Zeros of random power series with stationary Gaussian coefficients

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Random Matrices and Related Topics in Jeju

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Contents

Gaussian power series

- $\{a_k\}$: a deterministic (non-random) sequence of complex numbers
- $\{\zeta_k(\omega)\}$: **i.i.d.** $\sim N_{\mathbb{C}}(0,1)$, standard complex normal.
- The random power series

$$X(z) = X(z,\omega) = \sum_{k=0}^{\infty} a_k \zeta_k(\omega) z^k$$

defines a Gaussian analytic function (**GAF**) in the same circle of convergence for the deterministic power series $X(z) = \sum_{k=0}^{\infty} a_k z^k$.

- **Covariance kernel**: $S_X(z,w) = E[X(z)\overline{X(w)}] = \sum_{k=0}^{\infty} |a_k|^2 (z\bar{w})^k$ determines GAF.

Important example (hyperbolic GAF): $a_k \equiv 1$ $(\forall k = 0, 1, \dots)$

$$X_{hyp}(z) := \sum_{k=0}^{\infty} \zeta_k z^k \text{ on } \mathbb{D} \Longrightarrow S(z,w) = \frac{1}{1 - z\overline{w}} \quad \text{(Szegő kernel)}$$

Fact. For each $z \in \mathbb{D} = \{|z| < 1\}$, $X_{hyp}(z) \sim N_{\mathbb{C}}(0, (1 - |z|^2)^{-1})$.

Stationary AR(p) model

Autoregressive model AR(p)

$$Y_t = c + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \cdots + \varphi_p Y_{t-p} + \zeta_t \quad (t \in \mathbb{Z})$$

where $\{\zeta_t\}_{t\in\mathbb{Z}}$ are i.i.d. noise.

- AR(1): For $|z| < 1$ and $c = 0$,

$$Y_t = zY_{t-1} + \zeta_t \quad (t \in \mathbb{Z}) \tag{*}$$

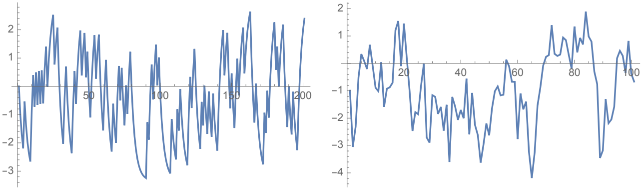

Figure: AR(1) with $z = 0.7$: Bernoulli noise (left) and $\mathbb{R}$-Gaussian noise (right)

- $X_{hyp}(z)$ is the stationary solution to (*) with $\mathbb{C}$-Gaussian noise.

Gaussian power series from AR(1)-model

- AR(1): For $|z| < 1$,

$$Y_t = zY_{t-1} + \zeta_t \quad (t \in \mathbb{Z})$$

- By introducing the shift operator $(Sx)_t = x_{t-1}$ $(t \in \mathbb{Z})$, we have

$$Y_t = (zSY)_t + \zeta_t \Longrightarrow Y_t = \{(1 - zS)^{-1}\zeta\}_t \Longrightarrow Y_t = \sum_{k=0}^{\infty} z^k (S^k \zeta)_t$$

- By expanding the RHS of the equation, we have

$$Y_t = Y_t(z) = \sum_{k=0}^{\infty} z^k \zeta_{t-k} \stackrel{d}{=} X_{hyp}(z) \quad (\forall t \in \mathbb{Z})$$

- $\mathbb{Y} = \{Y_t\}_{t \in \mathbb{Z}}$ forms a stationary GAF-valued process.

Peres-Virág's theorem

Theorem (Peres-Virág (2005))

The zeros of the hyperbolic GAF

$$X_{hyp}(z) = \sum_{k=0}^{\infty} \zeta_k z^k \quad \text{on } \mathbb{D}$$

is the determinantal point process associated with Bergman kernel

$$K(z, w) = \frac{1}{\pi(1 - z\overline{w})^2}.$$

Determinantal point process (DPP)

A point process is said to be a determinantal point process if there exists a kernel $K(z, w)$ such that the n-th correlation function is given by

$$\rho^n(z_1, \dots, z_n) = \det(K(z_i, z_j))_{i,j=1}^n,$$

In particular, the density of points is $\rho^1(z) = K(z, z)$.

