Fluctuations in various regimes of non-Hermiticity

(joint work with G. Akemann and M. Duits)

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- The complex elliptic Ginibre ensemble with parameter $0 \leq \tau < 1$ consists of $n \times n$ complex matrices M distributed by

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM_n, \qquad dM_n = \prod_{1 \leq i,j \leq n} d \operatorname{Re} M_{ij} \, d \operatorname{Im} M_{ij},$$

where $V(z) = \dfrac{|z|^2 - \tau \operatorname{Re}(z^2)}{1-\tau^2} = \dfrac{z\bar{z} - \frac{\tau}{2}(z^2 + \bar{z}^2)}{1-\tau^2}$

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- The elliptic Ginibre ensemble interpolates between the Ginibre ensemble ($\tau = 0$) and the Gaussian unitary ensemble ($\tau \uparrow 1$).
- The eigenvalues of the (elliptic) Ginibre ensemble live in the complex plane while those of the Gaussian unitary ensemble are real.
- The eigenvalues are distributed by

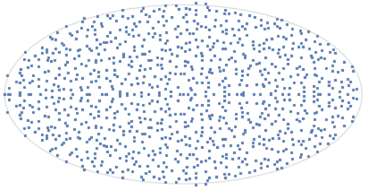
$$\rho_n(z_1, \ldots, z_n) = \frac{1}{c_n} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-nV(z_j)}, \qquad z_1, \ldots, z_n \in \mathbb{C}.$$

- (Elliptic law) Its eigenvalues accumulate on an elliptic **droplet**

$$\lim_{n\to\infty} \rho_n^{(1)}(z) = \begin{cases} \frac{1}{\pi(1-\tau^2)}, & z \in \mathscr{E}_\tau, \\ 0, & z \in \mathbb{C} \setminus \overline{\mathscr{E}_\tau}, \end{cases}$$

where

$$\mathscr{E}_\tau = \left\{ z \in \mathbb{C} : \left(\frac{\operatorname{Re} z}{1+\tau} \right)^2 + \left(\frac{\operatorname{Im} z}{1-\tau} \right)^2 < 1 \right\}.$$



- The eigenvalues of the elliptic Ginibre ensemble form a DPP

$$\rho_n^{(k)}(z_1, \ldots, z_k) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} \rho_n(z_1, \ldots, z_n) d^2 z_{k+1} \cdots d^2 z_n$$
$$= \det\left(\mathscr{K}_n(z_i, z_j)\right)_{1 \leq i,j \leq k},$$

with correlation kernel constructed with planar orthogonal polynomials

$$\mathscr{K}_n(z, w) = \sqrt{\omega(z)\omega(w)} \sum_{j=0}^{n-1} P_j(z)\overline{P_j(w)}, \qquad z, w \in \mathbb{C},$$

where $\omega(z) = e^{-nV(z)}$ and $\int_{\mathbb{C}} P_i(z)\overline{P_j(z)}\omega(z) d^2 z = \delta_{ij}$.

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- Explicitly, we have

$$P_j(z) = \begin{cases} \sqrt{\frac{n}{\pi j!}}\, z^j, & \tau = 0,\\ \sqrt{\frac{n}{\pi j!}}(1-\tau^2)^{\frac14}\left(\frac{\tau}{2}\right)^{\frac{j}{2}} H_j\left(\sqrt{\frac{n}{2\tau}}z\right), & \tau\in(0,1),\end{cases}$$

where $H_j(z) = (-1)^n e^{z^2}\frac{d^j}{dz^j}e^{-z^2}$ are Hermite polynomials of degree j.

Local scaling limits

- (Bulk scaling limit: Ginibre kernel) For $z \in \mathscr{E}_\tau$ and $u, v \in \mathbb{C}$

$$\lim_{n\to\infty} \frac{1-\tau^2}{n} \mathscr{K}_n\left(z + \sqrt{\frac{1-\tau^2}{n}}u, z + \sqrt{\frac{1-\tau^2}{n}}v\right) \equiv \frac{1}{\pi}\exp\left(u\overline{v} - \frac{|u|^2+|v|^2}{2}\right).$$

