# Logarithmically correlated fields in non-Hermitian random matrices

Giorgio Cipolloni, Princeton University

May 10, 2024

Random Matrices and Related Topics in Jeju

# Logarithmically correlated fields

A logarithmically (log–)correlated field $X(z)$ is a Gaussian random distribution on $\mathbf{R}^d$.

# Logarithmically correlated fields

A logarithmically (log–)correlated field $X(z)$ is a Gaussian random distribution on $\mathbf{R}^d$.

The law of $X(z)$ is determined by

$$\mathrm{Cov}\big[X(z_1), X(z_2)\big] = -\log|z_1 - z_2|.$$

# Logarithmically correlated fields

A logarithmically (log–)correlated field $X(z)$ is a Gaussian random distribution on $\mathbf{R}^d$.

The law of $X(z)$ is determined by

$$\mathrm{Cov}\big[X(z_1), X(z_2)\big] = -\log|z_1 - z_2|.$$

Some examples: 2D Gaussian Free Field (GFF), branching random walks.

# Logarithmically correlated fields

A logarithmically (log–)correlated field $X(z)$ is a Gaussian random distribution on $\mathbf{R}^d$.

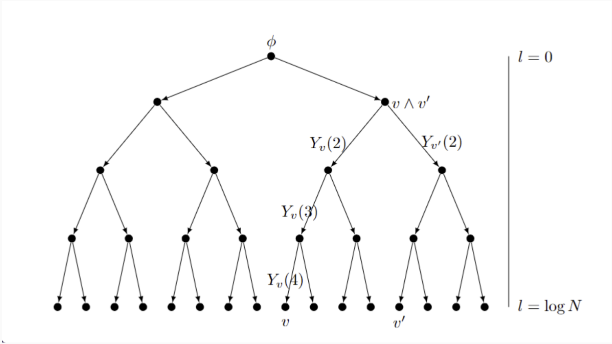
The law of $X(z)$ is determined by

$$\mathrm{Cov}\big[X(z_1), X(z_2)\big] = -\log|z_1 - z_2|.$$

Some examples: 2D Gaussian Free Field (GFF), branching random walks.

Emergence of log–correlated fields: conformal field theory, random surfaces, random matrices, number theory, statistical physics, SPDEs, . . .

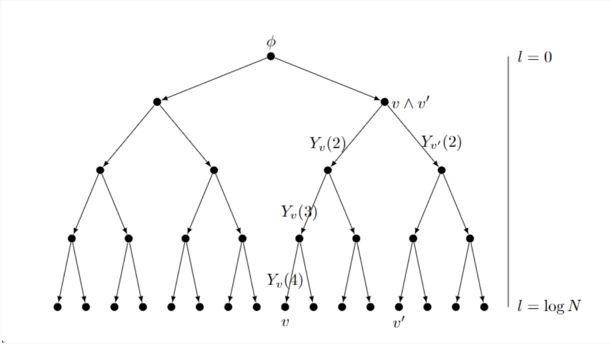
# (Gaussian) Branching Random Walk




Binary branching random walk (BRW) with increments $Y_v(l)$.

Here $Y_v(l)$ are i.i.d. $\mathcal{N}(0, \sigma^2)$.

# (Gaussian) Branching Random Walk




Binary branching random walk (BRW) with increments $Y_v(l)$.

Here $Y_v(l)$ are i.i.d. $\mathcal{N}(0, \sigma^2)$.

BRW:

$$X_v(N) = \sum_{l=1}^{\log N} Y_v(l), \qquad \mathbf{E}\left[Y_v(l) Y_{v'}(l)\right] = \begin{cases} \sigma^2 & \text{if} \quad l \leq v \wedge v', \\ 0 & \text{if} \quad l > v \wedge v'. \end{cases}$$

# Model: I.I.D. Random Matrices

# I.I.D. Random Matrices

We consider random matrices

$$X = \begin{pmatrix} x_{11} & \dots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NN} \end{pmatrix}$$

with independent identical distributed (i.i.d.) complex or real entries $x_{ab} \stackrel{d}{=} N^{-1/2}\chi$:

(i) $\mathbf{E}\,\chi = 0$,

(ii) $\mathbf{E}|\chi|^2 = 1$, $\mathbf{E}\,\chi^2 = 0$, or $\mathbf{E}\,\chi^2 = 1$,

(iii) $\mathbf{E}|\chi|^p \le C_p < \infty$.