Several extensions

- Krishnapur(2009): $\{G_k\}_{k=0}^{\infty}$ are i.i.d. $p \times p$ Ginibre matrices $\Longrightarrow$ DPP:

$$X_{matrix}(z) = \det\left(\sum_{k=0}^{\infty} G_k z^k\right)$$

- Forrester(2010), Matsumoto-S.(2013): $\{\zeta_k^{\mathbb{R}}\}_{k=0}^{\infty}$ are i.i.d. *real* Gaussian random variables $\Longrightarrow$ Pfaffian:

$$X_{real}(z) = \sum_{k=0}^{\infty} \zeta_k^{\mathbb{R}} z^k$$

- Katori-S.(2022): the i.i.d. Gaussian *Laurant series* on the annulus $\mathbb{A}_q$:

$$X_{\mathbb{A}_q}(z) = \sum_{k\in\mathbb{Z}} \zeta_k \frac{z^k}{\sqrt{1+q^{k+1}}}$$

- Noda-S.(2022): $\{\xi_k\}_{k=0}^{\infty}$ are *finitely dependent*, stationary Gaussian process coefficients. Expected number of points inside the ball:

$$X_{dep}(z) = \sum_{k=0}^{\infty} \xi_k z^k$$

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Density of zeros of GAF

Theorem (Edelman-Kostlan)

Let $X(z)$ be a GAF on D with covariance function $S_X(z,w)$. Then, the 1-correlation function of the zero process $\mathcal{Z}_X := \sum_{z \in D : X(z)=0} \delta_z$ of $X(z)$ (= the density of zeros) at z with $S_X(z,z) > 0$ is given by

$$\rho_X^1(z) = \frac{1}{4\pi} \Delta \log S_X(z,z) = \frac{1}{\pi} \partial_z \partial_{\overline{z}} \log S_X(z,z).$$

Ex.(hyperbolic GAF): $X_{hyp}(z) := \sum_{k=0}^{\infty} \zeta_k z^k$ on $\mathbb{D}$ ζ_k i.i.d. $\sim N_{\mathbb{C}}(0,1)$.
Then,

$$S_{X_{hyp}}(z,w) = \sum_{k=0}^{\infty} (z\overline{w})^k = \frac{1}{1 - z\overline{w}} \qquad \textbf{(Szegő kernel)}$$

and then

$$\rho_X^1(z) = \frac{1}{\pi} \partial_z \partial_{\overline{z}} \log \frac{1}{1-|z|^2} = \frac{1}{\pi(1-|z|^2)^2} \qquad \textbf{(hyperbolic volume)}$$

Calabi's rigidity for GAF

By analyticity of X, the information of the diagonal $S_X(z,z)$ recovers the off-diagonal $S_X(z,w)$. From this fact, we have the following:

Theorem (Sodin)

Let X and Y be GAF on D. If the 1-correlation functions $\rho^1_X(z)$ and $\rho^1_Y(z)$ of the zero processes $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ coincide, then there exists a non-vanishing, non-random analytic function h such that

$$Y \stackrel{d}{=} hX.$$

In particular, $\mathcal{Z}_X \stackrel{d}{=} \mathcal{Z}_Y$.

Example: This theorem implies that GAF on $\mathbb{D}$ whose density of zeros is the hyperbolic volume $\frac{1}{\pi(1-|z|^2)^2}$ is essentially unique in law, which is nothing but $X_{hyp}(z)$.

Our setting

- $\Xi = \{\xi_k\}_{k\in\mathbb{Z}}$ is a stationary, centered, complex Gaussian process with the covariance function $\gamma : \mathbb{Z} \to \mathbb{C}$, i.e.,

$$\gamma(k-\ell) = \mathbb{E}[\xi_k \overline{\xi_\ell}]$$

  with $\gamma(0) = 1$. In particular, $\xi_k \sim N_{\mathbb{C}}(0,1)$ for each $k \in \mathbb{Z}$.
- We consider the Gaussian power series with the covariance above:

$$X(z) = X_{\Xi}(z) := \sum_{k=0}^{\infty} \xi_k z^k$$

- If $\{\xi_k\}_{k\in\mathbb{Z}}$ are i.i.d., i.e., $\gamma(k) = \delta_{k,0}$, the GAF is $X_{hyp}(z)$.

