Kakeya sets in three dimensions

They have Hausdorff dimension strictly more than 5/2

Nets Katz (joint with J. Zahl)

May 10, 2017

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Our takeaway from Wolff's argument is that for essentially each T in a near $\frac{5}{2}$ dimensional Kakeya set, we have $H(T)$ consisting of about $\delta^{-\frac{3}{2}}$ tubes whose union essentially covers the Kakeya set.

K-Laba-Tao argument I: structure

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Stickiness There are $\frac{1}{\delta}$ Fat tubes (of dimension $\delta^{\frac{1}{2}} \times 1$) each containing $\frac{1}{\delta}$ many δ tubes. (Otherwise could use Wolff X-Ray result at scale $\delta^{\frac{1}{2}}$. But today, we don't get to change scale like that.)

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Graininess At each δ-cube of the Kakeya set, there is a $\delta^{\frac{1}{2}} \times \delta^{\frac{1}{2}} \times \delta$ flat containing all nearby points of the Kakeya set.

K-Laba-Tao argument II: The Heisenberg Group

The main idea of K-Laba-Tao: Even with stickiness/planyness/graininess, there is still an enemy. It is a $\frac{5}{2}$ dimensional Kakeya set which almost exists. "The Heisenberg group".

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With Bourgain's discretized sum-product theorem, it became possible also to use that the reals don't have (even approximately) positive dimensional subrings.

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BKT idea: Take $H(L_1, L_2)$ for L_1, L_2 fixed lines and call its elements points. Now take $H(L_3, L_4)$ for L_3, L_4 lines, and call its elements lines. A typical element of $H(L_3, L_4)$ intersects about $p^{\frac{1}{2}}$ lines of $H(L_1, L_2)$. Call such an intersection an incidence. Now we have an impossible point-line incidence configuration.

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Why is it a point-line configuration? OK, it's points and quadratic curves. Lines of $H(L_3, L_4, L)$ lie in one ruling of a doubly-ruled quadratic surface, the regulus.

Dvir's argument

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Suppose K is that small. We can find a nontrivial polynomial which vanishes on K of degree at most $1 - \frac{\epsilon}{n}$. The polynomial restricted to each line of the Kakeya set $q + tv$ is identically zero. Thus the highest order part vanishes on the direction v. This is true for every v so we reach a contradiction.

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Conclusion: The Kakeya problem is an algebra problem not an analysis problem.

Can we use reguli?

There are serious issues with using reguli in such a way in the real Kakeya problem. For three tubes to generate the δ neighborhood of a regulus, they have to be pairwise skew? Can we find pairwise skew triples of tubes which intersect many tubes in a near Wolff Kakeya set?

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Begin by looking in a particular hairbrush $H(T)$. If there are no pairs T_1, T_2 which are skew, then all tubes of $H(T)$ are close to the plane spanned by T and T_1 for some $\mu > 0$. The tubes of $H(T)$ cover the whole Kakeya set, so this is true of every tube in the Kakeya set.

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Run the same argument again with triples. If there are no T_1, T_2, T_3 in $H(T)$ which are pairwise skew then T_3 is either near the plane spanned by T and T_1 or near the plane spanned by T and T_2. Conclusion: there are $\delta^{-6.5}$ quadruples (T, T_1, T_2, T_3) with (T_1, T_2, T_3) pairwise skew and each of T_1, T_2, T_3 intersecting T. For each such quadruple T is in the δ neighborhood of $R(T_1, T_2, T_3)$.

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There are certainly δ^{-8} quintuples (T, T_1, T_2, T_3, T_4) with the T_j's in $H(T)$ and pairwise skew. But there is no guarantee that T_4 intersects $R(T_1, T_2, T_3)$ at large angle. If not, the set of T's for a fixed (T_1, T_2, T_3) may be in a tight range of the ruling so that T_4 may intersect many of these T's at large angle.

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By dyadic pigeonholing, we can restrict attention to one angle of intersection δ^{α} and it turns out $0 \leq \alpha \leq \frac{1}{2}$. As long as $\alpha < \frac{1}{2}$, a small part of the obsolete argument survives. But $\alpha = \frac{1}{2}$ is a genuinely new case.

A new enemy

For the remainder of the talk, we restrict our attention to the case $\alpha = \frac{1}{2}$. We define a regulus strip to be the intersection of the δ neighborhood of a regulus with a $\delta^{\frac{1}{2}}$ cylinder. Our Kakeya set contains a lot of regulus strips.

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Each regulus strip is contained in a $\delta^{\frac{1}{2}}$ fat tube. But at the same time, each $\delta^{\frac{1}{2}}$ tube contains just one regulus strip. Our example is far from sticky. There are $\delta^{-\frac{3}{2}}$ fat tubes.

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Our example is however plany at the scale $\delta^{\frac{1}{2}}$. Each regulus strip is intersected by each tube in its hairbrush at angle $\delta^{\frac{1}{2}}$. Thus the plane map is determined by the tangent plane to the regulus strip. It has degree 1 when restricted to a fat tube.

Can the plane maps exist?

The hairbrush of each fat tube T_1 covers each other fat tube T_2. But each plane containing T_1 intersects T_2 in only one place and is the plane of that place. In that way, the plane map of T_1 determines the plane map of T_2.

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Are there $\delta^{-\frac{3}{2}}$ fat tubes, each equipped with a linear plane map so that each fat tube infects each other fat tube with its own plane map? Can the plane maps be consistent.

Yes. Denote a line in $\mathbf{R}^3$ as $(a, b, 0) + t(c, d, 1)$. Now take the fat tubes of the form

$$ad - bc = 1 + O(\delta^{\frac{1}{2}}).$$

We call this the $SL(2)$ example.

Must the enemy be $SL(2)$?

Amazingly yes. A hint of this comes from the fact that the set of lines lying in a plane and going through a point are a line in line space. Our example in line space must be a hypersurface containing a three dimensional set of lines. Among algebraic surfaces only quadratics do this.

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To make a rigorous argument, cleverly select 14 points in the set of lines. (I.e cleverly select 14 fat tubes.) Now find a quadratic vanishing on these. It must vanish on every fat tube in our example.

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Is there really an almost-example corresponding to $SL(2)$? Sort of, yes. You can build one working over Z mod p^2. Yes I know it isn't a field.

Can $SL(2)$ exist in $\mathbf{R}$?

We can now try to contradict the existence of an example through incidence theory. Recall any tube that intersects a regulus strip touches all tubes in a strip. Pick one thin tube in each fat tube. We find $\delta^{-\frac{3}{2}}$ many δ tubes which are $\delta^{\frac{1}{2}}$ separated so that each one intersects δ^{-1} of the others.

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Obvious idea is to cut down one parameter. Essentially it is what we did before. Problem: lines in $\mathbf{R}^3$ are a four parameter family, not three.

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Key point: all our lines automatically have

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Moreover, we have a choice of one tube per regulus strip. A calculation shows that travelling along the strip is transverse to $SL(2)$. Just pick a line so that

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Now we have a genuine 3-parameter family of lines. Reduce by 1 parameter and we get an easy point-line incidence problem. These were the main ideas of the argument.

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It is quite a coincidence that both Heisenberg and $SL(2)$ are quadratic. Is there a unification of the two theories which is blind to the underlying field.

Alternatively, Tao has a heuristic argument for sticky Kakeya. Can that argument be extended to work slightly for anything even slightly sticky, as here? Then can one find useful algebraic structure in the plane maps of anything maximally anti-sticky?

Happy Birthday, Peter!

May the candles on your cake,
Burn like cities in your wake.