhon-R. '16. Given $\alpha \in (1,2)$, $p \in (1,\infty]$ with $\alpha p > 2$ there is a compact metric space X with Ahlfors regular μ and densely defined, non-positive definite, self-adjoint, Markovian, strongly local operator $\mathcal{L}$ such that neither $W^{\alpha,p} \cap L^\infty$ nor $\dot{W}^{\alpha,p} \cap L^\infty$ is an algebra.

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Remarks on the proof

- Estimates of local behavior of functions in $\dot{W}^{\alpha,p}$ near points of the graph G_m to give a two-term Taylor-type expansion. Roughly, if $F = (-\mathcal{L})^{-\alpha/2} f$ for $f \in L^p$ then at $x_0 \in G_m$

$$F(x) - F(x_0) \sim C_1 d(x, x_0)^{\beta_1} + C_2 d(x, x_0)^{\beta_2}$$

  (difference quotients are actually estimated in l^q of graph).
- The values β_1, β_2 depend on the measure scaling and Laplacian scaling of the fractal. We arrange that $2\beta_1 < \beta_2$.
- Then if $C_1 \neq 0$ one has $F^2 \notin \dot{W}^{\alpha,p}$ because

$$\begin{aligned} F^2(x) - F^2(x_0) &= 2F(x_0)\big(F(x) - F(x_0)\big) + \big(F(x) - F(x_0)\big)^2 \\ &\sim 2C_1 F(x_0) d(x, x_0)^{\beta_1} + C_1^2 d(x, x_0)^{2\beta_1} \end{aligned}$$

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- Estimates of local behavior of functions in $\dot{W}^{\alpha,p}$ near points of the graph G_m to give a two-term Taylor-type expansion. Roughly, if $F = (-\mathcal{L})^{-\alpha/2} f$ for $f \in L^p$ then at $x_0 \in G_m$

$$F(x) - F(x_0) \sim C_1 d(x,x_0)^{\beta_1} + C_2 d(x,x_0)^{\beta_2}$$

  (difference quotients are actually estimated in l^q of graph).
- The values β_1, β_2 depend on the measure scaling and Laplacian scaling of the fractal. We arrange that $2\beta_1 < \beta_2$.
- Then if $C_1 \neq 0$ one has $F^2 \notin \dot{W}^{\alpha,p}$ because

$$\begin{aligned} F^2(x) - F^2(x_0) &= 2F(x_0)\big(F(x) - F(x_0)\big) + \big(F(x) - F(x_0)\big)^2 \\ &\sim 2C_1 F(x_0) d(x,x_0)^{\beta_1} + C_1^2 d(x,x_0)^{2\beta_1} \end{aligned}$$

- But one can prove that a non-constant $F \in \dot{W}^{\alpha,p}$ has a non-zero C_1 at some x_0, so $\dot{W}^{\alpha,p}$ is not an algebra.

The class of fractals

- On most known fractals with Laplacian one gets some restriction (based on specific scaling properties of measure and Laplacian).
- To obtain the maximal range for which our argument precludes $W^{\alpha,p} \cap L^{\infty}$ from being an algebra we use generalized Vicsek sets for which Barlow proved one has control of the scalings by choosing dimension and number of cells.

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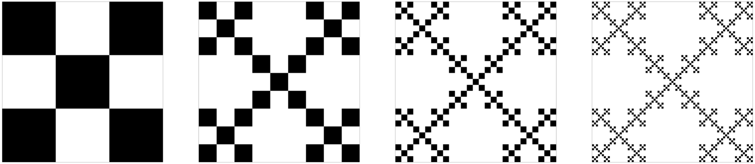

There is still work to do

- Red: Our counterexamples
- Green: Positive results under assumption of intrinsic metric, volume doubling, on-diagonal heat kernel estimates with $\sqrt{t}$, L^q estimates for the gradient of the heat kernel, etc. (Bernicot-Coulhon-Frey '15)