Fact: All such GAFs are on $\mathbb{D}$

The convergence radius of X_{Ξ} is almost surely 1. Then, $X_{\Xi}(z)$ is defined on $\mathbb{D}$ and its zeros are located inside $\mathbb{D}$ if exists.

Covariance function and spectral function

- Covariance kernel: There is a special covariance structure:

$$S_X(z,w) = S_{X_{hyp}}(z,w)G_2(z,w) = \underbrace{\frac{1}{1 - z\overline{w}}}_{\text{Szegő kernel}} \times \underbrace{G_2(z,w)}_{\text{spectral density}}$$

where

$$G_2(z,w) = 1 + G(z) + \overline{G(w)}, \quad G(z) = \sum_{k=1}^{\infty} \overline{\gamma(k)} z^k.$$

- **Spectral measure** $dF(\theta)$**:** Since $\gamma(k)$ is positive definite,

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\theta} dF(\theta)$$

- If $dF(\theta) = F'(\theta)\frac{d\theta}{2\pi}$, then $F'(\theta)$ is called the spectral density.

Example 1: 1-dependent case

$$\gamma(k) = \begin{cases} 1 & k = 0 \\ a & k = \pm 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } |a| \le 1/2$$

- $G(z) = az, \quad G_2(z, w) = 1 + a(z + \overline{w})$
- It is easy to see that

$$\int_{-\pi}^{\pi} e^{ik\theta}(1 + ae^{i\theta} + ae^{-i\theta})\frac{d\theta}{2\pi} = \gamma(k)$$

- This means that

$$F'(\theta) = G_2(e^{i\theta}, e^{i\theta}) = 1 + 2a\cos\theta.$$

Spectral density

When $G(z)$ is nice enough, we have

$$F'(\theta) = G_2(e^{i\theta}, e^{i\theta})$$

Example 2: Gaussian Markov case

For $0 \le \rho < 1$ and $\{\zeta_n\}_{n\in\mathbb{Z}}$ i.i.d. $\sim N_{\mathbb{C}}(0,1)$,

$$\xi_n := \sqrt{1-\rho^2}\sum_{k=0}^{\infty}\rho^k\zeta_{n-k} \left(\stackrel{d}{=} \sqrt{1-\rho^2}X_{hyp}(\rho)\right)$$

$$\gamma(k) = \rho^{|k|} \quad (0 < \rho < 1)$$

- $G(z) = \dfrac{\rho z}{1-\rho z}, \quad G_2(z,z) = \dfrac{1-\rho^2 z\bar{z}}{(1-\rho z)(1-\rho\bar{z})}.$
- $G(z)$ is analytic in $|z| < \rho^{-1}$,

$$F'(\theta) = \frac{1-\rho^2}{(1-\rho e^{i\theta})(1-\rho e^{-i\theta})} = \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2} > 0$$

Expected number of zeros of X_{Ξ}

- $N_X(D) = \#\{z \in D : X(z) = 0\}$: the number of zeros inside D.
- From the Edelman-Kostlan formula and the Stokes formula,

$$\mathbb{E}[N_X(D)] = \frac{1}{4\pi}\int_D \Delta \log S_X(z,z)dm(z) = \frac{1}{2\pi i}\oint_{\partial D} \partial_z \log S_X(z,z)dz$$

- In the present setting, since $S_X(z,z) = S_{X_{hyp}}(z,z)G_2(z,z)$

$$\mathbb{E}[N_X(D)] = \underbrace{\frac{1}{2\pi i}\oint_{\partial D}\frac{\overline{z}}{1-|z|^2}dz}_{\text{main term}} + \underbrace{\frac{1}{2\pi i}\oint_{\partial D}\frac{G'(z)}{G_2(z,z)}dz}_{\text{error term}}$$

- We focus on the case where $D = \mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$. We write $N_X(r)$ for $N_X(\mathbb{D}_r)$.

$$\mathbb{E}[N_X(r)] = \underbrace{\frac{r^2}{1-r^2}}_{\mathbb{E}[N_{X_{hyp}}(r)]} + \underbrace{\mathcal{J}(r)}_{\text{error term}}$$

Examples: the error term is $O(1)$

- **Example 0.** (i.i.d. case, hypberbolic GAF) When $\gamma(k) = \delta_{k,0}$,

$$\mathbb{E}[N_{X_{hyp}}(r)] = \frac{r^2}{1-r^2}.$$

- **Example 1.** (1-dependent) When $|a| < 1/2$,

$$\mathbb{E}[N_X(r)] = \frac{r^2}{1-r^2} - \frac{1}{2}\left(\frac{1}{\sqrt{1-4a^2}} - 1\right) + O(1-r^2) \quad \text{as } r \to 1$$

