A limit theorem for random games

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KIAS, Seoul 2017

Game

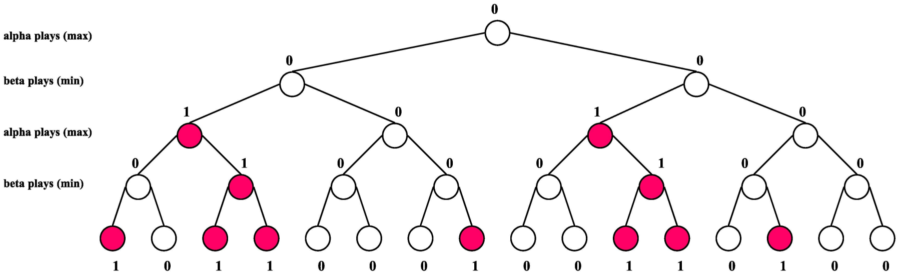
- Game with 2 players: α and β.
- Each of them alternately move R (right) or L (left) to form a string
- The game ends when each player placed N moves, for some predetermined $N \geq 1$.
- The collection of strings of length $2N$ is partitioned into two subsets A and B, known before the game starts.
- **WINNING RULE:** Player α wins if the final string ends in A and player β wins if the final string ends in B.

Zermelo's algorithm

Zermelo's Theorem dictates that either player α has a winning strategy or player β has a winning strategy.

Zermelo's algoritm

- Label the string that ends in A with 1, and the ones that end in B with 0.
- Proceed backwards filling all the nodes with 1's or 0's.
- If the value at the root of the game V_N is 1, then α has a winning strategy. If $V_N = 0, \beta$ has a winning strategy.


alpha plays (max)
beta plays (min)
alpha plays (max)
beta plays (min)
0
0 0
1 0 1 0
0 1 0 0 0 1 0 0
1 0 1 1 0 0 0 1 0 0 1 1 0 1 0 0

Randomizing the game

- Fix a probability $p \in (0, 1)$, and consider a coin:

  $X = 0$ with prob $\mathbf{p}$
  $X = 1$ with prob $1 - p$.

- For each of the strings, toss the coin to decide if the string ends in A or in B.

The value at the root of the tree V_N becomes a Bernoulli variable with

$$\mathbf{P}(V_N = 0) = h^{(N)}(p)$$

where

$$h(p) = \mathbf{P}\left(\mathrm{Max}(\min(x_1, x_2), \min(x_3, x_4)) = 0\right) = (1 - (1 - p)^2)^2.$$

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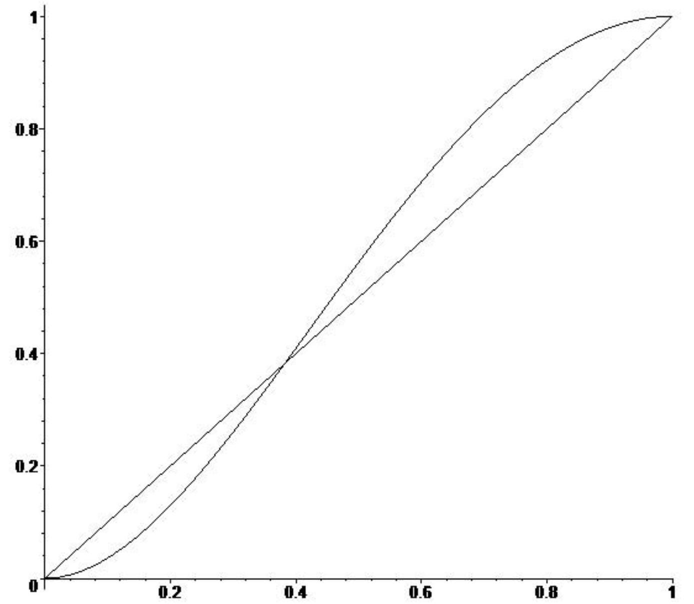
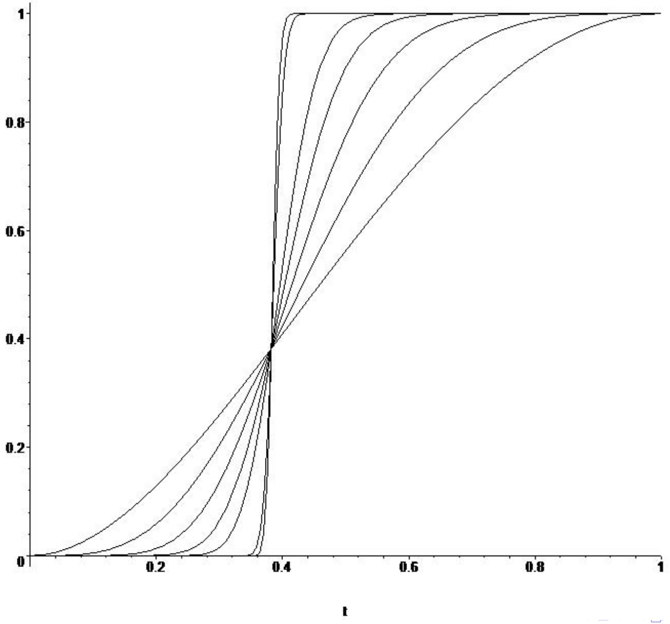

