Free energy expansion for determinantal and Pfaffian Coulomb gases

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Random Matrices and Related Topics in Jeju

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(With Byun and Kang) Partition functions of determinantal and Pfaffian Coulomb gases with radially symmetric potentials, Comm. Math. Phys. (2023)

Ginibre ensemble (Ginibre, 1965)

Consider a random matrix $X = (X_{i,j})_{i,j=1}^{N}$ whose entries are independent and identically distributed Gaussian random variables.

- Complex Ginibre ensemble : $X_{i,j}$ takes values in $\mathbb{C}$.

  - The system of eigenvalues has an interpretation as a 2D Coulomb gas system at a specific temperature in a specific external field.

- Symplectic Ginibre ensemble : $X_{i,j}$ takes values in the field of real quaternions.

  - The system of $2N$ complex eigenvalues can be interpreted as a 2D Coulomb gas system with complex conjugation symmetry.

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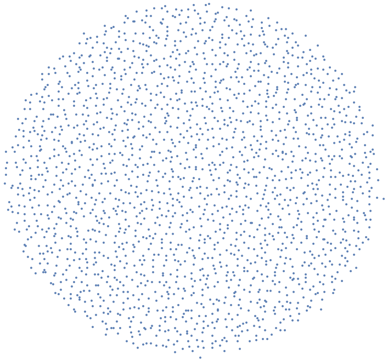
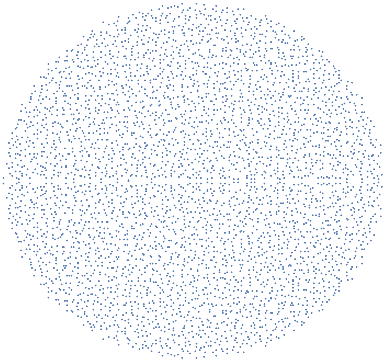
System of points with 2D Coulomb interactions



Complex Ginibre ensemble
$(N = 2000)$



Symplectic Ginibre ensemble
$(N = 2000)$

2D Coulomb gas system

We consider a system of N charged particles in $\mathbb{C}$ with pairwise logarithmic interaction, subject to a confining external potential $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$.

- The Hamiltonian of the system:

$$H_N(z_1, \cdots, z_N) = -\sum_{i \neq j} \log|z_i - z_j| + N \sum_{j=1}^{N} Q(z_j), \quad z_i \in \mathbb{C}$$

- The Boltzmann factor of the system is proportional to

$$e^{-\frac{\beta}{2} H_N(z_1, \cdots, z_N)} = \prod_{i<j} |z_i - z_j|^\beta \prod_{j=1}^{N} e^{-\frac{\beta}{2} N Q(z_j)}, \quad z_i \in \mathbb{C}.$$

  Here, $\beta > 0$ is the inverse temperature.

- $Q(z) = |z|^2$ and $\beta = 2$: Complex Ginibre ensemble.

2D Coulomb gas with complex conjugation symmetry

A system of charged particles with complex conjugation symmetry subject to a confining external potential $Q(z) = Q(\bar{z})$.

- Hamiltonian of the system:

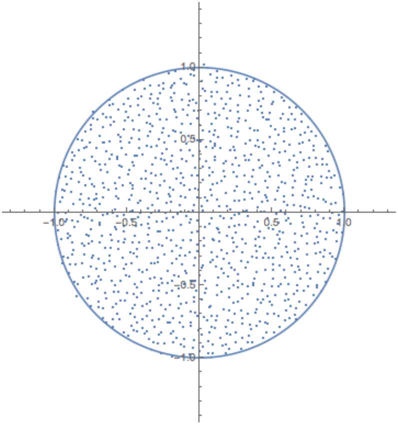
$$\tilde{H}_N = -\sum_{j \neq k} \log\left(|z_j - z_k||z_j - \bar{z}_k|\right) - \sum_{j=1}^{N} \log|z_j - \bar{z}_j|^2 + 2N \sum_{j=1}^{N} Q(z_j)$$