- **Example 2.** (Gaussian Markov) When $\gamma(k) = \rho^{|k|}$ $(\rho \in (0,1))$,

$$\mathbb{E}[N_X(r)] = \frac{r^2}{1-r^2} - \frac{\rho^2}{1-\rho^2} + O(1-r^2) \quad \text{as } r \to 1$$

Remark. For all the above cases, the spectral measures are absolutely continuous and their spectral density are strictly positive.

Examples: the error term is $O((1-r^2)^{-1/2})$ or more

- **Example 1.** (1-dependent) When $|a| = 1/2$,

$$\mathbb{E}[N_X(r)] = \frac{r^2}{1-r^2} - \frac{1}{2}\frac{1}{\sqrt{1-r^2}} + O(1) \quad \text{as } r \to 1$$

When $|a| = 1/2$, the spectral measure has zeros on the unit circle, a.e.

$$F'(\theta) = 1 + 2a\cos\theta = 1 \pm \cos\theta \geq 0.$$

For a certain kernel related to fractional BM, S. Mukeru et al. shows that

$$\frac{r^2}{1-r^2} - K_1\left(\frac{1}{2\sqrt{1-r^2}} - \frac{1}{2}\right) \leq \mathbb{E}[N_X(r)] \leq \frac{r^2}{1-r^2} - K_2\left(\frac{1}{2\sqrt{1-r^2}} - \frac{1}{2}\right)$$

for $0 < r < 1$.

Result A: Comparison of the expected number of zeros

Proposition (Noda-S.)

- $D \subset \mathbb{D}$: ∂D: a domain with smooth boundary
- $N_X(D) = \#\{z \in D : X(z) = 0\}$: the number of zeros inside D

$$\mathbb{E}[N_X(D)] \leq \mathbb{E}[N_{X_{hyp}}(D)].$$

The equality "=" holds for some (also any) domain D if and only if $X(z) \overset{d}{=} X_{hyp}(z)$.

When $D = \mathbb{D}_r$,

$$\mathbb{E}[N_X(r)] = \underbrace{\frac{r^2}{1-r^2}}_{\mathbb{E}[N_{X_{hyp}}(r)]} + \underbrace{\mathcal{J}(r)}_{\text{error term}},$$

where

$$\mathcal{J}(r) = \frac{1}{\pi}\int_{\mathbb{D}_r} \partial_z \partial_{\overline{z}} \log G_2(z,z) dm(z) = -\frac{1}{\pi}\int_{\mathbb{D}_r} \frac{|G'(z)|^2}{G_2(z,z)^2} dm(z) \leq 0.$$

Error term coming from modified spectral function

The error term can be expressed as

$$\mathcal{J}(r) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_r} \frac{G'(z)}{G_2(z,z)} dz = \frac{r}{2\pi i} \oint_{\partial \mathbb{D}} \frac{G'(rz)}{\Theta_r(z)} dz.$$

Note that, since $\overline{z} = r^2/z$ on $\mathbb{D}_r$,

$$\Theta_r(z) := \left(G_2(z,z)\big|_{\overline{z}=\frac{r}{z}} \right)\Big|_{z \to rz} = \sum_{k \in \mathbb{Z}} \gamma(k) r^{|k|} z^k, \quad G(z) = \sum_{k=1}^{\infty} \overline{\gamma(k)} z^k.$$

Remark. $\Theta_1(e^{i\theta})$ on $\partial \mathbb{D}$ is equal to the spectral density $F'(\theta)$ when the spectral measure is absolutely continuous. Roughly speaking, as $r \to 1$, the poles of the integrand in $\mathcal{I}(r)$ approach to the zeros of $F'(\theta)$ if exist.

Example 2: Gaussian Markov case. O(1)-error

$$\mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial \mathbb{D}} \frac{G'(rz)}{\Theta_r(z)} dz$$

where

$$G'(rz) = \frac{\rho}{(1-\rho r z)^2}, \quad \Theta_r(z) = \frac{(1-\rho^2 r^2) z}{(1-\rho r z)(z-\rho r)}.$$

- Note that $F'(\theta)$ is strictly positive on $\partial \mathbb{D}$.