1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1

$$h(p) = \mathbf{P}\left(\mathrm{Max}(\min(x_1, x_2), \min(x_3, x_4)) = 0\right) = (1 - (1-p)^2)^2.$$

$h(p) \nearrow$ in$(0,1)$
$h(0) = 0,\ \ h(1) = 1$
$h'(0) = h'(1) = 0$

$$h(p) = p \Leftrightarrow p = p^* = \frac{3-\sqrt{5}}{2} \approx 0,382$$


1
0.8
0.6
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0.2
0
0.2
0.4
0.6
0.8
1
t

$$\mathbf{P}(V_N = 0) = h^{(N)}(p)$$

Therefore, as $N \to \infty$:

- If $p < p^*$, then $\mathbf{P}(V_N = 0) = h^{(N)}(p) \to 0$, and α is almost certain to win. $V_N \to 1$
- If $p = p^*$, then $\mathbf{P}(V_N = 0) = h^{(N)}(p) = p^*$. $V_N \to X$, where X is a Bernoulli variable with prob. of success $(1 - p^*)$.
- If $p > p^*$, then $\mathbf{P}(V_N = 0) = h^{(N)}(p) \to 1$, and β is almost certain to win. $V_N \to 0$

In terms of quantiles: $V_N \to Q_X(p^*)$

Generalization

Consider a *monotone* Boolean function $H : \{0,1\}^n \to \{0,1\}$. ***(Voting rule)***

Ex: n=3

H(1,1,1)=1	H(0,1,1)=1	H(1,0,1)=1	**H(0,0,1)=1**
H(1,1,0)=1	H(0,1,0)=0	H(1,0,0)=0	H(0,0,0)=0

- Identify the subsets of $\{1,2,3\}$ with elemets of $\{0,1\}^3$.
- Look for the *minimal* subsets A under the action of H, i.e. $H(A) = 1$, and for any B with $B \subset A$, $H(B) = 0$.

Note that the minimal subsets are $\{1,2\}, \{3\}$ and $H = \max\,(\min(x_1, x_2), x_3)$

Sperner family

We can associate to each **monotone Boolean function** H a family of subsets $S = \{A_1, A_2, ..., A_k\}$ of $\{1, 2, ..., n\}$, such that no A_i is contained in any other A_j. We will call it a **Sperner family.**

In fact H can be represented as the **Sperner statistic** associated to S:

$$H = \max\left(\min_{A_1}, \min_{A_2}, ..., \min_{A_k}\right).$$

where

$$\min_A(x_1, x_2, ..., x_n) = \min(x_i; x_i \in A).$$

Examples

- *Projection (or dictatorship)*: $H(x_1, x_2, ..., x_n) = x_i$
- *Majority rule:* $H(x_1, x_2, ..., x_n) = \mathcal{X}_{(\frac{x_1+x_2+...+x_n}{n} > 1/2)}$
- *Order statistics:* $H_{(n;r)}(x_1, x_2, ..., x_n) = x_{i_r}$ where $x_{i_1} \leq x_{i_2} \leq ...x_{i_r} \leq ...x_{i_n}$. The Sperner family are all the subsets of size $n - r + 1$.
- *Zermelo statistics:* The Sperner family associated is such that all the subsets A_i are pairwise **disjoint**.
  The one associated to the game is a Zermelo statistic.

Problem

Consider Bernoulli variables X that take values 0 and 1 with probabilities p and $1 - p$ respectively.

Define the operator **H(X)** acting on Bernoulli variables X by **H(X)** $= \mathbf{H}(X_1, ..., X_n)$ where X_i are **independent** copies of X.

Problem: Understand the convergence (in distribution) of the iterates $\mathbf{H}^{(N)}$.