# I.I.D. Random Matrices

We consider random matrices

$$X = \begin{pmatrix} x_{11} & \dots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NN} \end{pmatrix}$$

with independent identical distributed (i.i.d.) complex or real entries $x_{ab} \overset{d}{=} N^{-1/2}\chi$:

(i) $\mathbf{E}\,\chi = 0$,

(ii) $\mathbf{E}|\chi|^2 = 1$, $\mathbf{E}\,\chi^2 = 0$, or $\mathbf{E}\,\chi^2 = 1$,

(iii) $\mathbf{E}|\chi|^p \leq C_p < \infty$.

Normalization guarantees that $\|X\| \sim 1$ as $N \to \infty$.

We consider random matrices

$$X = \begin{pmatrix} x_{11} & \dots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NN} \end{pmatrix}$$

with independent identical distributed (i.i.d.) complex or real entries $x_{ab} \stackrel{d}{=} N^{-1/2}\chi$:

(i) $\mathbf{E}\,\chi = 0$,

(ii) $\mathbf{E}|\chi|^2 = 1$, $\mathbf{E}\,\chi^2 = 0$, or $\mathbf{E}\,\chi^2 = 1$,

(iii) $\mathbf{E}|\chi|^p \leq C_p < \infty$.

Normalization guarantees that $\|X\| \sim 1$ as $N \to \infty$.

Remark: No integrable structure if $\chi \neq$Gaussian.

