An Analyst's Traveling Salesman Theorem for sets of dimension larger than one

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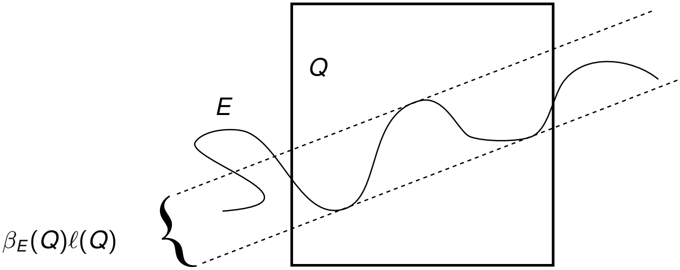
Analyst's traveling salesman theorem

For a cube $Q \subseteq \mathbb{R}^n$ of sidelength $\ell(Q)$ and $E \subseteq \mathbb{R}^n$ compact, let

$$\beta_E(Q) = \frac{\text{width of smallest tube containing } E \cap Q}{\ell(Q)}.$$



Analyst's Traveling Salesman Theorem

Theorem (Jones '90; Okikiolu, '92; Schul, '07)

Let $E \subseteq \mathbb{R}^n$.

1. *There is a curve Γ containing E so that*

$$\mathscr{H}^1(\Gamma) \lesssim \left(\operatorname{diam} E + \sum_{\substack{Q \text{ dyadic} \\ Q\cap E\neq\emptyset}} \beta_E(3Q)^2 \ell(Q) \right).$$

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2. *Conversely, if Γ is a curve, then*

$$\operatorname{diam} \Gamma + \sum_{\substack{Q \text{ dyadic} \\ Q \cap \Gamma \neq \emptyset}} \beta_\Gamma(3Q)^2 \ell(Q) \lesssim \mathscr{H}^1(\Gamma).$$

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Hence, for curves Γ, we have

$$\mathscr{H}^1(\Gamma) \sim \operatorname{diam} \Gamma + \sum_{\substack{Q \text{ dyadic} \\ Q \cap \Gamma \neq \emptyset}} \beta_\Gamma(3Q)^2 \ell(Q).$$

Applications of the TST

Theorem (Bishop, Jones, '90)

Harmonic measure on $\Omega \subseteq \mathbb{C}$ *simply connected is absolutely continuous w.r.t. arclength on* $\partial\Omega \cap \Gamma$, Γ *any rectifiable curve.*

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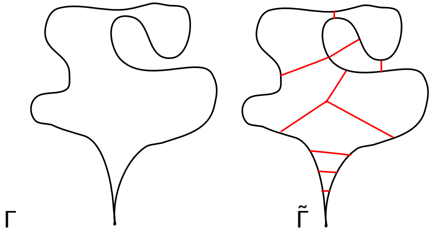
Theorem (A., Schul, '12)

There is $C > 0$ so that if $\Gamma \subseteq \mathbb{R}^n$ is a connected set, there is $\tilde{\Gamma} \supseteq \Gamma$ C-quasiconvex so that $\mathscr{H}^1(\tilde{\Gamma}) \lesssim_n \mathscr{H}^1(\Gamma)$.



Applications of the TST: ℓ^2-flatness

Corollary (Bishop, Jones '90)

If Γ is a compact connected set, it is 1-rectifiable if and only if

$$\sum_{\substack{x \in Q \\ Q \text{ dyadic}}} \beta_\Gamma(3Q)^2 < \infty \textit{ for } \mathscr{H}^1\textit{-a.e. } x \in \Gamma.$$

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- However, this isn't enough to *imply* rectifiability.
- The corollary tells us that the flatness must be decaying fast enough to characterize rectifiability.
- It also gives us more information for rectifiable sets: Not only does $\beta_\Gamma(3Q) \downarrow 0$, but at a square summable rate!

