Uniqueness and almost periodicity in time of solutions of the KdV equation with certain almost periodic initial conditions.

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joint work with D. Damanik (Rice), M. Goldstein (Toronto) and M. Lukic (Rice).

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  $$\partial_t H(t) = P(t)H(t) - H(t)P(t).$$

  This means that the family of unitary operators $U(t)$ which solves $\frac{d}{dt}U = PU$, $U(0) = I$ obeys $U(t)^* H(t)U(t) = H(0)$, so the operators $H(t)$ are mutually unitarily equivalent for all values of t.

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- Just having a Lax pair is not enough to deduce stronger statements of integrability, such as almost periodicity in t!

The KdV equation: integrability.

Integrability of the KdV equation was first established by Gardner, Greene, Kruskal and Miura in the setting of rapidly decaying initial data $u(x, 0) = V(x)$ using the inverse scattering transform linearization of the KdV evolution.

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In the 1970s, it was proved that for periodic initial data, the KdV equation is a completely integrable Hamiltonian system, with action-angle variables:

Theorem (McKean-Trubowitz, 1976).

If $V \in H^n(\mathbb{T})$, then there is a global solution $u(x,t)$ on $\mathbb{T} \times \mathbb{R}$. This solution is $H^n(\mathbb{T})$-almost periodic in $\mathbb{T}$.

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This means that $u(\cdot, t) = F(\zeta t)$ for some continuous $F : \mathbb{T}^\infty \mapsto H^n(\mathbb{T})$ and $\zeta \in \mathbb{R}^\infty$.

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Conjecture (Deift, 2008).

If $V : \mathbb{R} \mapsto \mathbb{R}$ is almost periodic, then there is a global solution $u(x,t)$ that is almost periodic in t.

Even short time existence of solutions is not known in this generality.

Global existence, uniqueness, and almost periodicity

We say that an almost periodic $V : \mathbb{R} \mapsto \mathbb{R}$ is a *Sodin-Yuditskii* function if the Schrödinger operators $H_V := -\partial_x^2 + V$ has purely absolutely continuous spectrum S which satisfies

1. S is a Carleson homogeneous subset of $\mathbb{R}$:
   $\exists \epsilon > 0 \ : \ |(x-\delta, x+\delta) \cap S| \geq \epsilon\delta, \ x \in S.$
2. $|\mathbb{R}_+ \setminus S| < \infty$

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Theorem (B.-Damanik-Goldstein-Lukic).

If V is a Sodin-Yuditskii function (plus some additional restrictions on the thickness of the spectrum, unfortunately), then

1. *(existence) there is a global solution $u(x,t)$ of KdV with $u(x,0) = V(x)$;*
2. *(uniqueness) if $\tilde{u}$ is another solution on $\mathbb{R} \times [-T, T]$, and $\tilde{u}, \partial_{xxx}\tilde{u} \in L^\infty(\mathbb{R} \times [-T, T])$, then $\tilde{u} = u$;*
3. *(x-dependence) for each t, $x \mapsto u(x,t)$ is almost periodic in x (with the same frequency vector);*
4. *(t-dependence) $t \mapsto u(\cdot, t)$ is $W^{4,\infty}(\mathbb{R})$-almost periodic in t.*

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- Let $\mathcal{P}(\omega, \varepsilon, \kappa)$ denote the space of functions of the form
  $V(x) = U(\omega x)$
  for a sampling function $U : \mathbb{T}^d \mapsto \mathbb{R}$ which can be written as
  $U(\theta) = \sum_{n \in \mathbb{Z}^d} c(n) e^{2\pi i n\theta}, \quad |c(n)| \leq \varepsilon \exp(-\kappa|n|)$.

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- We also assume that ω satisfies the following Diophantine condition

$$|n\omega| \geq a_0 |n|^{-b_0}, \quad n \in \mathbb{Z}^d \setminus \{0\}$$

for some $0 < a_0 < 1$, $d < b_0 < \infty$.

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Theorem (B.-Damanik-Goldstein-Lukic).