# Eigenvalues of random matrices with independent entries

$X$ is an $N \times N$ matrix with i.i.d. entries

$$\mathbf{E}\, x_{ab} = 0, \qquad \mathbf{E}|x_{ab}|^2 = \frac{1}{N}.$$

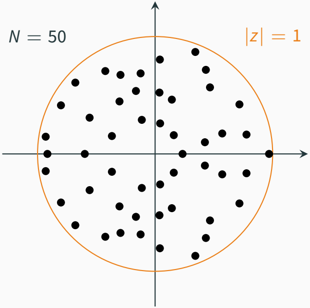



**Figure 1:** Real entries

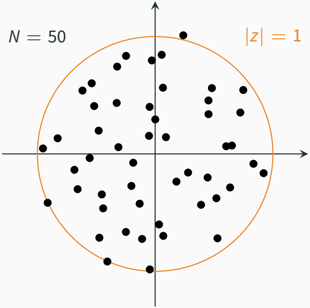



**Figure 2:** Complex entries

# Eigenvalues of random matrices with independent entries

$X$ is an $N \times N$ matrix with i.i.d. entries

$$\mathbf{E}\, x_{ab} = 0, \qquad \mathbf{E}|x_{ab}|^2 = \frac{1}{N}.$$

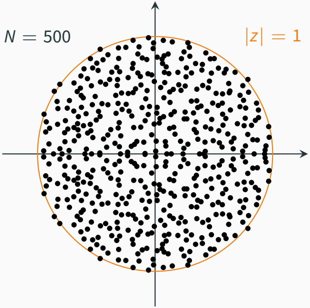



**Figure 1:** Real entries

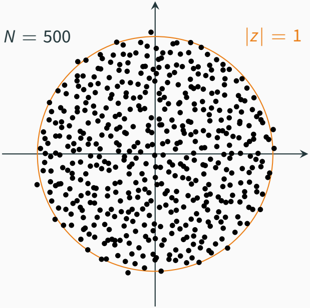



**Figure 2:** Complex entries

# Eigenvalues of random matrices with independent entries

$X$ is an $N \times N$ matrix with i.i.d. entries

$$\mathbf{E}\, x_{ab} = 0, \qquad \mathbf{E}|x_{ab}|^2 = \frac{1}{N}.$$

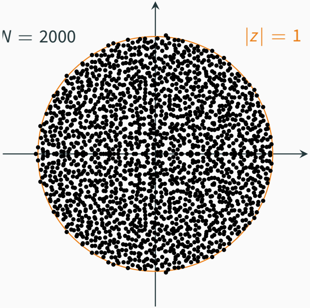



**Figure 1:** Real entries

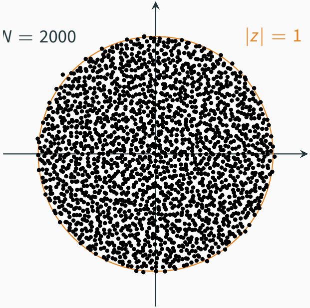



**Figure 2:** Complex entries

# Eigenvalues of random matrices with independent entries

$X$ is an $N \times N$ matrix with i.i.d. entries

$$\mathbf{E}\, x_{ab} = 0, \qquad \mathbf{E}|x_{ab}|^2 = \frac{1}{N}.$$

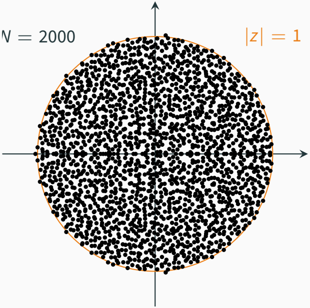



**Figure 1:** Real entries

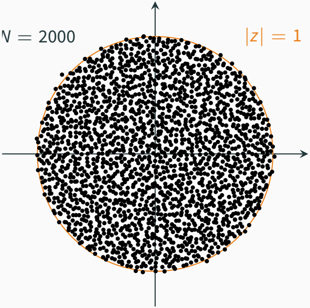



**Figure 2:** Complex entries

- Circular law: Convergence to the uniform distribution on the unit disk.
- Eigenvalues spacing $\sim N^{-1/2}$.
- Accumulation of $\sim \sqrt{N}$ eigenvalues on the real axis for real matrices.

CLT: Let $\sigma_1, \ldots, \sigma_N$ be the eigenvalues of $X$, then

$$\Gamma_N := N \left[ \frac{1}{N} \sum_{i=1}^{N} \delta_{\sigma_i} - \frac{1}{\pi} \mathbf{1}(|z| \leq 1) \, \mathrm{d}^2 z \right] \to GFF.$$

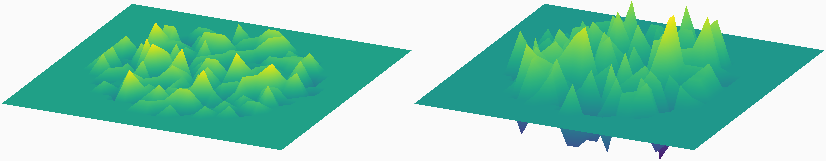


**Figure 3:** (Smoothed) fluctuations of 1000 eigenvalues of i.i.d. random matrices (left) vs. 1000 independent points uniformly distributed in the unit disk (right). Eigenvalues fluctuate much less.

# Log–correlated fields in NH Random Matrices

# Log–correlated field in 2D

**Theorem (Rider–Virag 2007)**

Let $X$ be a complex Ginibre matrix (i.e. $\chi$ is Gaussian), then for $z_1, z_2 \in \mathbf{D}$:

$$\operatorname{Cov}\left[\log\left|\det(X - z_1)\right|, \log\left|\det(X - z_2)\right|\right] \approx -\log|z_1 - z_2|.$$

# Log–correlated field in 2D

**Theorem (Rider–Virag 2007)**

Let $X$ be a complex Ginibre matrix (i.e. $\chi$ is Gaussian), then for $z_1, z_2 \in \mathbf{D}$:

$$\mathrm{Cov}\left[\log\left|\det\left(X - z_1\right)\right|, \log\left|\det\left(X - z_2\right)\right|\right] \approx -\log|z_1 - z_2|.$$

More precisely: Let $f, g \in H^1$, and let ($\sigma_i \in \mathrm{Spec}(X)$) $L_N(f) := \sum_i f(\sigma_i) - \mathbf{E}\sum_i f(\sigma_i)$. Then

$$\mathbf{E}\, L_N(f)L_N(g) \approx -\int_{\mathbf{D}}\int_{\mathbf{D}} \Delta f(z_1)\Delta g(z_2) \log|z_1 - z_2|\; \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2.$$

# Log–correlated field in 2D

**Theorem (Rider–Virag 2007)**

Let $X$ be a complex Ginibre matrix (i.e. $\chi$ is Gaussian), then for $z_1, z_2 \in \mathbf{D}$:

$$\mathrm{Cov}\left[\log\left|\det(X - z_1)\right|, \log\left|\det(X - z_2)\right|\right] \approx -\log|z_1 - z_2|.$$

More precisely: Let $f, g \in H^1$, and let $(\sigma_i \in \mathrm{Spec}(X))$ $L_N(f) := \sum_i f(\sigma_i) - \mathbf{E}\sum_i f(\sigma_i)$. Then

$$\mathbf{E}\, L_N(f) L_N(g) \approx -\int_{\mathbf{D}}\int_{\mathbf{D}} \Delta f(z_1)\Delta g(z_2) \log|z_1 - z_2|\, \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2.$$

Remark 1: Proven for radial $f$ by **(Forrester 1999)**. Also predicted general formula.