Higher dimensional β's are not adequate for TST

If $E \subseteq \mathbb{R}^n$, $d \leq n$, and we define

$$\beta^d_{E,\infty}(Q) = \inf\left\{\sup_{x \in E \cap Q} \frac{\mathrm{dist}(x,P)}{\ell(Q)} : P \text{ is a d-plane}\right\}$$

then the first direction of the TST holds for $d = 2$ (Pajot, '96) and for $d > 2$ under some assumptions (David-Toro, '12), but not the other.

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Theorem (Jones, Fang)

There is a 3*-dimensional Lipschitz graph* Γ *in* $[0,1]^4$ *so that*

$$\sum_{\substack{Q \subseteq [0,1]^4 \\ Q \cap E \neq \emptyset}} \beta^3_{\Gamma,\infty}(3Q)^2 \ell(Q)^3 = \infty.$$

Thus, in generalizing either the TST or the Bishop-Jones corollary, we need a new β-number.

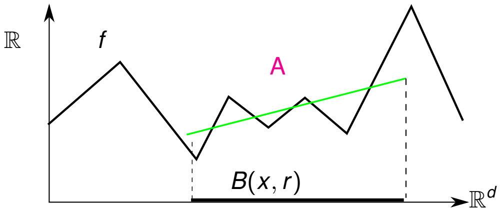
Dorronsoro's theorem, '85

Let $f \in W^{1,2}(\mathbb{R}^d)$ and define (recall $\fint_B f d\mu = \frac{1}{\mu(B)} \int_B f d\mu$)

$$\alpha(x,r) = \inf\left\{ \left(\fint_{B(x,r)} \left(\frac{f(y) - A(y)}{r} \right)^2 dy \right)^{\frac{1}{2}} : A \text{ is linear} \right\}$$

Then

$$||\nabla f||_2^2 \sim \int_{\mathbb{R}^d} \int_0^\infty \alpha(x,r)^2 \frac{dr}{r} dx.$$



TST for graphs

For $S, E \subseteq \mathbb{R}^n$, define

$$\beta_{E,2}^{d}(S) = \inf_{P \text{ a d-plane}} \left(\frac{1}{(\operatorname{diam} S)^d} \int_{S \cap E} \left(\frac{\operatorname{dist}(y, P)}{\operatorname{diam} S} \right)^2 d\mathscr{H}^d(y) \right)^{1/2}$$

Theorem (Dorronsoro)

Let $f : \mathbb{R}^d \to \mathbb{R}^{n-d}$ be L-Lipschitz (L very small) and $\Gamma \subseteq \mathbb{R}^n$ be its graph. Then

$$\int_\Gamma \int_0^\infty \beta_{\Gamma,2}^{d}(B(x,r))^2 \frac{dr}{r} d\mathscr{H}^d(x) \sim_L ||\nabla f||_2^2$$

or equivalently,

$$\sum_{Q \cap \Gamma \neq \emptyset} \beta_{\Gamma,2}^{d}(3Q)^2 \ell(Q)^d \sim_L ||\nabla f||_2^2.$$

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For $E \subseteq \mathbb{R}^n$ and S a cube or ball, define

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$$\begin{aligned} ||\nabla f||_2^2 = \int_{B(0,R)} |\nabla f|^2 &\sim \int_{B(0,R)} (\sqrt{1+|\nabla f|^2} - 1) \\ &= \mathscr{H}^d(\Gamma) - w_d R^d \end{aligned}$$

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Hence,

$$\mathscr{H}^d(\Gamma) \sim ||\nabla f||_2^2 + w_d R^d \sim \int_\Gamma \int_0^\infty \beta_{\Gamma,2}^d(B(x,r))^2 \frac{dr}{r} d\mathscr{H}^d(x) + (\operatorname{diam} \Gamma)^d.$$

David-Semmes Theorem

Theorem

Let $E \subseteq \mathbb{R}^n$ be Ahlfors d-regular, meaning

$$\mathscr{H}^d(B(x,r) \cap E) \sim r^d \quad \text{for all } x \in E, \quad r > 0.$$

Then the following are equivalent:

1. $d\sigma := \beta^d_{E,2}(B(x,r))^2 dx \frac{dr}{r}$ *is a* **Carleson measure**, *meaning* $\sigma(B(x,r) \times (0,r)) \leq Cr^d$ *for all* $x \in E$ *and* $r > 0$.