- The Boltzmann factor of the system is proportional to

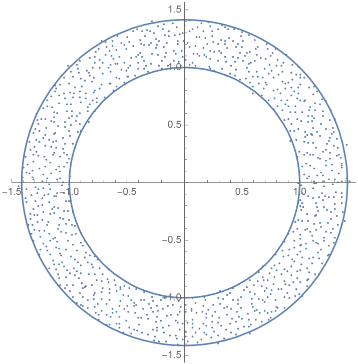
$$e^{-\frac{\beta}{2}\tilde{H}_N(z_1,\cdots,z_N)} = \prod_{j<k} |z_j - z_k|^\beta |z_j - \bar{z}_k|^\beta \prod_{j=1}^{N} |z_j - \bar{z}_j|^\beta e^{-\beta N Q(z_j)}$$

- $Q(z) = |z|^2$ and $\beta = 2$: Symplectic Ginibre ensemble.

2D Coulomb gas ensemble



$Q(z) = |z|^2$



$Q(z) = |z|^2 - 2\log|z|$

Equilibrium measures and droplets

- Q is a smooth potential satisfying $Q(z) \gg 2\log|z|$ near infinity.
- Convergence of the empirical measure for 2D Coloumb gas system [Johansson 98, Hedenmalm-Makarov 13, Chafaï-Gozlan-Zitt 14]

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{z_j} \to \sigma_Q, \quad N \to \infty.$$

- Equilibrium measure σ_Q: a unique probability measure that minimizes the weighted logarithmic energy

$$I_Q[\mu] = \iint_{\mathbb{C}^2} \log\frac{1}{|z-t|}\,d\mu(z)\,d\mu(t) + \int_{\mathbb{C}} Q\,d\mu.$$

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- σ_Q is of the form $d\sigma_Q(z) = 1_{S_Q}(z)\Delta Q(z)\, dA(z), \quad dA(z) = d^2z/\pi$, where $\Delta = \partial\bar{\partial}$ and S_Q is the compact support called the droplet.

Droplets with radially symmetric potentials

- For a radially symmetric potential $Q(z) = q(|z|)$ that is strictly subharmonic,
$$S = \{z \in \mathbb{C} : r_0 \leq |z| \leq r_1\},$$
where the pair of constant (r_0, r_1) is characterized by
$$r_0 q'(r_0) = 0, \qquad r_1 q'(r_1) = 1.$$

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$$d\sigma_Q(z) = 1_S(z)\, dA(z), \quad S = \{z \in \mathbb{C} : |z| \leq 1\}$$

Droplets with radially symmetric potentials

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- If $Q(z) = |z|^2$

$$d\sigma_Q(z) = 1_S(z)\, dA(z), \quad S = \{z \in \mathbb{C} : |z| \leq 1\}$$

- If $Q(z) = |z|^{2\lambda} - 2c \log|z|$ $(c > 0)$,

$$d\sigma_Q(z) = \lambda^2 |z|^{2\lambda - 2}\, 1_S(z)\, dA(z), \quad S = \{\left(\tfrac{c}{\lambda}\right)^{\frac{1}{2\lambda}} \leq |z| \leq \left(\tfrac{c+1}{\lambda}\right)^{\frac{1}{2\lambda}}\}.$$

Partition function for the Coulomb gas

- Partition functions of the 2D Coulomb gas system:

$$Z_{N,Q}^{(\beta)} = \int_{\mathbb{C}^N} \prod_{i<j} |z_i - z_j|^\beta \prod_{j=1}^{N} e^{-\frac{\beta}{2} NQ(z_j)} dA(z_j),$$

$$\tilde{Z}_{N,Q}^{(\beta)} = \int_{\mathbb{C}^N} \prod_{j<k} |z_j - z_k|^\beta |z_j - \bar{z}_k|^\beta \prod_{j=1}^{N} |z_j - \bar{z}_j|^\beta e^{-\beta NQ(z_j)} dA(z_j).$$