The zero of $\Theta_r(z)$ is $z = 0$ independent of r.

$$\begin{aligned}\mathcal{J}(r) &= \frac{r}{2\pi i} \oint_{\partial \mathbb{D}} \frac{\rho(z-\rho r)}{1-\rho r z} \frac{1}{(1-\rho^2 r^2) z} dz \\ &= \frac{-\rho^2 r^2}{1-\rho^2 r^2} \\ &= -\frac{\rho^2}{1-\rho^2} + O(1-r^2)\end{aligned}$$

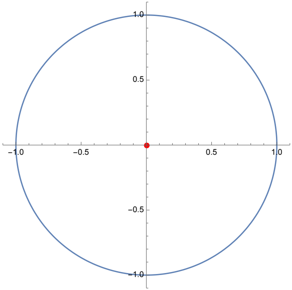


Figure: Zero of $\Theta_r(z)$ = Pole

Example 1: 1-dependent case. $O(1)$ or $O((1-r^2)^{-1/2})$ error.

$$\mathcal{J}(r) = \frac{r\nu_r}{\nu_r - \nu_r^{-1}}$$

1. When $|a| < 1/2$, $F'(\theta) > 0$.
   As $r \to 1$,
   $$\nu_r \to \frac{-1+\sqrt{1-4a^2}}{2a} \in (-1,1).$$
   $$\mathcal{J}(r) = -\frac{1}{2}\Big(\frac{1}{\sqrt{1-4a^2}} - 1\Big) + O(1).$$
2. $|a| = 1/2$. When $a = 1/2$,
   $F'(\pi) = 0$.
   As $r \to 1$, $\nu_r, \nu_r^{-1} \to e^{i\pi} = -1$.
   $$\nu_r - \nu_r^{-1} = \frac{\sqrt{1-4a^2r^2}}{ar} = \frac{2\sqrt{1-r^2}}{r}$$
   $$\mathcal{J}(r) = -\frac{1}{2}\frac{1}{(1-r^2)^{1/2}} + O(\sqrt{1-r^2}).$$

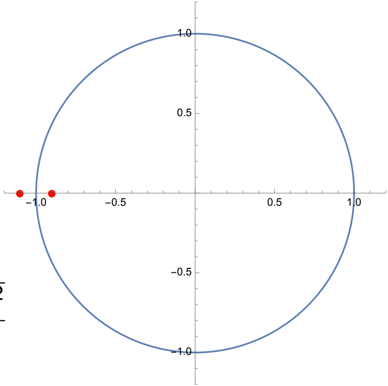


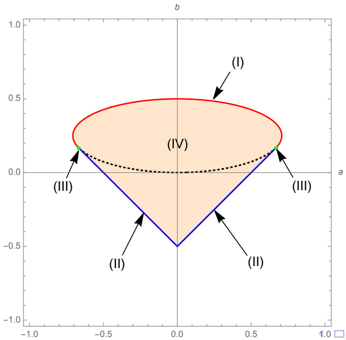
Figure: Zeros of $\Theta_r(z)$ = Poles

Example 3. 2-dependent case

We consider the coefficient Gaussian process $\{\xi_k\}_{k\in\mathbb{Z}}$ with the following 2-dependent covariance matrix of the form:

$$\gamma_{a,b}(k) = \begin{cases} 1 & (k=0) \\ a & (k=\pm 1) \\ b & (k=\pm 2) \\ 0 & (\text{otherwise}). \end{cases}$$

$\{\gamma_{a,b}(k)\}_{k\in\mathbb{Z}}$ is positive definite iff $(a,b)\in\mathcal{P}$, where $\mathcal{P}$ is drawn below.



Result B: 2-dependent case

Theorem (Noda-S.)