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- (Edge scaling limit: Faddeeva plasma kernel) For $z \in \partial\mathscr{E}_\tau$ and $u, v \in \mathbb{C}$

$$\lim_{n\to\infty} \frac{1-\tau^2}{n} \mathscr{K}_n \left(z + \sqrt{\frac{1-\tau^2}{n}} u\vec{n}(z), z + \sqrt{\frac{1-\tau^2}{n}} v\vec{n}(z) \right) \equiv \frac{1}{\pi} \exp\left(u\overline{v} - \frac{|u|^2 + |v|^2}{2} \right) \operatorname{erfc}\left(\frac{u + \overline{v}}{\sqrt{2}} \right).$$

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- These local scaling limits are universal (bulk: Ameur, Hedenmalm, Makarov 2010, edge: Tao, Vu 2015, Cipolloni, Erdős, Schröder 2021, Hedenmalm, Wennman 2021.)

Weak non-Hermiticity regime

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- (edge) For $\alpha = \frac{1}{3}$ one finds a deformation of the Airy kernel near the edge $x = 2$ (Bender 2008).
- Weak non-Hermiticity regime can be considered for other models (e.g., Ameur, Byun 2023).

Global scaling limits: linear statistics

- We focus on linear statistics

$$S_n[f] = f(z_1) + \ldots + f(z_n)$$

where $z_1, \ldots, z_n$ are eigenvalues of the elliptic Ginibre ensemble (with parameter $0 \le \tau < 1$).

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- Case 1: Smooth linear statistics.
  We assume that $f : \mathbb{C} \to \mathbb{R}$ is smooth (or at least C^2).
- Case 2: Rough linear statistics (number variance).
  We assume that $f : \mathbb{C} \to \{0, 1\}$ is an indicator function $f(z) = 1_A(z)$ where $A \subset \mathbb{C}$.

Theorem (CLT smooth linear statistics, strong non-Hermiticity)

Let $0 \leq \tau < 1$ be fixed, and let $f : \mathbb{C} \to \mathbb{R}$ be C^2 and L^2. Let $S_n[f] = f(z_1) + \ldots + f(z_n)$, where $z_1, \ldots, z_n$ are picked from the elliptic Ginibre ensemble. Then, as $n \to \infty$

$$S_n[f] - \mathbb{E}S_n[f] \to N\left(0, \sigma^2 + \tilde{\sigma}^2\right)$$

in distribution, where, with $\phi(z) = \frac{1}{2}(z + \sqrt{z^2 - 4\tau})$, we have

$$\sigma^2 = \frac{1}{4\pi}\|f \circ \phi^{-1}\|_{H^1(\mathbb{D})} = \frac{1}{4\pi}\int_{\mathscr{E}_\tau} |\nabla f(z)|^2 d^2 z,$$

$$\tilde{\sigma}^2 = \frac{1}{2}\|f \circ \phi^{-1}\|_{H^{1/2}(\partial\mathbb{D})}$$

$$= \frac{1}{4\pi^2}\int_{\partial\mathscr{E}_\tau}\int_{\partial\mathscr{E}_\tau}\left|\frac{f(z) - f(w)}{\phi(z) - \phi(w)}\right|^2 |\phi'(z)dz||\phi'(w)dw|.$$

- For $\tau = 0$ this was first proved by Rider and Virag 2007.
- A general result for random normal matrices by Ameur, Hedenmalm, Makarov 2011.

- Rider and Virag 2007 ($\tau = 0$) proved that this implies that

$$h_n(z) = \log \left| \prod_{j=1}^{n} (z - z_j) \right| - \mathbb{E} \log \left| \prod_{j=1}^{n} (z - z_j) \right|$$

converges weakly to a Gaussian free field h^* (the planar Gaussian free field, conditioned to be harmonic outside the unit disk).

$$\int_{\mathbb{C}} h_n(z) f(z) d^2 z \to \int_{\mathbb{C}} h^*(z) f(z) d^2 z$$

for test functions $f : \mathbb{C} \to \mathbb{R}$ with continuous partial derivatives in open neighborhood of unit disk and at most exponential growth outside.
(and similar for k-point correlation functions.)

- Proved in generality for random normal matrices by Ameur, Hedenmalm, Makarov 2011.