There exists $\varepsilon_0(a_0, b_0, \kappa) > 0$ *such that if* $V \in \mathcal{P}(\omega, \varepsilon, \kappa)$, $\varepsilon < \varepsilon_0$, *then*

1. *(existence) there is a global solution* $u(x, t)$;
2. *(uniqueness) if* $\tilde{u}$ *is another solution on* $\mathbb{R} \times [-T, T]$, *and* $\tilde{u}, \partial_{xxx}\tilde{u} \in L^\infty(\mathbb{R} \times [-T, T])$, *then* $\tilde{u} = u$;
3. *(x-dependence) for each* t, $x \mapsto u(x, t)$ *is quasiperiodic in* x *and* $u(\cdot, t) \in \mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4)$;
4. *(t-dependence)* $t \mapsto u(\cdot, t)$ *is* $\mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4)$*-almost periodic in* t.

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On $\mathcal{P}(\omega, \varepsilon, \kappa)$, the L^∞-norm is equivalent with the norm

$$\|V - \tilde{V}\|_r = \left(\sum_{n \in \mathbb{Z}^d} |c(n) - \tilde{c}(n)|^2 e^{2|n|r} \right)^{1/2}$$

for any $r < \kappa$, and with the Sobolev norm inherited from $W^{k,\infty}(\mathbb{R})$ for any $k \in \mathbb{N}$. So the derivatives of u are also almost periodic in t, and so is each Fourier coefficient $c(n,t)$ of $u(x,t)$.

Weyl solutions and Green function

- For $z \in \mathbb{C} \setminus \sigma(H_V)$, the second order differential equation
$$-y'' + Vy = zy$$
has nontrivial solutions $\psi_\pm(x; z)$, called *Weyl solutions*, such that $\psi_\pm(x; z) \in L^2([0, \pm\infty), dx)$.

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- *The half-line m-functions* associated with the half-line restrictions of $H_V = -\partial_x^2 + V$ to $[x, \pm\infty)$ with a Dirichlet boundary condition at x are given by
$$m_\pm(x; z) = \frac{\psi'_\pm(x; z)}{\psi_\pm(x; z)}.$$
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For each x, these are meromorphic functions of $z \in \mathbb{C} \setminus \sigma(H_V)$.
- The *Green function* of the Schrödinger operator H_V is the integral kernel of $(H_V - z)^{-1}$; formally
$$G(x, y; z, V) = \langle \delta_x, (H_V - z)^{-1} \delta_y \rangle.$$

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In terms of the Weyl functions, the diagonal Green function is:

$$G(x,x;z,V) = \frac{1}{m_-(x;z) - m_+(x;z)},$$

and it is an analytic function of $z \in \mathbb{C} \setminus \sigma(H_V)$ for each x.

Reflectionless operators

- The Schrödinger operator is called *reflectionless* if

$$\lim_{\epsilon \downarrow 0} \mathrm{Re}\, G(x,x;E+i\epsilon,V) = 0 \text{ for Lebesgue-a.e. } E \in \sigma(H_V) =: S$$

for some $x \in \mathbb{R}$ (and therefore all $x \in \mathbb{R}$; the definition is x-independent). Notation: $V \in \mathcal{R}(S)$.

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- Equivalently, V is reflectionless if the Weyl functions are *pseudocontinuable*:

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If V is almost periodic and $\sigma_{\mathrm{ac}}(H_V) = \sigma(H_V) = S$, then $V \in \mathcal{R}(S)$.

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Theorem (Rybkin, 2008).

Let $V \in \mathcal{R}(S)$ and $\sigma_{\mathrm{ac}}(H_V) = S$. Assume that $u(x,t)$ is a solution of KdV with $u, \partial_{xxx} u \in L^\infty(\mathbb{R} \times [-T, T])$, for some $T > 0$. Then, $u(\cdot, t) \in \mathcal{R}(S)$ for all $t \in [-T, T]$.

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- Define $\mu_j(x) := \begin{cases} E \in (E_j^-, E_j^+) & G(x,x;E) = 0 \\ E_j^+ & G(x,x;E) < 0 \text{ for all } E \in (E_j^-, E_j^+) \\ E_j^- & G(x,x;E) > 0 \text{ for all } E \in (E_j^-, E_j^+) \end{cases}$.