# Log–correlated field in 2D

**Theorem (Rider–Virag 2007)**

Let $X$ be a complex Ginibre matrix (i.e. $\chi$ is Gaussian), then for $z_1, z_2 \in \mathbf{D}$:

$$\mathrm{Cov}\left[\log\left|\det\left(X - z_1\right)\right|, \log\left|\det\left(X - z_2\right)\right|\right] \approx -\log|z_1 - z_2|.$$

More precisely: Let $f, g \in H^1$, and let $(\sigma_i \in \mathrm{Spec}(X))$ $L_N(f) := \sum_i f(\sigma_i) - \mathbf{E}\sum_i f(\sigma_i)$. Then

$$\mathbf{E}\, L_N(f) L_N(g) \approx -\int_{\mathbf{D}}\int_{\mathbf{D}} \Delta f(z_1)\Delta g(z_2) \log|z_1 - z_2|\; \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2.$$

Remark 1: Proven for radial $f$ by **(Forrester 1999)**. Also predicted general formula.

Remark 2: Similar result holds for general i.i.d. matrices **(C.–Erdős–Schröder 2019)**.

**Theorem (Rider–Virag 2007)**

Let $X$ be a complex Ginibre matrix (i.e. $\chi$ is Gaussian), then for $z_1, z_2 \in \mathbf{D}$:

$$\mathrm{Cov}\left[\log\left|\det\left(X - z_1\right)\right|, \log\left|\det\left(X - z_2\right)\right|\right] \approx -\log|z_1 - z_2|.$$

More precisely: Let $f, g \in H^1$, and let $(\sigma_i \in \mathrm{Spec}(X))$ $L_N(f) := \sum_i f(\sigma_i) - \mathbf{E}\sum_i f(\sigma_i)$. Then

$$\mathbf{E}\, L_N(f) L_N(g) \approx -\int_{\mathbf{D}}\int_{\mathbf{D}} \Delta f(z_1)\Delta g(z_2) \log|z_1 - z_2| \, \mathrm{d}^2 z_1 \, \mathrm{d}^2 z_2.$$

Remark 1: Proven for radial $f$ by **(Forrester 1999)**. Also predicted general formula.

Remark 2: Similar result holds for general i.i.d. matrices **(C.–Erdős–Schröder 2019)**.

For general i.i.d. matrices dependence on the fourth cumulant $\kappa_4$ of the entries.

# Log–correlated field in 2D

**Theorem (Rider–Virag 2007)**

Let $X$ be a complex Ginibre matrix (i.e. $\chi$ is Gaussian), then for $z_1, z_2 \in \mathbf{D}$:

$$\mathrm{Cov}\left[\log\left|\det(X - z_1)\right|, \log\left|\det(X - z_2)\right|\right] \approx -\log|z_1 - z_2|.$$

More precisely: Let $f, g \in H^1$, and let $(\sigma_i \in \mathrm{Spec}(X))$ $L_N(f) := \sum_i f(\sigma_i) - \mathbf{E}\sum_i f(\sigma_i)$. Then

$$\mathbf{E}\, L_N(f) L_N(g) \approx -\int_{\mathbf{D}}\int_{\mathbf{D}} \Delta f(z_1)\Delta g(z_2) \log|z_1 - z_2| \, \mathrm{d}^2 z_1 \, \mathrm{d}^2 z_2.$$

Remark 1: Proven for radial $f$ by **(Forrester 1999)**. Also predicted general formula.

Remark 2: Similar result holds for general i.i.d. matrices **(C.–Erdős–Schröder 2019)**.

For general i.i.d. matrices dependence on the fourth cumulant $\kappa_4$ of the entries.

Remark 3: Similar result for normal matrices **(Ameur–Hedenmalm–Makarov 2008, 2011)**, **(Ameur, Kang, Seo 2018)**.