Partition function for the Coulomb gas

- Partition functions of the 2D Coulomb gas system (complex)

$$Z_{N,Q}^{(\beta)} = \int_{\mathbb{C}^N} \prod_{i<j} |z_i - z_j|^\beta \prod_{j=1}^{N} e^{-\frac{\beta}{2} N Q(z_j)} dA(z_j).$$

- Asymptotic up to the $O(N)$-term:
  [Leblé and Serfaty, 17], [Bauerschmidt, Bourgade, Nikula, and Yau, 19]

$$\log Z_{N,Q}^{(\beta)} \sim -\frac{\beta}{2} I_Q[\sigma_Q] N^2 + \frac{\beta}{4} N \log N - \Big(C(\beta) + \Big(1 - \frac{\beta}{4}\Big) E_Q[\sigma_Q] \Big) N,$$

  where $C(\beta)$ is a constant independent of the potential Q.

  - $I_Q[\sigma_Q]$: the weighted logarithmic energy.
  - $E_Q[\sigma_Q] = \int_{\mathbb{C}} \log \Delta Q \, d\sigma_Q$: the entropy associated with σ_Q.

Conjecture on the partition functions

- An explicit formula for the large N expansion of $Z_{N,Q}^{(\beta)}$ was predicted by Zabrodin and Wiegmann (2006).

  It was predicted that

  $$\log Z_{N,Q}^{(\beta)} \sim C_0 N^2 + C_1 N \log N + C_2 N + C_3 \log N + C_4$$

  and they proposed the explicit formulas for the constants $C_i = C_i(Q, \beta)$ $(i = 1, \cdots, 4)$.

Conjecture on the partition functions

- Zabrodin-Wiegmann conjecture: [Zabrodin-Wiegmann, 06]

$$\log Z_{N,Q}^{(\beta)} \sim C_0 N^2 + C_1 N \log N + C_2 N + C_3 \log N + C_4.$$

- Jancovici et al. conjecture:

$$\log Z_{N,Q}^{(\beta)} \sim C_0 N^2 + C_1 N \log N + C_2 N + \alpha_\beta \sqrt{N} + C_{3,\chi} \log N$$

⋆ $C_{3,\chi} = \left(\frac{1}{2} - \frac{\chi}{12}\right)$ where χ is the Euler index of the droplet.
($\chi = 1$ for a disk, $\chi = 2$ for a sphere)
[Jancovici, Manificat, and Pisani, 94], [Téllez and Forrester, 99]

⋆ For the disk geometry, $\alpha_\beta = -\frac{4}{3\sqrt{\pi}} \log(\beta/2)$ (surface tension).
Lutsyshin, [Can, Forrester, Téllez, and Wiegmann, 15]

Results on the partition functions ($\beta = 2$, complex)

It is conjectured that if the droplet is connected, the partition function has the asymptotic expansion of the form

$$\log Z_{N,Q}^{(2)} = -I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}E_Q[\sigma_Q]\Big)N + \frac{6-\chi}{12}\log N + \frac{\log(2\pi)}{2} + \chi\zeta'(-1) + F_Q + o(1),$$

where χ is the Euler characteristic of the droplet and ζ is the Riemann zeta function.

Results on the partition functions ($\beta = 2$, complex)

It is conjectured that if the droplet is connected, the partition function has the asymptotic expansion of the form

$$\log Z^{(2)}_{N,Q} = -I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}E_Q[\sigma_Q]\Big)N + \frac{6-\chi}{12}\log N + \frac{\log(2\pi)}{2} + \chi\zeta'(-1) + F_Q + o(1),$$

where χ is the Euler characteristic of the droplet and ζ is the Riemann zeta function.