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- If $\mu_j(x) \in (E_j^-, E_j^+)$, then exactly one of the half-line Schrödinger operators $-\partial_x^2 + V$ on the half-lines $(-\infty, x)$ and (x, ∞), with Dirichlet boundary condition at x, has an eigenvalue at $\mu_j(x)$. Equivalently, the exactly one Weyl function $m_\pm(t, x)$ has a pole at $\mu_j(x)$. The sign $\sigma_j(x) \in \{+, -\}$ labels that half-line.
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- View $(\mu_j(x), \sigma_j(x))_{j \in J}$ as an element of the torus $\mathcal{D}(S) = \prod_{j \in J} \mathbb{T}_j$.
- Introduce an angular variable φ_j on $\mathbb{T}_j$ by
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- Introduce an angular variable φ_j on $\mathbb{T}_j$ by $\mu_j = E_j^- + (E_j^+ - E_j^-) \cos^2(\varphi_j/2), \qquad \sigma_j = \operatorname{sgn} \sin \varphi_j.$
- The metric on $\mathcal{D}(S)$ is given by $\|\varphi - \tilde\varphi\|_{\mathcal{D}(S)} = \sup_{j \in J} \gamma_j^{1/2} \|\varphi_j - \tilde\varphi_j\|_{\mathbb{T}}$.

Craig type conditions

- Set $\gamma_j := E_j^+ - E_j^-$. Set $\eta_{j,l} := \mathrm{dist}((E_j^-, E_j^+), (E_l^-, E_l^+))$ for $j, l \in J$ and $\eta_{j,0} := \mathrm{dist}((E_j^-, E_j^+), \underline{E})$ for $j \in J$. Denote

$$C_j = (\eta_{j,0} + \gamma_j)^{1/2} \prod_{\substack{l \in J \\ l \neq j}} \left(1 + \frac{\gamma_l}{\eta_{j,l}}\right)^{1/2}.$$

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- We need to assume the Craig-type conditions

$$\sum_{j \in J} \gamma_j^{1/2} < \infty, \qquad \sum_{j \in J} \gamma_j^{1/2} \frac{1 + \eta_{j,0}}{\eta_{j,0}} C_j < \infty,$$

$$\sup_{j \in J} \sum_{\substack{l \in J \\ l \neq j}} \left(\frac{\gamma_j^{1/2} \gamma_l^{1/2}}{\eta_{j,l}}\right)^a (1 + \eta_{j,0}) C_j < \infty \qquad \text{for } a \in \{\tfrac{1}{2}, 1\},$$

$$\sum_{j \in J} (1 + \eta_{j,0}^2) \gamma_j < \infty.$$

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- Set $\gamma_j := E_j^+ - E_j^-$. Set $\eta_{j,l} := \operatorname{dist}((E_j^-, E_j^+), (E_l^-, E_l^+))$ for $j, l \in J$ and $\eta_{j,0} := \operatorname{dist}((E_j^-, E_j^+), \underline{E})$ for $j \in J$. Denote

$$C_j = (\eta_{j,0} + \gamma_j)^{1/2} \prod_{\substack{l \in J \\ l \neq j}} \left(1 + \frac{\gamma_l}{\eta_{j,l}}\right)^{1/2}.$$

- We need to assume the Craig-type conditions

$$\sum_{j \in J} \gamma_j^{1/2} < \infty, \qquad \sum_{j \in J} \gamma_j^{1/2} \frac{1 + \eta_{j,0}}{\eta_{j,0}} C_j < \infty,$$

$$\sup_{j \in J} \sum_{\substack{l \in J \\ l \neq j}} \left(\frac{\gamma_j^{1/2} \gamma_l^{1/2}}{\eta_{j,l}}\right)^a (1 + \eta_{j,0}) C_j < \infty \qquad \text{for } a \in \{\tfrac{1}{2}, 1\},$$

$$\sum_{j \in J} (1 + \eta_{j,0}^2) \gamma_j < \infty.$$

- The conditions imply that the spectrum S is Carleson homogeneous (Sodin, using ideas from Jones-Marshall).