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2. E *is* **uniformly rectifiable**: *there is* $L > 0$ *so that for every* $x \in E$ *and* $r > 0$, *there is* $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^n$ L*-Lipschitz so that* $f(A) \subseteq E \cap B(x,r)$ *and* $\mathscr{H}^d(f(A)) \geq L^{-1} r^d$.

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This is like the TST in the sense that it gives a condition for when **a big piece of** E is contained in a Lipschitz surface, rather than all of it.

How far we can get with $\beta^d_{E,2}$

Theorem

Let $E \subseteq \mathbb{R}^n$ have $0 < \mathscr{H}^d(E) < \infty$. Then E is d-rectifiable if (Tolsa, '15) and only if (A., Tolsa, '15) $\int_0^1 \beta^2_{E,2}(B(x,r))^2 \frac{dr}{r} < \infty$.

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Theorem (Edelen, Naber, Valtorta, '16)

If μ is Radon on $\mathbb{R}^n$, $\theta^d_*(\mu, x) \leq b$ and $\int_0^1 \beta^d_{\mu,2}(B(x,r))^2 \frac{dr}{r} \leq M$ for μ-a.e. $x \in B(0,1)$, then $\mu(B(x,r)) \lesssim (b + M)r^d$ for $x \in B(0,1)$, $r > 0$.

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There are also other β-numbers defined for measures and results that characterize the 1-rectifiable structure of the measure (Badger and Schul,'16).

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4. we can't use β_∞.

A new β-number

Recall that

$$\beta^d_{E,2}(S)^2 = \inf_{P \text{ a d-plane}} \frac{1}{(\operatorname{diam} S)^d} \int_{S\cap E} \left(\frac{\operatorname{dist}(y,P)}{\operatorname{diam} S} \right)^2 d\mathscr{H}^d(y)$$

$$= \inf_{P \text{ a d-plane}} \frac{1}{(\operatorname{diam} S)^d} \int_0^\infty \mathscr{H}^d \left(\left\{ x \in S \cap E : \left(\frac{\operatorname{dist}(y,P)}{\operatorname{diam} S} \right)^2 > t \right\} \right) dt.$$

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A TST for "nice" surfaces

Theorem (A., Schul)

Let $E \subseteq \mathbb{R}^n$ *be so that for all* $x \in E$ *and* $r \in (0, \operatorname{diam} E)$*,*

$$\mathscr{H}^d_\infty(E \cap B(x,r)) \geq c r^d.$$

Then

$$\mathscr{H}^d(E) \lesssim (\operatorname{diam} E)^d + \sum_{Q \cap E \neq \emptyset} \hat{\beta}^d_{E,2}(3Q)^2 \ell(Q)^d.$$

Moreover, for "nice" surfaces,

$$(\operatorname{diam} E)^d + \sum_{Q \cap E \neq \emptyset} \hat{\beta}^d_{E,2}(3Q)^2 \ell(Q)^d \lesssim \mathscr{H}^d(E).$$

Nice surfaces

ϵ-Reifenberg flat sets: Let $\delta > 0$, $E \subseteq \mathbb{R}^n$ be so that, for all $x \in E$ and $0 < r < \delta \operatorname{diam} E$, there is a d-plane $P_{x,r}$ so that

$$\operatorname*{dist}_{Haus}(E \cap B(x,r), P_{x,r} \cap B(x,r)) \leq \epsilon r.$$

For these kinds of sets, we have

$$(\operatorname{diam} E)^d + \sum_{Q \cap E \neq \emptyset} \hat{\beta}^d_{E,2}(3Q)^2 \ell(Q)^d \sim_\delta \mathscr{H}^d(E)$$

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Proof: Let $\mathscr{D}_k$ denote Christ cubes on E of sidelength k. Let $\mathscr{D} = \bigcup \mathscr{D}_k$, and for $Q \in \mathscr{D}$,let B_Q be a large ball around Q.