# Log–correlated field in 3D

Consider the Ornstein-Uhlenbeck flow (here $B_t$ is matrix Brownian motion)

$$\mathrm{d}X_t = -\frac{1}{2}X_t\,\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad X_0 = X.$$

# Log–correlated field in 3D

Consider the Ornstein-Uhlenbeck flow (here $B_t$ is matrix Brownian motion)

$$\mathrm{d}X_t = -\frac{1}{2}X_t\,\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad X_0 = X.$$

**Theorem (Bourgade–C.–Huang 2024)**

Let $X$ be an i.i.d. matrix, then

$$\mathrm{Cov}\left[\log\left|\det\left(X_t - z_1\right)\right|, \log\left|\det\left(X_s - z_2\right)\right|\right] \approx -\frac{1}{2}\log\left[|t-s| + |z_1 - z_2|^2\right].$$

# Log–correlated field in 3D

Consider the Ornstein-Uhlenbeck flow (here $B_t$ is matrix Brownian motion)

$$\mathrm{d}X_t = -\frac{1}{2}X_t\,\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad X_0 = X.$$

**Theorem (Bourgade–C.–Huang 2024)**

Let $X$ be an i.i.d. matrix, then

$$\mathrm{Cov}\left[\log\left|\det(X_t - z_1)\right|, \log\left|\det(X_s - z_2)\right|\right] \approx -\frac{1}{2}\log\left[|t-s| + |z_1 - z_2|^2\right].$$

Remark 1: Not known even for Ginibre! No space–time determinantal structure.

# Log–correlated field in 3D

Consider the Ornstein-Uhlenbeck flow (here $B_t$ is matrix Brownian motion)

$$\mathrm{d}X_t = -\frac{1}{2}X_t\,\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad X_0 = X.$$

**Theorem (Bourgade–C.–Huang 2024)**

Let $X$ be an i.i.d. matrix, then

$$\mathrm{Cov}\left[\log\left|\det\left(X_t - z_1\right)\right|, \log\left|\det\left(X_s - z_2\right)\right|\right] \approx -\frac{1}{2}\log\left[|t-s| + |z_1 - z_2|^2\right].$$

Remark 1: Not known even for Ginibre! No space–time determinantal structure.

Remark 2: Same regularity $2D$ additive stochastic heat equation (Edwards–Wilkinson universality class).

# Log–correlated field in 3D

Consider the Ornstein-Uhlenbeck flow (here $B_t$ is matrix Brownian motion)

$$\mathrm{d}X_t = -\frac{1}{2}X_t\,\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad X_0 = X.$$

**Theorem (Bourgade–C.–Huang 2024)**

Let $X$ be an i.i.d. matrix, then

$$\mathrm{Cov}\left[\log\left|\det\left(X_t - z_1\right)\right|, \log\left|\det\left(X_s - z_2\right)\right|\right] \approx -\frac{1}{2}\log\left[|t-s| + |z_1 - z_2|^2\right].$$

Remark 1: Not known even for Ginibre! No space–time determinantal structure.

Remark 2: Same regularity $2D$ additive stochastic heat equation (Edwards–Wilkinson universality class).

Remark 3: The limiting Gaussian field is not Markovian.

# The Fyodorov–Hiary–Keating (FHK) conjecture

# The Fyodorov–Hiary–Keating (FHK) conjecture

Let $e^{i\omega_1}, \dots, e^{i\omega_N}$ the eigenvalues of a CUE matrix (Haar unitary).

# The Fyodorov–Hiary–Keating (FHK) conjecture

Let $e^{i\omega_1}, \ldots, e^{i\omega_N}$ the eigenvalues of a CUE matrix (Haar unitary). Consider the characteristic polynomial

$$X_n(z) = \prod_{j=1}^{N}(1 - e^{i\omega_j} z).$$

# The Fyodorov–Hiary–Keating (FHK) conjecture

Let $e^{i\omega_1}, \dots, e^{i\omega_N}$ the eigenvalues of a CUE matrix (Haar unitary). Consider the characteristic polynomial

$$X_n(z) = \prod_{j=1}^{N} (1 - e^{i\omega_j} z).$$

FHK conjecture:

$$\max_{|z|=1} \log |X_N(z)| = m_N + \xi_N, \qquad m_N := \log N - \frac{3}{4} \log \log N,$$

with $\xi_N$ expected to converge to the sum of two independent Gumbel, i.e. $F(x) = e^{-e^{-x}}$.

# The Fyodorov–Hiary–Keating (FHK) conjecture

Let $e^{i\omega_1}, \dots, e^{i\omega_N}$ the eigenvalues of a CUE matrix (Haar unitary). Consider the characteristic polynomial

$$X_n(z) = \prod_{j=1}^{N} (1 - e^{i\omega_j} z).$$

FHK conjecture:

$$\max_{|z|=1} \log |X_N(z)| = m_N + \xi_N, \qquad m_N := \log N - \frac{3}{4} \log \log N,$$

with $\xi_N$ expected to converge to the sum of two independent Gumbel, i.e. $F(x) = e^{-e^{-x}}$.