- [Byun-Kang-S. 23] The asymptotic expansion was obtained up to $O(1)$ for radially symmetric Q with $\Delta Q > 0$ in $\mathbb{C}$.
  - The formulae for the $O(\log N)$ and $O(1)$ terms depend on whether the droplet is an annulus($\chi = 0$) or a disk($\chi = 1$).
  - The coefficient of $\log N$ agrees with the Jancovici et al. conjecture.

Results on the partition functions ($\beta = 2$, complex)

- [Byun-Kang-S. 23] The asymptotic expansion is obtained up to $O(1)$ for radially symmetric Q with $\Delta Q > 0$ in $\mathbb{C}$.

- The droplet: $S_Q = \{z \in \mathbb{C} : r_0 \leq |z| \leq r_1\}$.
- $r_0 = 0$: disk ($\chi = 1$), $r_0 > 0$: annulus ($\chi = 0$).

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- The droplet: $S_Q = \{z \in \mathbb{C} : r_0 \leq |z| \leq r_1\}$.
- $r_0 = 0$: disk ($\chi = 1$), $r_0 > 0$: annulus ($\chi = 0$).
- $O(1)$ term:

$$\frac{1}{2}\log(2\pi) + F_Q + \chi\,\zeta'(-1),$$

$$F_Q = \frac{1}{12}\log\left(\frac{1}{r_1^2\Delta Q(r_1)}\right) - \frac{1}{16}r_1\frac{(\partial_r\Delta Q)(r_1)}{\Delta Q(r_1)} + \frac{1}{24}\int_{r_0}^{r_1}\left(\frac{\partial_r\Delta Q(r)}{\Delta Q(r)}\right)^2 r\,dr$$
$$+\begin{cases} 0 & \text{if } r_0 = 0 \ (\chi = 1), \\ \frac{1}{12}\log(r_0^2\Delta Q(r_0)) + \frac{1}{16}r_0\frac{(\partial_r\Delta Q)(r_0)}{\Delta Q(r_0)} & \text{if } r_0 > 0 \ (\chi = 0). \end{cases}$$

Results on the partition functions ($\beta = 2$, complex)

- [Webb-Wong 19], [Deaño-Simm 22], [Byun-Charlier 22]
  Characteristic polynomials for complex random matrices.

- [Charlier 23]
  Large gap asymptotics on annuli in the random normal matrix model.

- [Ameur-Charlier-Cronvall 23]

  The asymptotic expansion has been studied for radially symmetric Q when the droplet has multiple components.

$$\log Z_{N,Q}^{(2)} = -I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}E_Q[\sigma_Q]\Big)N + \frac{6-\chi}{12}\log N + \frac{\log(2\pi)}{2} + \chi\zeta'(-1) + F_Q + \mathcal{G}_n + o(1)$$

- [Byun-S.-Yang 24] The asymptotic expansion is obtained for the non-radially symmetric potential

$$Q(z) = |z|^2 - 2c\log|z-a|, \quad c > 0 \text{ and } a \in \mathbb{C}.$$

Results on the partition functions ($\beta = 2$, symplectic)

- Partition function of the 2D Coulomb gas system (symplectic case)

$$\tilde{Z}_{N,Q}^{(\beta)} = \int_{\mathbb{C}^N} \prod_{j<k} |z_j - z_k|^\beta |z_j - \bar{z}_k|^\beta \prod_{j=1}^{N} |z_j - \bar{z}_j|^\beta e^{-\beta N Q(z_j)} \, dA(z_j)$$

- [Byun-Kang-S. 23] The asymptotic expansion for $\tilde{Z}_{N,Q}^{(2)}$ was obtained up to $O(1)$ for radially symmetric Q with $\Delta Q > 0$ in $\mathbb{C}$.

Results on the partition functions ($\beta = 2$, symplectic)

- [Byun-Kang-S. 23] The asymptotic expansion for $\tilde{Z}^{(2)}_{N,Q}$ was obtained up to $O(1)$ for radially symmetric Q with $\Delta Q > 0$ in $\mathbb{C}$.