NOTE: $\mathbf{H}^{(N)}$ is a Bernoulli variable that take values 0 and 1

$$\{\mathbf{H}(X_1, ..., X_n) = 0\} = \bigcap_{i=1}^{k} \{\min_{A_i}(X_1, ..., X_n = 0\}$$

Therefore, $\mathbf{P}(\mathbf{H(X)} = 0)$ is the polynomial

$$h(p) = 1 - \sum_{i} (1-p)^{|A_i|} + \sum_{i<j} (1-p)^{|A_i \cup A_j|} - ...$$

and

$$\mathbf{P}(\mathbf{H}^{(N)}(\mathbf{X}) = 0) = h^{(N)}(p)$$

Easy cases

h is $\nearrow$ in $[0,1]$ with $h(0)=0, h(1)=1$, $h'(0)= |\bigcap_1^k A_i|$ and $h'(1)=$ number of singletons.

- *Projection:* $h(p)=\mathbf{P}(\mathbf{H(X)}=0)=\mathbf{P}(X_i=0)=p$. Therefore $\mathbf{H}^{(N)}(\mathbf{X})=X$.

- *Upper case:* The family S contains no singleton and $\bigcap_1^k A_i \neq \emptyset$, then $h(p)>p$. Therefore $h^{(N)}(p)\to 1$ if $p\neq 0$.

$$\mathbf{H}^{(N)}(\mathbf{X}) \to Q_X(0)$$

- *Lower case:* The family S contains a singleton and $\bigcap_1^k A_i = \emptyset$, then $h(p)<p$.. Therefore $h^{(N)}(p)\to 0$ if $p\neq 1$.

$$\mathbf{H}^{(N)}(\mathbf{X}) \to Q_X(1)$$

Sperner polynomial

To study the remaining case ($h'(0) = h'(1) = 0$) we consider the **Sperner polynomial**

$$g(p) = 1 - h(1-p)$$

Then $g(p) = \mathbf{P}(\mathbf{H(X)} = 1)$ where the Bernoulli variable X has probability of **success p.**

Note also that

$$\mathbf{E}_p(\mathbf{H}) = g(p)$$

$$\mathbf{Var}_p(\mathbf{H}) = g(p)(1 - g(p))$$

Fourier Analysis: Influences

We define the *influence* of the variable i; $1 \leq i \leq n$ as

$$\mathbf{I}_i(\mathbf{H}) = \mathbf{P}_p(H(X) \neq H(X \otimes e_i))$$

where $X \otimes e_i$ means X with the i-th bit flipped.

The *total influence* of H, $\mathbf{I}_p(H)$, is the sum of all the influences.

Russo's Lemma: $g'(p) = \mathbf{I}_p(H)$

Efron-Stein inequality *(Isoperimetric inequality):*

$$\mathbf{I}_p(H) \geq \frac{1}{p(1-p)} \mathbf{Var}_p(H)$$

with equality if and only if H is a projection.

Sperner point

As a consequence we obtain

$$g'(p) = \mathbf{I}_p(H) \geq \frac{g(p)(1 - g(p))}{p(1 - p)}$$

So, if p is a fixed point of the Sperner polynomial $g(p)$, then $g'(p) \geq 1$. In fact. $g'(p) > 1$ unless H is a projection.

Theorem

Let $S = \{A_1, ..., A_k\}$ be a Sperner family with $k \geq 2$ each $A_j \geq 2$ and $\bigcap A_j = \emptyset$, then the polynomial h_S has a unique fixed point (***Sperner point***) $\omega_H \in (0, 1)$ that happens to be repellent.

Continuous selectors

A **continuous selector** H is a continuous function defined in $\mathbf{R}^n$ such that

$$H(x_1, x_2, \dots, x_n) \in \{x_1, x_2, \dots, x_n\}$$

Theorem: *Any continuous selector is a Sperner statistic, and conversely.*

Key point: Continuous selectors are monotone and they are determined by their restriction to the Boolean cube.

Note: If H is a continuous selector, then for all t,

$$\mathbf{1}_{\{H(x_1,\ldots,x_n)>t\}} = H(\mathbf{1}_{\{x_1>t\}},\ldots,\mathbf{1}_{\{x_n>t\}})$$

Consequently:

Let X be a random variable with *distribution function* F_X, and $X_1,\ldots,X_n$ *independent copies* of X. Then

$$\mathbf{P}(H(X_1,\ldots,X_n)\leq t) = h\left(F_X(t)\right)$$

Write $\mathbf{H(X)} = H(X_1,\ldots,X_N)$, iterating the expression above, we get

$$\mathbf{P}(\mathbf{H}^{(N)}(\mathbf{X})\leq t) = h^{(N)}(F_X(t))$$

Theorem

Let H be a ***continuous selector***. Then, except for projections, there is a unique point $\omega_H \in [0,1]$ so that for any random variable X

$$\mathbf{H}^{(N)}(\mathbf{X}) \to Q_X(\omega_H)$$