For any set A the number variance $V_n[A]$ is defined as the variance of the linear statistic $S_n[f] = f(z_1) + \ldots + f(z_n)$ where $f(z) = 1_A(z)$.

Theorem (Number variance, strong non-Hermiticity)

Let $0 \leq \tau < 1$ be fixed. Write $z = x + iy$. Suppose that A is a simple region that is strictly inside $\mathscr{E}_\tau$. Suppose that it is parametrized as $|y| \leq \phi(|x|)$, with $-a < x < a$, for some strictly decreasing C^1 function $\phi : [0, a] \to [0, b]$ that satisfies $\phi'(0) = 0$. Then we have as $n \to \infty$

$$\frac{2\pi\sqrt{\pi}}{\sqrt{\Delta V(z) n}} V_n[A] = |\partial A| + \mathscr{O}(1/\sqrt{n}).$$

- In the case $\tau = 0$ (Ginibre), the result is true for any Cacciopoli set (Lin 2023), see also Levi, Marzo, Ortega-Cerdà 2023.

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- Universality results are known for radial symmetric potentials and radial symmetric sets A (Akemann, Byun, Ebke 2023).

Theorem (Number variance, strong non-Hermiticity)

Let $0 \leq \tau < 1$ *and define*

$$A = A_n(S) = \begin{cases} \mathscr{E}_\tau \cup \left\{ [z + \frac{2}{\sqrt{\Delta V(z) n}} \vec{n}(z) S, z] : z \in \partial \mathscr{E}_\tau \right\}, & S \geq 0, \\ \mathscr{E}_\tau \setminus \left\{ [z + \frac{2}{\sqrt{\Delta V(z) n}} \vec{n}(z) S, z] : z \in \partial \mathscr{E}_\tau \right\}, & S < 0. \end{cases}$$

Here $\vec{n}(z)$ *denotes the outward unit normal vector at* z *on* $\partial \mathscr{E}_\tau$. *Then*

$$\lim_{n \to \infty} \frac{2\pi\sqrt{\pi}}{\sqrt{\Delta V(z) n}} \frac{V_{n,A}}{|\partial \mathscr{E}_\tau|} = f(S),$$

where

$$f(S) = \sqrt{2\pi} \int_{-\infty}^{S} \frac{\operatorname{erfc}(t) \operatorname{erfc}(-t)}{4} dt.$$

- For radial symmetric potentials and radial symmetric A, this was proved by Akemann, Byun, Ebke 2023.

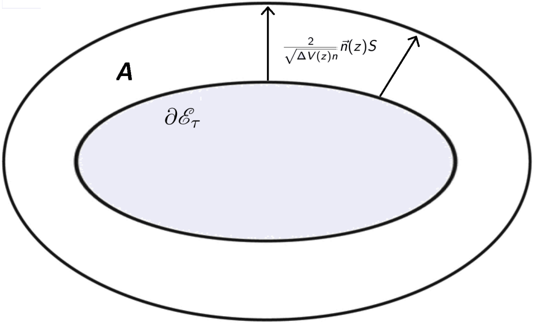

Figure: The set A where $S > 0$.

The Rényi entropy with parameter $q > 1$ is given by

$$s_q[A] = \frac{1}{q-1} \operatorname{Tr} \log(\mathbb{A}^q + (\mathbb{I} - \mathbb{A})^q),$$

where the overlap matrix $\mathbb{A}$ is given by

$$\mathbb{A}_{jk} = \int_A P_j(z)\overline{P_k(z)} e^{-nV(z)} d^2 z.$$

Theorem (Holography for random matrices)

Let $0 < \tau \leq 1$. For any $q > 1$ the Rényi entropy satisfies the bound

$$s_q[A] \leq \frac{4q \log 2}{q-1} V_{n,A}.$$

In particular $s_q[A] \leq C_q^\tau \sqrt{n} |\partial A|$ for some constant $C_q^\tau > 0$, uniformly for sets $A \subset \mathscr{E}_\tau$ (as before).