The Dubrovin flow and the trace formula

Theorem. (Craig 1989)

Under (a weaker form) of Craig-type conditions on S, the $\varphi_j(x)$ evolve according to the Dubrovin flow

$$\frac{d}{dx}\varphi(x) = \Psi(\varphi(x))$$

which is given by a Lipshitz vector field Ψ,

$$\Psi_j(\varphi) = \sigma_j \sqrt{4(\underline{E} - \mu_j)(E_j^+ - \mu_j)(E_j^- - \mu_j)\prod_{k\neq j}\frac{(E_k^- - \mu_j)(E_k^+ - \mu_j)}{(\mu_k - \mu_j)^2}},$$

and the trace formula recovers the potential,

$$V(x) = Q_1(\varphi(x)) := \underline{E} + \sum_{j\in J}(E_j^+ + E_j^- - 2\mu_j(x)).$$

KdV evolution on Dirichlet data

Add time dependence: consider a solution $u(x,t)$ and its Dirichlet data $\phi(x,t)$.

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Proposition.

If S obeys the Craig-type conditions, then

$$\partial_x \varphi(x, t) = \Psi(\varphi(x, t)), \qquad \partial_t \varphi(x, t) = \Xi(\varphi(x, t)),$$

where Ξ is a Lipshitz vector field given by

$$\Xi_j = -2(Q_1 + 2\mu_j)\Psi_j,$$

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$$u(x, t) = Q_1(\varphi(x, t)) = \underline{E} + \sum_{j \in J} (E_j^+ + E_j^- - 2\mu_j(x, t)).$$

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An important step: For $E \in \left\{E_\pm^j\right\}$, there exists a nontrivial eigensolution which is a normalized limit of Weyl solutions at the gap edges $E \in \left\{E_\pm^j\right\}$:

$$\lim_{z\in(E_j^-,E_j^+);\ z\to E} c_\pm(z)\psi_\pm(x;z) = \tilde{\psi}(x)$$

uniformly on compacts, for some normalizing $c_\pm(z)$.

Dirichlet data determines a reflectionless potential and its derivatives

Under the Craig-type conditions on S, we prove

Proposition.

Let $f \in \mathcal{D}(S)$. There exists unique $\varphi : \mathbb{R} \to \mathcal{D}(S)$ such that $\varphi(0) = f$ and

$$\partial_x \varphi(x,t) = \Psi(\varphi(x,t)).$$

If we define $V : \mathbb{R}^2 \to \mathbb{R}$ by

$$V(x) = Q_1(\varphi(x))$$

then $V(x) \in \mathcal{R}(S) \cap C^4(\mathbb{R}) \cap W^{4,\infty}$ and $B(V(x)) = f$.
If we define $Q_k = \underline{E}^k + \sum_{j \in J}((E_j^-)^k + (E_j^+)^k - 2\mu_j^k)$, then V obeys the higher order trace formulas

$$Q_2 \circ \varphi = -\tfrac{1}{2} V'' + V^2$$

$$Q_3 \circ \varphi = \frac{3}{16} V^{(4)} - \frac{3}{2} V V'' - \frac{15}{16}(V')^2 + V^3$$

Existence of solutions

Now we add the time dependence to obtain a solution of the KdV equation:

Proposition.

Let S satisfy Craig-type conditions and let $V(x) \in \mathcal{R}(S)$. Let $f = B(V) \in \mathcal{D}(S)$.
Then there exists $\varphi : \mathbb{R}^2 \to \mathcal{D}(S)$ such that $\varphi(0,0) = f$ and

$$\partial_x \varphi(x,t) = \Psi(\varphi(x,t)), \qquad \partial_t \varphi(x,t) = \Xi(\varphi(x,t)).$$

If we define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x,t) = Q_1(\varphi(x,t))$, then the function $u(x,t)$ obeys the KdV equation with $u(x,0) = V(x)$.
Moreover, for each $t \in \mathbb{R}$, we have $u(\cdot,t) \in \mathcal{R}(S)$ and $B(u(\cdot,t)) = \varphi(0,t)$, and

$$Q_2 \circ \varphi = -\tfrac{1}{2}\partial_x^2 u + u^2$$
$$Q_3 \circ \varphi = \frac{3}{16}\partial_x^4 u - \frac{3}{2} u \partial_x^2 u - \frac{15}{16}(\partial_x u)^2 + u^3$$

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The two results are proven by showing convergence of approximants with finite gap spectra $S^N = [\underline{E}, \infty) \setminus \bigcup_{j=1}^N (E_j^-, E_j^+)$, for which the above statements were known.