If $Q_0 \in \mathscr{D}_0$ and $\varepsilon > 0$ small enough, we'll show

$$\sum_{R \subseteq Q_0} \beta_E(B_Q)^2 \ell(R)^d \lesssim \mathscr{H}^d(E).$$

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- For each $R \in Stop(N)$, make a stopping-time region S_R by putting $R \in S_R$ and adding cubes Q to S_R if Q's parent is in S_R and if $\angle(P_Q, P_R) < \alpha$. Again, remove a cube to smoothen out minimal cubes, then let

$$Stop(N+1) = \bigcup_{R \in Stop(N)} \text{ minimal cubes in } S_R.$$

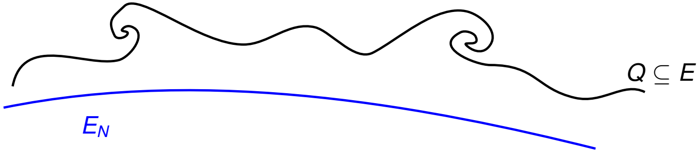
$$Total(N+1) = \text{ cubes not contained in any cube from } Stop(N+1).$$

- We can use David-Toro to construct a surface E_N so that

$$\mathrm{dist}(x, E_N) \lesssim \inf_{x \in Q \in Total(N)} \epsilon \ell(Q) \quad \text{for all } x \in E$$

  and E_N is a $C\varepsilon$-Lip graph near Q (i.e. in B_Q) over P_Q.

- If we never stop over x in our N-th stopping time, $x \in E_N \cap E$.



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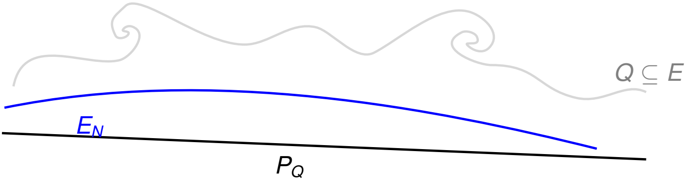

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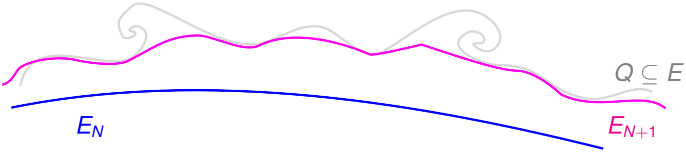

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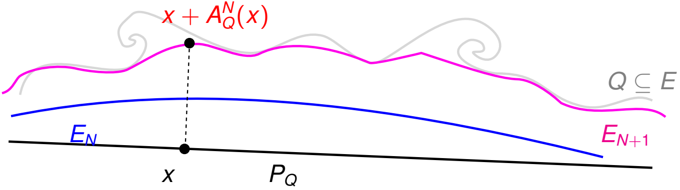

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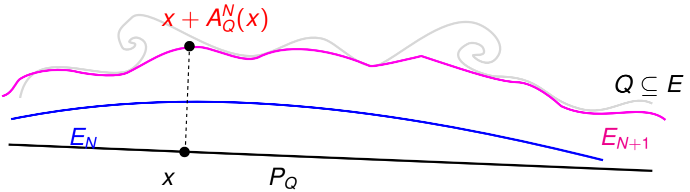
$$|DA_Q^N| \gtrsim \alpha \chi_{\pi_{P_Q}(R)} \text{ when } R \in Stop(N+1).$$



Sketch of proof for Reifenberg flat sets

Lemma

$$\sum_{N} \sum_{Q \in Stop(N)} \ell(Q)^d \lesssim \mathscr{H}^d(E).$$

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Then for $\varepsilon \ll \alpha$,

$$\sum_{\substack{R \in Stop(N+1) \\ R \subseteq Q}} \ell(R)^d \lesssim \int_{P_Q} \chi_{\pi_{P_Q}(R)} \lesssim \alpha^{-2} \int_{P_Q} |DA_Q^N|^2$$