**Theorem (Paquette–Zeitouni 2022)**

There exists a deterministic $C$ such that

$$\max_{|z|=1} \log |X_N(z)| - \log N - \frac{3}{4} \log \log N - C \Longrightarrow \xi,$$

with $\xi$ being the sum of two independent random variables.

# The Fyodorov–Hiary–Keating (FHK) conjecture

Let $e^{\mathrm{i}\omega_1}, \dots, e^{\mathrm{i}\omega_N}$ the eigenvalues of a CUE matrix (Haar unitary). Consider the characteristic polynomial

$$X_n(z) = \prod_{j=1}^{N} (1 - e^{\mathrm{i}\omega_j} z).$$

FHK conjecture:

$$\max_{|z|=1} \log |X_N(z)| = m_N + \xi_N, \qquad m_N := \log N - \frac{3}{4} \log\log N,$$

with $\xi_N$ expected to converge to the sum of two independent Gumbel, i.e. $F(x) = e^{-e^{-x}}$.

**Theorem (Paquette–Zeitouni 2022)**

There exists a deterministic $C$ such that

$$\max_{|z|=1} \log |X_N(z)| - \log N - \frac{3}{4} \log\log N - C \Longrightarrow \xi,$$

with $\xi$ being the sum of two independent random variables.

Previous results: $\log N$–term **(Arguin–Belius–Bourgade 2015)**; $\log\log N$–term **(Paquette–Zeitouni 2016)**; $\max_{|z|=1} \log |X_N(z)| - m_N$ is tight **(Chhaibi–Madaule–Najnudel 2018)**.

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log\left(\prod_{i=1}^{N} |\sigma_i - z|\right) - \mathbf{E}(\cdots) = \log\left|\det\big(X - z\big)\right| - \mathbf{E}(\cdots).$$

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log\left(\prod_{i=1}^{N} |\sigma_i - z|\right) - \mathbf{E}(\cdots) = \log\left|\mathrm{det}\big(X - z\big)\right| - \mathbf{E}(\cdots).$$

Conjecture (Non–Hermitian FHK):

$$\max_{|z|\leq 1} \Psi_N(z) = \frac{1}{\sqrt{2}}\left(\log N - \frac{3}{4}\log\log N + \xi_N\right).$$

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log\left(\prod_{i=1}^{N} |\sigma_i - z|\right) - \mathbf{E}(\cdots) = \log\left|\det\bigl(X - z\bigr)\right| - \mathbf{E}(\cdots).$$

Conjecture (Non–Hermitian FHK):

$$\max_{|z|\le 1} \Psi_N(z) = \frac{1}{\sqrt{2}}\left(\log N - \frac{3}{4}\log\log N + \xi_N\right).$$

No conjectures about $\xi_N$ for any 2D ensemble!

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log\left(\prod_{i=1}^{N} |\sigma_i - z|\right) - \mathbf{E}(\cdots) = \log\left|\det\big(X - z\big)\right| - \mathbf{E}(\cdots).$$

Conjecture (Non–Hermitian FHK):

$$\max_{|z|\le 1} \Psi_N(z) = \frac{1}{\sqrt{2}}\left(\log N - \frac{3}{4}\log\log N + \xi_N\right).$$

No conjectures about $\xi_N$ for any 2D ensemble!

---

**Theorem (C., Landon 2024)**

Let $X$ be a real or complex i.i.d. matrix then for any $\epsilon > 0$ we have

$$\lim_{N\to\infty} \mathbf{P}\left(\left(\frac{1}{\sqrt{2}} - \epsilon\right)\log N \le \max_{|z|\le 1} \Psi_N(z) \le \left(\frac{1}{\sqrt{2}} + \epsilon\right)\log N\right) = 1.$$

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log \left( \prod_{i=1}^{N} |\sigma_i - z| \right) - \mathbf{E}(\cdots) = \log \left|\det\big(X - z\big)\right| - \mathbf{E}(\cdots).$$

Conjecture (Non–Hermitian FHK):

$$\max_{|z|\le 1} \Psi_N(z) = \frac{1}{\sqrt{2}} \left( \log N - \frac{3}{4} \log\log N + \xi_N \right).$$

No conjectures about $\xi_N$ for any 2D ensemble!