The asymptotic behavior of the expected number of zeros $\mathbb{E}[N_{X_{a,b}}(r)]$ is given by the following: as $r \to 1$,

1. $(a, b) \in \mathcal{P}_{(I)}$

   $$\mathbb{E}[N_{X_{a,b}}(r)] = \frac{r^2}{1-r^2} - \sqrt{\frac{2b}{6b-1}}\frac{1}{(1-r^2)^{1/2}} + O(1)$$

2. $(a, b) \in \mathcal{P}_{(II)}$

   $$\mathbb{E}[N_{X_{a,b}}(r)] = \frac{r^2}{1-r^2} - \frac{1}{2}\sqrt{\frac{1-2b}{1-6b}}\frac{1}{(1-r^2)^{1/2}} + O(1)$$

3. $(a, b) = (\pm 2/3, 1/6) = \mathcal{P}_{(III)}$

   $$\mathbb{E}[N_{X_{a,b}}(r)] = \frac{r^2}{1-r^2} - \frac{1}{2^{5/4}}\frac{1}{(1-r^2)^{3/4}} + O\left(\frac{1}{(1-r^2)^{1/4}}\right)$$

4. $(a, b) \in \mathcal{P}_{(IV)}$

   $\exists C(a, b) \geq 0$ s.t. $\mathbb{E}[N_{X_{a,b}}(r)] = \frac{r^2}{1-r^2} - C(a, b) + O\left(1 - r^2\right)$

Remark. $X_{0,0}(z) = X_{hyp}(z)$ is in the case $\mathcal{P}_{(IV)}$.

Zeros of $\Theta_r(z)$ as $r \to 1$

The error term can be expressed as

$$\mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial\mathbb{D}} \frac{G'(rz)}{\Theta_r(z)} dz.$$

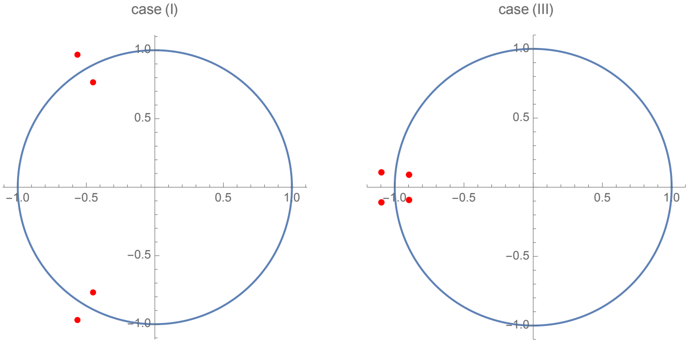


Figure: case(i) and (ii) $F'(\theta)$ has two zeros of multiplicity 2. case(iii) $F'(\theta)$ has a zero of multiplicity 4.

n-dependent case

- We consider the n-dependent, stationary complex Gaussian process.

$$\gamma_n(k) = \binom{2n}{n+k}\binom{2n}{n}^{-1} \text{ for } |k| = 0, 1, 2, \ldots, n; \quad 0 \text{ otherwise.}$$

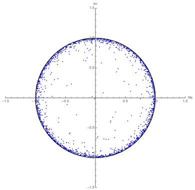


Figure: i.i.d. case

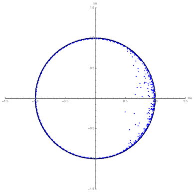


Figure: $\{\gamma_{30}(k)\}_{k\in\mathbb{Z}}$ case

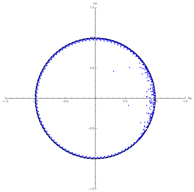


Figure: $\{\gamma_{60}(k)\}_{k\in\mathbb{Z}}$ case

$$F'(\theta) = \binom{2n}{n}^{-1}\left(2\cos\frac{\theta}{2}\right)^{2n} \quad \theta \in [0, 2\pi),$$

where $\theta = \pi$ is the zeros of multiplicity $2n$.

Zeros of $\Theta_r(z)$ as $r \to 1$

The error term can be expressed as

$$\mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial\mathbb{D}} \frac{G'(rz)}{\Theta_r(z)} dz.$$

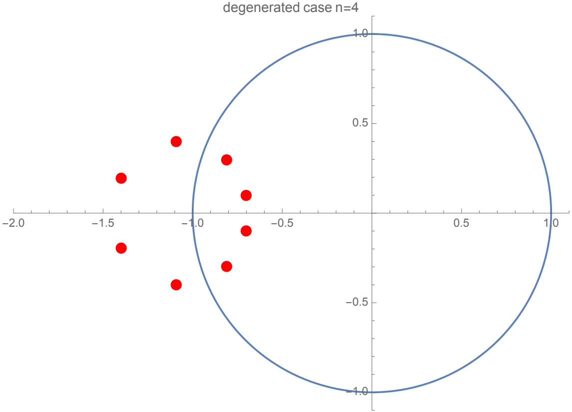


Figure: The degenerated case for $n = 4$. Red points are the zeros of $\Theta_r(z)$

Result C: n-dependent case

Theorem (Noda-S.)