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- If $Z(Q) \subseteq Q$ are points not contained in a cube from $Stop(N+1)$,

$$\sum_N \sum_{Q \in Type(1,N)} \ell(Q)^d \lesssim \sum_N \sum_{Q \in Type(1,N)} \mathscr{H}^d(Z(Q)) \leq \mathscr{H}^d(E).$$

Sketch of proof for Reifenberg flat sets

- Say $Q \in \mathit{Type}(2, N)$ if $Q \in \mathit{Stop}(N)$ and
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- $\mathscr{H}^d(\Gamma_Q^N) \approx \mathscr{H}^d(F_N(B_Q \cap E_N))$ and $|B_Q \cap P_Q| \approx \mathscr{H}^d(B_Q \cap E_N)|$ for $\varepsilon > 0$ small, and so for C large

$$\mathscr{H}^d(F_N(B_Q \cap E_N)) - \mathscr{H}^d(E_N \cap B_Q) > \varepsilon\ell(Q)^d.$$

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Then

$$\begin{aligned}\sum_{Q \in Type(2,N)} \ell(Q)^d &\lesssim_\varepsilon \sum_{Q \in Type(2,N)} (\mathscr{H}^d(F_N(B_Q \cap E_N)) - \mathscr{H}^d(E_N \cap B_Q)) \\ &= \sum_{Q \in Type(2,N)} \int_{B_Q \cap E_N} (J_{F_N} - 1) \lesssim \int_{E_N} (J_{F_N} - 1) \\ &= \mathscr{H}^d(E_{N+1}) - \mathscr{H}^d(E_N)\end{aligned}$$

Sketch of proof for Reifenberg flat sets

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Then

$$\sum_{Q \in \mathit{Type}(2,N)} \ell(Q)^d \lesssim_\varepsilon \sum_{Q \in \mathit{Type}(2,N)} \left(\mathscr{H}^d(F_N(B_Q \cap E_N)) - \mathscr{H}^d(E_N \cap B_Q)\right)$$

Maybe $J_{F_N} - 1 < 0!$

$$= \sum_{Q \in \mathit{Type}(2,N)} \int_{B_Q \cap E_N} (J_{F_N} - 1) \lesssim \int_{E_N} (J_{F_N} - 1)$$

$$= \mathscr{H}^d(E_{N+1}) - \mathscr{H}^d(E_N) + \mathit{Error}(N)$$

Sketch of proof for Reifenberg flat sets

- Use Dorronsoro to show

$$\sum_{R \in S_Q} \beta_{\Gamma_Q^N}(3R)^2 \ell(R)^d \lesssim \ell(Q)^d \text{ whenever } Q \in Stop(N).$$

Sketch of proof for Reifenberg flat sets

- Use Dorronsoro to show

$$\sum_{R\in S_Q} \beta_{\Gamma_Q^N}(3R)^2\ell(R)^d \lesssim \ell(Q)^d \text{ whenever } Q\in Stop(N).$$

- These approximate

$$\begin{aligned}
\sum_{R\in\subseteq Q_0} \beta_E(3R)^2\ell(R)^d &= \sum_N \sum_{Q\in Stop(N)} \sum_{R\in S_Q} \beta_E(3R)^2\ell(R)^d \\
&\lesssim \sum_N \sum_{Q\in Stop(N)} \sum_{R\in S_Q} \beta_{\Gamma_Q^N}(3R)^2\ell(R)^d + Error \\
&\lesssim \sum_N \sum_{Q\in Stop(N)} \ell(Q)^d + Error \\
&\lesssim \mathscr{H}^d(E).
\end{aligned}$$

Future Work

1. Most quantitative rectifiability results are for Ahlfors regular sets, but maybe we don't need this.
2. What other kinds of sets are ”nice”?

Thanks!

고맙습니다!

Och grattis på födelsedagen!