---

**Theorem (C., Landon 2024)**

Let $X$ be a real or complex i.i.d. matrix then for any $\epsilon > 0$ we have

$$\lim_{N\to\infty} \mathbf{P}\left( \left( \frac{1}{\sqrt{2}} - \epsilon \right) \log N \le \max_{|z|\le 1} \Psi_N(z) \le \left( \frac{1}{\sqrt{2}} + \epsilon \right) \log N \right) = 1.$$

Remark 1: Previously known only when $X$ is complex Ginibre **(Lambert 2019)**.

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log\left(\prod_{i=1}^{N} |\sigma_i - z|\right) - \mathbf{E}(\cdots) = \log\left|\det\big(X - z\big)\right| - \mathbf{E}(\cdots).$$

Conjecture (Non–Hermitian FHK):

$$\max_{|z|\le 1} \Psi_N(z) = \frac{1}{\sqrt{2}}\left(\log N - \frac{3}{4}\log\log N + \xi_N\right).$$

No conjectures about $\xi_N$ for any 2D ensemble!

---

**Theorem (C., Landon 2024)**

Let $X$ be a real or complex i.i.d. matrix then for any $\epsilon > 0$ we have

$$\lim_{N\to\infty} \mathbf{P}\left(\left(\frac{1}{\sqrt{2}} - \epsilon\right)\log N \le \max_{|z|\le 1} \Psi_N(z) \le \left(\frac{1}{\sqrt{2}} + \epsilon\right)\log N\right) = 1.$$

Remark 1: Previously known only when $X$ is complex Ginibre **(Lambert 2019)**.

Remark 2: Note that even the real Ginibre case was not known!

# Non–Hermitian Fyodorov–Hiary–Keating conjecture

Consider the (centered) log–characteristic polynomial (here $\sigma_i \in \mathrm{Spec}(X)$)

$$\Psi_N(z) := \log \left( \prod_{i=1}^{N} |\sigma_i - z| \right) - \mathbf{E}(\cdots) = \log \left| \det(X - z) \right| - \mathbf{E}(\cdots).$$

Conjecture (Non–Hermitian FHK):

$$\max_{|z| \le 1} \Psi_N(z) = \frac{1}{\sqrt{2}} \left( \log N - \frac{3}{4} \log \log N + \xi_N \right).$$

No conjectures about $\xi_N$ for any 2D ensemble!

---

**Theorem (C., Landon 2024)**

Let $X$ be a real or complex i.i.d. matrix then for any $\epsilon > 0$ we have

$$\lim_{N \to \infty} \mathbf{P} \left( \left( \frac{1}{\sqrt{2}} - \epsilon \right) \log N \le \max_{|z| \le 1} \Psi_N(z) \le \left( \frac{1}{\sqrt{2}} + \epsilon \right) \log N \right) = 1.$$

Remark 1: Previously known only when $X$ is complex Ginibre **(Lambert 2019)**.

Remark 2: Note that even the real Ginibre case was not known!

Remark 3: Maximum over $|\Im z| \sim N^{-\alpha}$ different in real and complex case ($\neq$ union bound).

# Sketch proof NH FHK (first order)

# Hermitization+Regularization

Step 1: Notice that (Girko's formula)

$$\log \left|\det(X - z)\right| = \frac{1}{2} \log \det[(X - z)(X - z)^*] = \sum_i \log \lambda_i^z,$$

with $\lambda_i^z$ the singular values of $X - z$.

# Hermitization+Regularization

Step 1: Notice that (Girko's formula)

$$\log \left|\det(X - z)\right| = \frac{1}{2} \log \det[(X - z)(X - z)^*] = \sum_i \log \lambda_i^z,$$

with $\lambda_i^z$ the singular values of $X - z$.

GOAL:

$$\left(\frac{1}{\sqrt{2}} - \epsilon\right) \log N \leq \sup_{|z| \leq 1} \left[\sum_i \log \lambda_i^z - \mathbf{E}(\cdots)\right] \leq \left(\frac{1}{\sqrt{2}} + \epsilon\right) \log N$$

# Hermitization+Regularization

Step 1: Notice that (Girko's formula)

$$\log\left|\det(X-z)\right| = \frac{1}{2}\log\det[(X-z)(X-z)^*] = \sum_i \log\lambda_i^z,$$

with $\lambda_i^z$ the singular values of $X-z$.