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- e.g., for $\tau = 0$ and radial symmetric sets Lacroix-A-Chez-Toine, Majumdar, Schehr 2018 showed

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} s_q[\{z \in \mathbb{C} : |z| \leq a\}] = \alpha_q \int_a^\infty \log\left(\frac{1}{2^q}\operatorname{erfc}(t)^q + \frac{1}{2^q}\operatorname{erfc}(-t)^q\right) dt, \qquad 0 \leq a < 1.$$

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- We suspect that a more explicit limiting relation between the entropy and ∂A exists (as $n \to \infty$).

Linear statistics in the weak non-Hermiticity regime

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- Our goal is to find a CLT interpolation as $n \to \infty$ (depending on κ) between the Ginibre ensemble and the GUE.
- This implies an interpolation between a 2D and 1D Gaussian free field.
- It appears that $0 < \alpha < 1$ is the right choice to achieve this.

Theorem (Interpolating variance, weak non-Hermiticity)

Assume that $0 < \alpha < 1$ *and* $\kappa > 0$ *are fixed. Let* $f : \mathbb{C} \to \mathbb{R}$ *be a* C^2 *function with compact support, and form the linear statistic* $S_n[f] = \sum_{j=1}^n f(n^\alpha z_j)$*, where the summation is over eigenvalues from the elliptic Ginibre ensemble with parameter* $\tau = 1 - \frac{\kappa}{n^\alpha}$*. Then we have*

$$\lim_{n\to\infty} \operatorname{Var} S_n[f] = \frac{1}{4\pi} \int_{|\operatorname{Im}(z)|\leq\kappa} |\nabla f(z)|^2 d^2 z + \frac{1}{8\pi^2} \iint_{\operatorname{Im} z = \operatorname{Im} w = \pm\kappa} \left(\frac{f(z) - f(w)}{z - w} \right)^2 dz dw.$$

- It turns out to be technically difficult to prove the CLT (and thus GFF interpolation).

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Assume that $0 < \alpha < 1$ *and* $\kappa > 0$ *are fixed. Let* $f : \mathbb{C} \to \mathbb{R}$ *be a* C^2 *function with compact support, and form the linear statistic* $S_n[f] = \sum_{j=1}^n f(n^\alpha z_j)$, *where the summation is over eigenvalues from the elliptic Ginibre ensemble with parameter* $\tau = 1 - \frac{\kappa}{n^\alpha}$. *Then we have*

$$\lim_{n\to\infty} \mathrm{Var}\, S_n[f] = \frac{1}{4\pi} \int_{|\operatorname{Im}(z)|\leq\kappa} |\nabla f(z)|^2 d^2 z + \frac{1}{8\pi^2} \iint_{\operatorname{Im} z = \operatorname{Im} w = \pm\kappa} \left(\frac{f(z) - f(w)}{z - w} \right)^2 dz dw.$$

- It turns out to be technically difficult to prove the CLT (and thus GFF interpolation).
- As $\kappa \to \infty$, we reobtain the result by Rider-Virag 2007 (without boundary term), while when $\kappa \downarrow \infty$ we obtain (a version of) the GUE counter part (Girko 2001).

Conjecture

Let $\tau = 1 - \frac{\kappa}{n}$ *where* $\kappa > 0$*. For* $a \in (0,2)$ *and* $T \in (0,\infty]$*, let* $A = [-a,a] \times [-T,T]$*. Then the number variance of eigenvalues taken from the elliptic Ginibre ensemble satisfies*

$$V_{n,A} = C_1(\kappa) n + C_{\frac{1}{2}}(\kappa)\sqrt{n} + C_0(\kappa)\log n + o(\log n)$$

as $n \to \infty$*, for certain positive constants* $C_1(\kappa), C_{\frac{1}{2}}(\kappa), C_0(\kappa)$*. Furthermore, we have*

$$\lim_{\kappa \downarrow 0} C_1(\kappa) = \lim_{\kappa \downarrow 0} C_{\frac{1}{2}}(\kappa) = 0,$$
$$\lim_{\kappa \to \infty} C_1(\kappa) = \frac{1}{2\pi\sqrt{\pi}} \left|\partial\{z \in 2\mathbb{D} : |\operatorname{Re} z| \leq a\}\right|.$$

- The $\log n$ behavior should be compared to Costin-Lebowitz 1995 (GUE, $\tau = 1$).
- A hint to such a result was already in Fyodorov, Khoruzhenko, and Sommers '97.

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감사합니다!