- Droplet: $S_Q = \{z \in \mathbb{C} : r_0 \le |z| \le r_1\}$.
- $r_0 = 0$ (disk, $\chi = 1$), $\quad r_0 > 0$ (annulus, $\chi = 0$)

$$\begin{aligned}\log \tilde{Z}^{(2)}_{N,Q} =& -2I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N \\ &+ \Big(\frac{\log(4\pi)}{2} - 1 + \int \log|w-\bar{w}|\,d\sigma_Q(w) - \frac{E_Q[\sigma_Q]}{2}\Big)N \\ &+ \Big(\frac{1}{2} - \frac{\chi}{24}\Big)\log N + \frac{\log(2\pi)}{2} + \frac{\chi}{2}\Big(\zeta'(-1) + \frac{5\log 2}{12}\Big) \\ &+ \frac{1}{2}F_Q + \frac{1}{8}\log\Big(\frac{\Delta Q(r_0)}{\Delta Q(r_1)}\Big) + o(1).\end{aligned}$$

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Large N expansions of partition functions

Theorem (Byun-Kang-S. 2023)

For a radially symmetric potential Q which is smooth, subharmonic in C, and strictly subharmonic in a neighborhood of the droplet, we have

$$\log Z_{N,Q}^{(2)} = -I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{E_Q[\sigma_Q]}{2}\Big)N + \Big(\frac{1}{2} - \frac{\chi}{12}\Big)\log N + \Big(\frac{\log(2\pi)}{2} + F_Q + \chi\,\zeta'(-1)\Big) + O(N^{-1/12}(\log N)^3),$$

$$\log \tilde{Z}_{N,Q}^{(2)} = -2I_Q[\sigma_Q]N^2 + \frac{1}{2}N\log N + \Big(\frac{\log(4\pi)}{2} - 1 + \int \log|w-\bar{w}|\,d\sigma_Q(w) - \frac{E_Q[\sigma_Q]}{2}\Big)N + \Big(\frac{1}{2} - \frac{\chi}{24}\Big)\log N + \frac{\log(2\pi)}{2} + \frac{1}{2}F_Q + \frac{1}{8}\log\Big(\frac{\Delta Q(r_0)}{\Delta Q(r_1)}\Big) + \frac{\chi}{2}\Big(\zeta'(-1) + \frac{5\log 2}{12}\Big) + O(N^{-1/12}(\log N)^3).$$

Determinantal and Pfaffian structures

- The k-point correlation function of a point process $\{\zeta_j\}$:

$$\mathbf{R}_{N,k}(\eta_1, \cdots, \eta_k) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2k}} \mathbf{P}_N \left[\mathcal{N}(D(\eta_j, \epsilon)) \geq 1 \text{ for all } 0 \leq j \leq k \right],$$

where $\mathcal{N}(D)$ is the number of particles in $D \subset \mathbb{C}$.

Determinantal and Pfaffian structures

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  where $\mathcal{N}(D)$ is the number of particles in $D\subset\mathbb{C}$.

- For $\beta=2$, the Coulomb gas forms a determinantal point process.

  $$\mathbf{R}_{N,k}(\eta_1,\cdots,\eta_k)=\det\left(\mathbf{K}_N(\eta_i,\eta_j)\right)_{i,j=1}^{k},$$

  where $\mathbf{K}_N:\mathbb{C}^2\to\mathbb{C}$ is called a correlation kernel.

- For $\beta=2$, the Coulomb gas system with complex conjugation symmetry forms a Pfaffian point process.

  $$\tilde{\mathbf{R}}_{N,k}(\zeta_1,\cdots,\zeta_k)=\prod_{j=1}^{k}(\bar{\zeta}_j-\zeta_j)\cdot \mathrm{Pf}_{1\leq i,j\leq k}[\boldsymbol{\mathcal{K}}_N(\zeta_i,\zeta_j)],$$

  where the 2×2 matrix-valued kernel $\boldsymbol{\mathcal{K}}_N$.