Sodin-Yuditskii theory

Define $\xi_j(z)$ as the harmonic measure of $S \cap \{y : y \geq E_j^+\}$ in $\mathbb{C} \setminus S$ evaluated at z, i.e. the solution of the Dirichlet problem on $\mathbb{C} \setminus S$ with boundary values on S given by

$$\xi_j(x) = \begin{cases} 1 & x \in S, x \geq E_j^+ \\ 0 & x \in S, x \leq E_j^- \end{cases}$$

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Sodin–Yuditskii map (an infinite dimensional version of Abel map) $A : \mathcal{D}(S) \to \mathbb{T}^J = \pi^* (\mathbb{C} \setminus S)$,

$$A_j(\varphi) = \pi \sum_{k \in J} \sigma_k \left(\xi_j(\mu_k) - \xi_j(E_k^-)\right) \pmod{2\pi\mathbb{Z}}$$

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Defined for an arbitrary Parreau–Widom subset of $\mathbb{R}$.

Theorem. (Sodin-Yuditskii, 1995)

Let S be a Sodin-Yuditskii set.
Then the map $\mathcal{M} := A \circ B$ is a homeomorphism between $\mathcal{R}(S)$ equipped with uniform topology and $\mathcal{D}(S)$.
It linearizes the translation flow:

$$\mathcal{M}(u(\cdot + x)) = \mathcal{M}(u(0)) + \delta x.$$

for some $\delta \in \mathbb{R}^J$.

Almost periodicity of the solution

Proposition.

Let S satisfies Craig-type conditions.
Then the map $\mathcal{M} := A \circ B$ is a homeomorphism between $\mathcal{R}(S)$ equipped with $W^{4,\infty}$ topology and $\mathcal{D}(S)$.
The map $\mathcal{M}$ linearizes the KdV flow: for some $\delta, \zeta \in \mathbb{R}^J$,

$$\mathcal{M}(u(x,t)) = \mathcal{M}(u(0,0)) + \delta x + \zeta t.$$

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- The first part follows from the higher order trace formulas.
- For the second part, we work with the flow on $\mathcal{D}(S)$.
- We use finite gap approximants, for which linearity of the Abel map is known,

$$A_j^N(\varphi^N(x,t)) = A_j^N(\varphi^N(0,0)) + \delta_j^N x + \zeta_j^N t,$$

  and uniform convergence on compacts.

Small quasiperiodic initial data.

Theorem. (Damanik-Goldstein-Lukic)

Let ω satisfies the Diophantine conditions. There exists $\varepsilon_0(a_0, b_0, \kappa) > 0$ such that if

$$V \in \mathcal{P}(\omega, \varepsilon, \kappa), \ \varepsilon < \varepsilon_0, \text{ and } S = \sigma(H_V),$$

then

$$\mathcal{R}(S) \subset \mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4).$$

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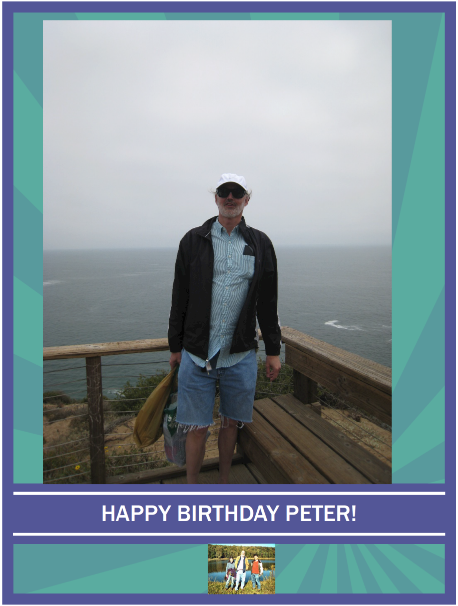
- A spectrum of any $V \in \mathcal{P}(\omega, \varepsilon, \kappa)$ satisfies the Craig-type conditions.
- So the unique solution of KdV for the initial data V satisfies

$$u(\cdot, t) \in \mathcal{P}(\omega, \sqrt{4\varepsilon}, \kappa/4) \text{ for any } t \geq 0$$


HAPPY BIRTHDAY PETER!