GOAL:

$$\left(\frac{1}{\sqrt{2}}-\epsilon\right)\log N \leq \sup_{|z|\leq 1}\left[\sum_i \log\lambda_i^z - \mathbf{E}(\cdots)\right] \leq \left(\frac{1}{\sqrt{2}}+\epsilon\right)\log N$$

Step 2: By comparison for extremal statistics **(Landon–Lopatto–Marcinek 2018)** it is enough

$$\Psi_N(z,t) := \sup_{|z|\leq 1}\sum_i \log\lambda_i^z(t) - \mathbf{E}(\cdots) \approx \frac{\log N}{\sqrt{2}}.$$

Here $\lambda_i^z(t)$ are the singular values of $X_t - z$, with $X_t = X + \sqrt{t}X_{\mathrm{Gin}}$ and $t = o(1)$.

# Branching random walk representation

Step 3: How to get the branching structure?

## Branching random walk representation

Step 3: How to get the branching structure?

Instead of convergence (of moments char. pol.) to Gaussian Multiplicative Chaos (GMC),

# Branching random walk representation

Step 3: How to get the branching structure?

Instead of convergence (of moments char. pol.) to Gaussian Multiplicative Chaos (GMC), we use Dyson Brownian Motion (DBM)!

---

The singular values of $X + \sqrt{t}X_{\mathrm{Gin}} - z$ are the solution of the DBM:

$$\mathrm{d}\lambda_i^z(t) = \frac{\mathrm{d}b_i^z(t)}{\sqrt{2N}} + \frac{1}{2N}\sum_{j\neq i}\frac{1}{\lambda_i^z(t) - \lambda_j^z(t)}\,\mathrm{d}t.$$

# Branching random walk representation

Step 3: How to get the branching structure?

Instead of convergence (of moments char. pol.) to Gaussian Multiplicative Chaos (GMC),

we use Dyson Brownian Motion (DBM)!

---

The singular values of $X + \sqrt{t}X_{\mathrm{Gin}} - z$ are the solution of the DBM:

$$\mathrm{d}\lambda_i^z(t) = \frac{\mathrm{d}b_i^z(t)}{\sqrt{2N}} + \frac{1}{2N}\sum_{j\neq i}\frac{1}{\lambda_i^z(t)-\lambda_j^z(t)}\,\mathrm{d}t.$$

Define $t_k := N^{ck}/N$, for a small $c > 0$. Using DBM:

$$\Psi_N(z,t) \approx \sum_{k=1}^{K} Y_k(z), \qquad Y_k(z) := \sum_i \int_{t_{k-1}}^{t_k} \frac{\mathrm{d}b_i^z(s)}{\lambda_i^z(s) - \mathrm{i}s}.$$

# Branching random walk representation

Step 3: How to get the branching structure?

Instead of convergence (of moments char. pol.) to Gaussian Multiplicative Chaos (GMC),

we use Dyson Brownian Motion (DBM)!

---

The singular values of $X + \sqrt{t}X_{\mathrm{Gin}} - z$ are the solution of the DBM:

$$d\lambda_i^z(t) = \frac{db_i^z(t)}{\sqrt{2N}} + \frac{1}{2N}\sum_{j\neq i}\frac{1}{\lambda_i^z(t)-\lambda_j^z(t)}\,dt.$$

Define $t_k := N^{ck}/N$, for a small $c > 0$. Using DBM:

$$\Psi_N(z,t) \approx \sum_{k=1}^{K} Y_k(z), \qquad Y_k(z) := \sum_i \int_{t_{k-1}}^{t_k} \frac{db_i^z(s)}{\lambda_i^z(s) - \mathrm{i}s}.$$

Remark: Branching structure coming from $\sum_k Y_k(z)$:

$$\mathrm{Var}[Y_k(z)] \approx \frac{1}{K}\log N, \qquad \mathrm{Cov}\big[Y_k(z_1), Y_k(z_2)\big] \approx \begin{cases} \frac{1}{K}\log N & \text{if} \quad |z_1-z_2|^2 \ll t_k, \\ 0 & \text{if} \quad |z_1-z_2|^2 \gg t_k. \end{cases}$$

**Main results:**

- **Emergence of a 3D log–correlated field with respect to parabolic distance.**
- **Computation first order $\max_z \log |\det(X - z)|$ for general i.i.d. matrices.**

## Summary

### Main results:

- Emergence of a **3D log–correlated field** with respect to **parabolic distance**.
- Computation first order $\max_z \log |\det(X - z)|$ for **general i.i.d. matrices**.

### Main technical inputs:

- Branching structure from DBM.
- Analysis of singular values via weakly correlated DBMs.
- Proof of local laws for products of resolvents.

THANK YOU VERY MUCH FOR YOUR ATTENTION!