Let $\Xi = \{\xi_k\}_{k\in\mathbb{Z}}$ be the Gaussian process with covariance function

$$\gamma_n(k) = \begin{cases} \binom{2n}{n+k}\binom{2n}{n}^{-1} & |k| = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise}, \end{cases}$$

and $X(z)$ be GAF with coefficients Ξ. Then,

$$\mathbb{E}[N_X(r)] = \frac{r^2}{1-r^2} - D_n(1-r^2)^{-\frac{2n-1}{2n}} + O\left((1-r^2)^{-\frac{2n-3}{2n}}\right) \quad \text{as } r \to 1,$$

where

$$D_n = \frac{1}{2n\sin\frac{\pi}{2n}}\left\{\binom{2(n-1)}{n-1}\right\}^{\frac{1}{2n}}.$$

Result D: Finitely dependent case

In general, if $F'(\theta)$ has a zero with multiplicity $2k$ on $(-\pi, \pi]$, the term $(1-r^2)^{-\frac{2k-1}{2k}}$ appears as $r \to 1$ in the asymptotics of $\mathbb{E}[N_X(r)]$. Hence we have the following:

Corollary (Noda-S.)

- $\Xi = \{\xi_k\}_{k\in\mathbb{Z}}$: finitely dependent, stationary, complex Gaussian process with mean 0 and variance 1.
- The spectral density $F'(\theta)$ of Ξ has zeros θ_j of multipicity $2k_j$ for $j = 1, 2, ..., p$.
- Set $N = \max_{1\le j\le p} k_j$.

Then, $\exists C_\Xi > 0$ s.t.

$$\mathbb{E}[N_X(r)] = \frac{r^2}{1-r^2} - C_\Xi (1-r^2)^{-\frac{2N-1}{2N}} + o\left((1-r^2)^{-\frac{2N-1}{2N}}\right) \quad \text{as } r \to 1.$$

Discussions so far

- The spectral measure of the coefficient Gaussian process plays an important role for zeros of GAF.
- So far we have seen finitely dependent cases, where the spectral function is a trigonometric polynomial.
- We should study the number of zeros on the sectorial domain $\{z \in \mathbb{D} : a < \arg z < b\}$ or its directional density.

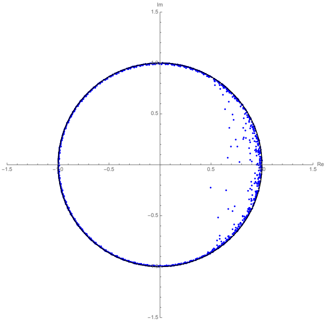


Figure: $\{\gamma_{30}(k)\}_{k\in\mathbb{Z}}$ case

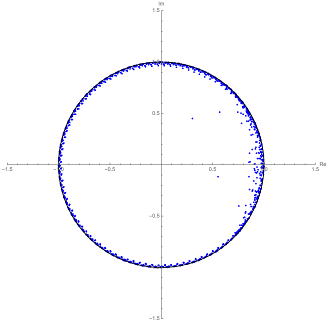


Figure: $\{\gamma_{60}(k)\}_{k\in\mathbb{Z}}$ case

Spectral measure point of view

- The spectral (probability) measure of $\Xi = \{\xi_k\}_{k\in\mathbb{Z}}$ is $dF(s)$, i.e.,

$$\gamma(k) = \int_{-\pi}^{\pi} e^{-iks} dF(s)$$

- In this case, Ξ, and thus, $X_\Xi(z)$ has a spectral representation

$$X_\Xi(z) = X_\Xi(z) = \int_{-\pi}^{\pi} \frac{1}{1 - ze^{-is}} dZ(s)$$

  where Z is a Gaussian random measure on $(-\pi, \pi]$ with independent increment and

$$\mathbb{E}[Z(A)\overline{Z(B)}] = \int_{A\cap B} dF(t).$$

- The covariance kernel of $X(z)$ is given by

$$S_X(z, w) = \int_{-\pi}^{\pi} \frac{1}{1 - ze^{-is}} \cdot \overline{\frac{1}{1 - we^{-is}}} dF(s)$$