Correlation kernel for a determinantal Coulomb gas

- Correlation kernel

$$\mathbf{K}_N(\zeta,\eta) = \mathbf{k}_N(\zeta,\eta)\, e^{-NQ(\zeta)/2 - NQ(\eta)/2}.$$

- $\mathbf{k}_N$: reproducing kernel for the space H_N of analytic polynomials of degree $< N$ equipped with the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f\,\bar{g}\, e^{-NQ}\, dA.$$

- $p_{N,j}$: orthonormal polynomial of degree j with respect to $e^{-NQ}\, dA$.

$$\mathbf{k}_N(\zeta,\eta) = \sum_{j=0}^{N-1} p_{N,j}(\zeta)\, \overline{p_{N,j}(\eta)}.$$

Matrix kernel for a Pfaffian Coulomb gas

- The 2×2 matrix-valued kernel $\boldsymbol{\mathcal{K}}_N$ is defined as

$$\boldsymbol{\mathcal{K}}_N(\zeta,\eta) = e^{-NQ(\zeta)-NQ(\eta)} \begin{pmatrix} \boldsymbol{\kappa}_N(\zeta,\eta) & \boldsymbol{\kappa}_N(\zeta,\bar{\eta}) \\ \boldsymbol{\kappa}_N(\bar{\zeta},\eta) & \boldsymbol{\kappa}_N(\bar{\zeta},\bar{\eta}) \end{pmatrix}.$$

- A skew-symmetric form

$$\langle f, g \rangle_s = \int_{\mathbb{C}} \Big(f(\zeta) g(\bar{\zeta}) - g(\zeta) f(\bar{\zeta}) \Big) (\zeta - \bar{\zeta}) e^{-2NQ(\zeta)} \, dA(\zeta)$$

- Skew-orthogonality condition: for monic q_m of degree m

$$\langle q_{2k+1}, q_{2l+1} \rangle_s = \langle q_{2k}, q_{2l} \rangle_s = 0, \quad \langle q_{2k}, q_{2l+1} \rangle_s = -\langle q_{2l+1}, q_{2k} \rangle_s = r_k \delta_{k,l}$$

  with $r_k > 0$ being their skew-norms.

$$\boldsymbol{\kappa}_N(\zeta,\eta) = \sum_{k=0}^{N-1} \frac{q_{2k+1}(\zeta) q_{2k}(\eta) - q_{2k}(\eta) q_{2k+1}(\eta)}{r_k}.$$

Cf. [Akemann-Ebke-Parra 22] Skew-orthogonal polynomials in the complex plane and their Bergman-like kernels – Construction of SOP in terms of OP

Integrable structures

Using the determinantal structure (the Pfaffian structure) and de Bruijin's type formula, we have

$$\log Z_{N,Q}^{(2)} = \log N! + \sum_{j=0}^{N-1} \log h_j, \qquad \log \tilde{Z}_{N,Q}^{(2)} = \log N! + \sum_{j=0}^{N-1} \log r_j.$$

For a radially symmetric potential Q,

- h_j is the norm of the monic orthogonal polynomial of degree j with respect to e^{-NQ}.
$$h_j = \int_{\mathbb{C}} |z|^{2j} e^{-NQ(z)}\, dA(z).$$
- r_j is the skew norm of monic skew-orthogonal polynomial:

$$r_j = \langle q_{2j}, q_{2j+1} \rangle_s = 2\tilde{h}_{2j+1}, \qquad \tilde{h}_j := \int_{\mathbb{C}} |z|^{2j} e^{-2NQ(z)}\, dA(z).$$

