Etudes for the inverse spectral problem

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An example

Let A_n and B_n be the $n \times n$ Toeplitz matrices

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \dots \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \dots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} & \dots \\ \frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Then

$$[S(A_{n+1}^{-1}) - S(A_n^{-1})] = [S(B_{n+1}^{-1}) - S(B_n^{-1})]^{-1}$$

where $S(\cdot)$ denotes the sum of all matrix elements.

Canonical systems

$$\Omega \dot{M} = zHM \quad \text{on} \quad [0, \infty)$$

- $H = H(t)$ is a given real 2×2 matrix-valued function
- $H \in L^1_{\mathrm{loc}}[0, \infty)$, $H \geq 0$ a.e., $H \neq 0$ a.e.
- $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $z \in \mathbb{C}$ is "spectral" parameter
- Fundamental solution ("matrizant")

  $M = M(t, z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $M(0, z) = I$

Why canonical systems?

- Easy to solve if $z = 0$
- Every "self-adjoint" system is $\simeq$ to a canonical system
- Roughly speaking:

$$\mathrm{Dirac}(L^1_{\mathrm{loc}}) \simeq \mathrm{CS}(W^{1,1}_{\mathrm{loc}}), \det H \equiv 1,$$

and

$$\mathrm{Schroedinger}(L^1_{\mathrm{loc}}) \simeq \mathrm{CS}(W^{2,1}_{\mathrm{loc}}), \det H \equiv 1$$

De Branges spaces

Recall: $M(t,z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Fact: $\forall t,\ E = A - iC$ is a dB function i.e.

$$\Theta_E = \frac{A + iC}{A - iC} \text{ is an inner function in } \mathbb{C}_+.$$

dB space $\mathcal{B}(E) = E(H^2 \ominus \Theta_E H^2)$ has reproducing kernel

$$K(z,w) = \frac{1}{\pi} \frac{C(z)\overline{A(w)} - A(z)\overline{C(w)}}{z - \bar{w}}$$

Fact (if no "jump intervals")

$$\mathcal{B}_t \equiv \mathcal{B}(E_t) = \mathcal{W} L^2_H(0,t),$$

where

$$\mathcal{W} : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto F(z) = \frac{1}{\pi} \int_0^\infty \langle H(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} A_t(\bar{z}) \\ C_t(\bar{z}) \end{pmatrix} \rangle \, dt$$

Spectral measures

Def: μ is a spectral measure of H if

$$\forall t,\ \mathcal{B}_t \overset{\text{iso}}{\subset} L^2(\mu).$$

Facts:

- spectral measures exist (any "limit point" of $|E_t|^{-2}$)
- uniqueness of $\mu = \mu_H$ (if $\int_0^\infty \mathrm{trace}(H) = \infty$)
- μ is regular, i.e. $\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty$
- $\mathcal{W} : L^2_H \to L^2(\mu)$ is unitary.

dB Theorem: if $\mathcal{B}_1, \mathcal{B}_2 \subset L^2(\mu)$ and μ is regular, then $\mathcal{B}_1 \subset \mathcal{B}_2$ or $\mathcal{B}_2 \subset \mathcal{B}_1$

By definition, two SAS are equivalent if they have the same dB spaces, or equivalently,

same spectral measures

Paley-Wiener spaces

If $H \equiv I$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos tz & -\sin tz \\ \sin tz & \cos tz \end{pmatrix},$$

so

$$E_t = e^{-itz}, \ \mu = m, \ \mathcal{B}_t = \mathrm{PW}_t,$$

$$K(z, w) = \frac{1}{\pi} \frac{\sin t(z - \bar{w})}{z - \bar{w}}.$$

Also, $\mathrm{PW}_t = \mathcal{F} L^2[-t; t]$

CS of Paley-Wiener type

Def: $H \in$(PW) if $\forall s \exists t,\ \mathcal{B}_t \doteq \mathrm{PW}_s$

Thm: If $H \in$(PW), then $\det(H) \neq 0$ a.e. and $\int_0^\infty \sqrt{\det H} = \infty$, so the CS is equivalent to a system with $\det H = 1$ a.e.

Remark: $\det H \equiv 1 \ \not\Rightarrow\ H \in (\mathrm{PW})$ but

$$\det H \equiv 1 \ \&\ H \in W^{1,1}_{\mathrm{loc}} \ \Rightarrow\ H \in (\mathrm{PW})$$

PW-sampling measures

Def: $\mu \in$(PW) if μ is sampling for all PW-spaces:

$$\forall t\ \exists C\ \forall f \in \mathrm{PW}_t,\ \|f\|_{\mathrm{PW}} \asymp \|f\|_{L^2(\mu)}$$

Lemma. Suppose $\det(H) = 1$ a.e. and $\mu = \mu_H$. Then $H \in$(PW) $\Leftrightarrow$ $\mu \in$(PW). We have $\mathcal{B}_t = \mathrm{PW}_t(\mu)$ in this case.

Examples of PW-measures:

- $\mu \asymp m$
- most periodic measures
- if $(\mu - m)\hat{} \in L^1_{\mathrm{loc}}$ (Gelfand-Levitan condition for Dirac systems)

Characterization of PW-measures

Theorem. $\mu \in$(PW) iff
(i) $\forall x \in \mathbb{R},\ \mu(x, x+1) \leq \mathrm{const}$
(ii) $\forall t\ \exists \delta\ \exists L,$

$$|I| \geq L \ \Rightarrow\ C_\delta(I) \geq t|I|,$$

where $C_\delta(I)$ is the maximal number of disjoint intervals α intersecting I and satisfying $|\alpha| \geq \delta$ and $\mu(\alpha) \geq \delta$.

[Logvinenko-Sereda, Ortega, Seip, ...]

Inverse spectral problem

dB Theorem: every regular measure is the spectral measure of some CS

Theorem. $\forall \mu \in (\mathrm{PW})\ \exists H$ such that $\det(H) = 1$ a.e. and $\mu = \mu_H$.

H is almost unique: H and $\check{H}$ have the same spectral measure iff

$$\exists k \in \mathbb{R},\ \check{H} = T_k^* H T_k, \quad T_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Krein's strings

Corollary: 1-to-1 correspondence between PW-measures μ symmetric wrt 0 and diagonal Hamiltonians $H = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \in (\mathrm{PW})$

Pf:

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \mapsto \begin{pmatrix} h_{11} & -h_{12} \\ -h_{12} & h_{22} \end{pmatrix}$$

descends to $\mu \mapsto \mu^\star$, $\mu^\star(e) = \mu(-e)$.

Truncated Toeplitz operators

If $\mu \in$(PW), then we can define bounded, invertible operators

$$A = A_{\mu,t} : \ \mathrm{PW}_t \to \mathrm{PW}_t$$

by the formula $(Af, g) = \int f\bar{g}\, d\mu$, (for all f and g).

Lemma. Let $k_t(z)$ be the kernel of $\mathcal{B}_t$ at $w = 0$, and $\overset{\circ}{k}_t (z) = \frac{\sin tz}{\pi z}$. Then

$$k_t = i_t \left[A^{-1}_{\mu,t} \overset{\circ}{k}_t \right]$$

where $i_t : \mathrm{PW}_t \to \mathcal{B}_t$, $f \mapsto f$.

[Pf: $A_t = i_t^* i_t$]

Recovering $h_{11}(t)$ (and the string)

Theorem. If $\mu \in$(PW) is the spectral measure of H, then

$$h_{11}(t) = h^{\mu}(t) := \pi \frac{d}{dt} k_t(0).$$

Pf: $\int_0^t \langle H(s)X_z(s), X_w(s)\rangle \; ds = \pi K_t(z,w)$ where $X = \begin{pmatrix} A \\ C \end{pmatrix}$. Apply to $z = w = 0$, so $X_0 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Corollary. If $\mu \in$(PW) is symmetric, then μ is the spectral measure of

$$H = \begin{pmatrix} h^{\mu} & 0 \\ 0 & \frac{1}{h^{\mu}} \end{pmatrix}$$

Hilbert transform

$$(T^\mu f)(z) = \frac{1}{\pi} \int \left[\frac{f(\zeta) - f(z)}{\zeta - z} + \frac{\zeta f(z)}{1+\zeta^2} \right] \, d\mu(\zeta).$$

Theorem. If $\mu \in$(PW), then μ is the spectral measure of

$$H = \begin{pmatrix} h^\mu & g^\mu \\ g^\mu & * \end{pmatrix}$$

where

$$g^\mu(t) = \pi \frac{d}{dt} (T^\mu k_t)(0)$$

Different initial conditions

$$\Omega \dot{M} = zHM, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

dB function, spaces, spectral measure for IC $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$E = A - iC, \quad \mathcal{B}_t, \quad \mu = \mu_H.$$

dB function, spaces, spectral measure for IC $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\tilde{E} = B - iD, \quad \tilde{\mathcal{B}}_t, \quad \tilde{\mu} = \tilde{\mu}_H.$$