Evolution of equilibrium measures

- Consider the potential $Q_\tau = Q/\tau$ for $\tau > 0$.
- Corresponding equilibrium measure: $d\sigma_\tau = \frac{1}{\tau} 1_{S_\tau} \Delta Q \, dA$
- Chain of S_τ: if $\tau_1 \leq \tau_2$ then $S_{\tau_1} \subset S_{\tau_2}$
- The movement of a boundary of the droplet S_τ follows a weighted Laplacian growth (or a weighted Hele-Shaw flow).
- The weighted orthogonal polynomial $|p_k(z)|^2 e^{-NQ}$ is concentrated near ∂S_τ with $\tau = k/N$.
- Asymptotic expansions of planar orthogonal polynomials for general potentials: [Hedenmalm-Wennman 21], [Hedenmalm 23]

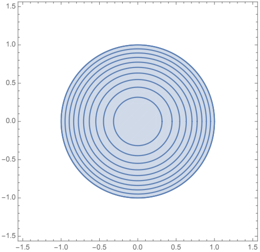
Droplets and Hele-Shaw flows

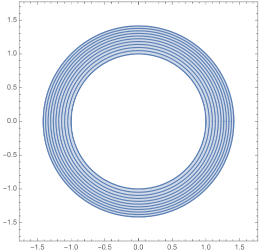
- The droplets S_τ for radially symmetric cases:
- $Q(z) = |z|^2$(disk case) and $Q(z) = |z|^2 - 2\log|z|$(annulus case).

  For $Q(z) = |z|^2$, $S_\tau = \{z \in \mathbb{C} : |z| \leq \sqrt{\tau}\}$ (disk).

  For $Q(z) = |z|^2 - 2\log|z|$, $S_\tau = \{z \in \mathbb{C} : 1 \leq |z| \leq \sqrt{1+\tau}\}$ (annulus).
- Consider $S_0 = \lim_{\tau \to 0} S_\tau$.

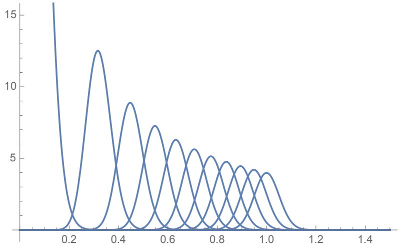
  For the disk case, S_0 is a point. For the annulus case, S_0 is a circle.



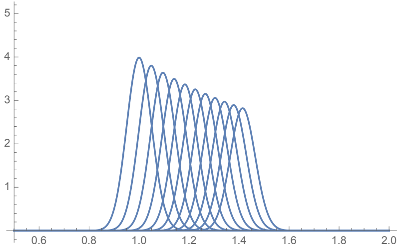


Weighted orthogonal polynomials

- $p_{N,j}$: an orthonormal polynomial of degree j with respect to e^{-NQ}.
- Graphs of $|p_{N,j}|^2 e^{-NQ}$ restricted on $\mathbb{R}$ ($N = 100$, $j = 0, 10, \cdots, 100$).
- For $\tau = j/N$, the graph has a pick near the outer boundary of S_τ.
- Except lower degree polynomials for the disk case, $|p_{N,j}|^2 e^{-NQ}$ is asymptotically gaussian.



$Q(z) = |z|^2$



$Q(z) = |z|^2 - 2\log|z|$

Weighted orthogonal polynomial norms

- For radially symmetric cases,

$$\log Z_N^{(2)} = \log N! + \sum_{j=0}^{N-1} \log h_j, \quad h_j = \int_{\mathbb{C}} |z|^{2j} e^{-NQ}\, dA.$$

- For $\tau = j/N$, let r_τ be the radius of outer boundary of S_τ.
- We distinguish the following two cases for $\epsilon > 0$.

- Case 1 $(r_\tau \gg N^{-\epsilon})$:

  If $r_0 > 0$ (annulus), this case covers all $j = 0, 1, \cdots, N-1$.

  If $r_0 = 0$ (disk), this case covers $j = N^\varepsilon, \cdots, N-1$ with some $\varepsilon > 0$.