Involution $\mu \mapsto \tilde{\mu}$:

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \mapsto \begin{pmatrix} h_{22} & -h_{12} \\ -h_{12} & h_{11} \end{pmatrix}$$

Duality

Corollary. If H is diagonal, and both μ and $\tilde{\mu}$ are PW-sampling, then

$$H = \begin{pmatrix} h^{\mu} & 0 \\ 0 & \frac{1}{h^{\mu}} \end{pmatrix} = \begin{pmatrix} \frac{1}{h^{\tilde{\mu}}} & 0 \\ 0 & h^{\tilde{\mu}} \end{pmatrix}$$

General case:
two relations between h^{μ}, g^{μ}, $h^{\tilde{\mu}}$, and $g^{\tilde{\mu}}$

Warning: $\mu \in$(PW) $\not\Rightarrow \tilde{\mu} \in$(PW)

Clark's measures

Lemma. μ and $\tilde{\mu}$ are dual (as spectral measures) iff there exists $\Theta \in H^\infty(\mathbb{C}_+)$, $\|\Theta\|_\infty \leq 1$, such that

$$\mathcal{P}\mu = \Re\frac{1-\Theta}{1+\Theta}, \quad \mathcal{P}\tilde{\mu} = \Re\frac{1+\Theta}{1-\Theta}$$

Hint: $\Theta = \frac{A-iB}{A+iB}$ for the measures of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Periodic spectral measures

- $M^+(\mathbb{T})$ finite positive measures (with infinite support) on the circle $[0, 2\pi]$. We identify them with 2π-periodic measures on $\mathbb{R}$.
- Fourier series $\mu \sim \sum_{-\infty}^{\infty} \gamma_k e^{ikx}$. The "moments" are $\gamma_k = \overline{\gamma_{-k}}$; μ is symmetric iff all moments are real.
- Two Toeplitz matrices, $\Gamma(\mu)$ and $\Delta(\mu)$:

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \dots \\ \gamma_{-1} & \gamma_0 & \gamma_1 & \dots \\ \gamma_{-2} & \gamma_{-1} & \gamma_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \dots \\ -\gamma_{-1} & 0 & \gamma_1 & \dots \\ -\gamma_{-2} & -\gamma_{-1} & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Solving the inverse problem for periodic measures

Theorem. If $\mu \in M^+(\mathbb{T})$, then the functions $h^\mu(t)$ and $g^\mu(t)$ are locally constant on $\mathbb{R}_+ \setminus \frac{1}{2}\mathbb{N}$, so

$$h^\mu(t) = h_0, h_1, h_2, \ldots, \quad g^\mu(t) = g_0, g_1, g_2, \ldots$$

on

$$(0.0, 0.5), (0.5, 1.0), (1.0, 1.5), \ldots$$

and we have

$$h_n = S\left[\Gamma_{n+1}^{-1}\right] - S\left[\Gamma_n^{-1}\right],$$

$$g_n = S\left[\Delta_{n+1}\Gamma_{n+1}^{-1}\right] - S\left[\Delta_n\Gamma_n^{-1}\right].$$

Back to our example

- $\mu = 1 + \cos$, so $\Gamma(\mu) = \begin{pmatrix} 1 & \frac{1}{2} & 0 & \dots \\ \frac{1}{2} & 1 & \frac{1}{2} & \dots \end{pmatrix}$
- We have $\Gamma_1^{-1} = (1)$, $\Gamma_2^{-1} = \frac{4}{3}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \dots$ and $S_1 = 1$, $S_2 = \frac{4}{3}, \dots$
- We conclude: $h_0 = 1$, $h_1 = \frac{1}{3}$, $h_2 = \frac{2}{3}, h_3 = \frac{2}{5}$ etc.
- Find symmetric dual measure $\tilde{\mu}$:

$$\mathcal{P}\mu = \Re(1 + e^{iz}), \quad \mathcal{P}\tilde{\mu} = \Re\frac{1}{1 + e^{iz}} = \frac{1}{2} + \frac{1}{2}\Re\frac{1 - S}{1 + S},$$

so

$$\tilde{\mu} = \frac{1}{2} + \pi\sum_{-\infty}^{\infty}\delta_{\pi + 2\pi k}, \quad \Gamma(\tilde{\mu}) = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \dots & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} & \dots \end{pmatrix}$$

- Computation gives $\tilde{h}_0 = 1$, $\tilde{h}_1 = 3$, $\tilde{h}_2 = \frac{3}{2}, \tilde{h}_3 = \frac{5}{2}$ etc.

Happy birthday, Peter!