- Case 2 $(r_\tau \ll N^{-\epsilon})$:

  This covers the lower degree terms of disk droplet case $(r_0 = 0)$
  $j = 0, \cdots, N^\varepsilon - 1$.

Radially symmetric cases

- Let r_τ be the radius of outer boundary of S_τ.
- If $\Delta Q > 0$, the function

$$V_\tau(r) = Q(r) - 2\tau \log r, \quad r \geq 0$$

  has a unique local minimum at $r = r_\tau$.
- Apply the Laplace method to

$$h_j = \int_{\mathbb{C}} |z|^{2j} e^{-NQ} \, dA = \int_0^\infty e^{-NV_{\tau(j)}(r)} r \, dr, \quad \tau = \frac{j}{N}$$

  for the case when $r_\tau \gg N^{-\epsilon}$.
- If $r_\tau \ll N^{-\epsilon}$, different estimates are used for h_j.

Radially symmetric case: annular droplet

- Annular case ($r_0 > 0$): for each j with $0 \le j \le N-1$,

$$\log h_j \sim -NV_\tau(r_\tau) + \frac{1}{2}\Big(\log(2\pi r_\tau^2) - \log N - \log \Delta Q(r_\tau)\Big) + \frac{1}{N}\mathcal{B}_1(r_\tau)$$

- Apply the Euler-Maclaurin formula

$$\sum_{j=m}^{n} f(j) = \int_m^n f(x)\,dx + \frac{f(m)+f(n)}{2} + \sum_{k=1}^{l-1} \frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(n) - f^{(2k-1)}(m)\Big) + \cdots .$$

to obtain the large N expansion of

$$\log Z_{N,Q}^{(2)} = \log N! + \sum_{j=0}^{N-1} \log h_j.$$

Radially symmetric case: disk droplet

- Disk case ($r_0 = 0$): we compute

$$\sum_{j=0}^{n-1} \log h_j = \sum_{j=0}^{m_N-1} \log h_j + \sum_{j=m_N}^{N-1} \log h_j,$$

  where $m_N = N^\epsilon$.

- For $j = 0, 1, \cdots, m_N - 1$,

$$\log h_j = -NQ(0) - (j+1)\log\Big(\frac{N}{2}\partial_r^2 Q(0)\Big) + \log j! + O\big(N^{-\frac{1}{2}}(j+1)^{\frac{3}{2}}(\log N)^3\big).$$

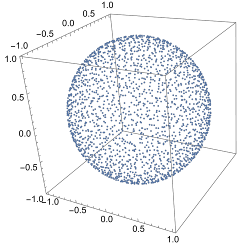
- For $j = m_N, \cdots, N-1$,

$$\log h_j = -NV_{\tau(j)}(r_{\tau(j)}) + \frac{1}{2}\Big(\log(2\pi r_{\tau(j)}^2) - \log N - \log \Delta Q(r_{\tau(j)})\Big) + \frac{1}{N}\mathcal{B}_1(r_{\tau(j)}) + O(j^{-\frac{3}{2}}(\log N)^{\alpha}).$$

Spherical one-component plasma

Complex spherical ensemble

- $Q(z) = \frac{N+1}{N} \log(1+|z|^2)$



- Partition function

$$\log Z_{N,Q} = -\frac{N^2}{2} + \frac{1}{2} N \log N + \Big(\frac{\log(2\pi)}{2} - 1\Big) N + \frac{6-\chi}{12} \log N + \frac{\log(2\pi)}{2} - \frac{1}{12} + \chi\zeta'(-1) + O(N^{-2}).$$

Here, the Euler characteristic is $\chi = 2$.

- [Byun-Park 24] Large gap probabilities of complex and symplectic spherical ensembles with point charges.
- [Byun-Kang-S.-Yang] Partition functions of determinantal and symplectic spherical one component plasma, in preparation

Thank you for your attention!