Spectral representation of 1-intensity

- When $z = re^{i\varphi}$,

$$S_X(z,z) = \int_{-\pi}^{\pi} \frac{1}{|1 - ze^{-it}|^2} dF(t) = \frac{1}{1-r^2} \int_{-\pi}^{\pi} P_r(\varphi - t) dF(t),$$

  where $P_r(\theta)$ is the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

- When $dF(t) = f(t)dt$, [O

$$S_X(z,z) = \frac{1}{1-|z|^2} \mathbb{E}_z[f(B_\tau)],$$

  where B_t is the complex Brownian motion and τ is the first hitting time to $\partial\mathbb{D}$.

- The variance of $X(z)$ grows according to the value of $\mathbb{E}_z[f(B_\tau)]$.

The i.i.d. case and the case $f(s) = \mathbf{1}_{-[\pi/2,\pi/2]}(s)$

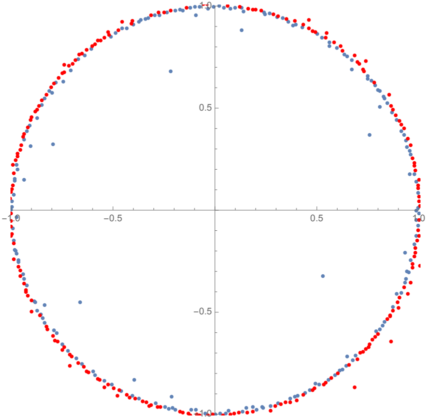


Figure: Zeros of finite approximations of degree 400. Left: i.i.d. case $X_{hyp}(z)$, whose spectral measure is Lebesgue measure and Right: $X_{\Xi}(z)$ for the spectral measure $f(s) = \mathbf{1}_{-[\pi/2,\pi/2]}(s)$. Zeros inside the disc is in blue and those outside the disc is in red.

The case $f(s) = \mathbf{1}_{-[\pi/2,\pi/2]}(s)$

$$\rho_1(re^{i\varphi}) = \frac{1}{\pi(1-r^2)^2} \times \begin{cases} 1 - \dfrac{\frac{(1-r^2)^2}{1+2r^2\cos(2\varphi)+r^4}}{\left(\pi - \arctan\left(\frac{(1-r^2)\sec\varphi}{2r}\right)\right)^2} & \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ 1 - (\frac{2}{\pi})^2 & \varphi = \pm\frac{\pi}{2} \\ 1 - \dfrac{\frac{(1-r^2)^2}{1+2r^2\cos(2\varphi)+r^4}}{\arctan^2\left(\frac{(1-r^2)\sec\varphi}{2r}\right)} & \varphi \in (-\pi,\pi) \setminus (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

$$= \begin{cases} \dfrac{1}{\pi(1-r^2)^2} - \dfrac{1}{4\pi^2\cos^2\varphi} + O(1-r^2) & \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \left(1 - (\frac{2}{\pi})^2\right) \dfrac{1}{\pi(1-r^2)^2} & \varphi = \pm\frac{\pi}{2} \\ \dfrac{1}{12\pi\cos^2\varphi} + O(1-r^2) & \varphi \in (-\pi,\pi) \setminus (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

Spectral representation of 1-intensity of zeros

Proposition (S.)

Let $dF(s)$ be the spectral measure of the coefficient Gaussian process $\Xi = \{\xi_k\}_{k\in\mathbb{Z}}$ and $P_r(s)$ be the Poisson kernel. Then,

$$\rho_1(re^{i\varphi}) = \frac{1}{\pi(1-r^2)^2} \frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\big(1-\cos(t-s)\big)P_r(s)^2 P_r(t)^2 dF_\varphi(s)dF_\varphi(t)}{\Big(\int_{-\pi}^{\pi} P_r(s)dF_\varphi(s)\Big)^2}$$

where dF_φ is the shifted spectral measure defined as

$$dF_\varphi(s) = dF(s+\varphi).$$

We need the asymptotics of the Poisson(-type) integrals

$$\mathcal{P}_r(\mu) = \int_{-\pi}^{\pi} P_r(s)d\mu(s), \quad \mathcal{Q}_r(\mu) = \int_{-\pi}^{\pi} P_r(s)^2 d\mu(s).$$

Boundary behavior of Poisson integrals

Lemma

When h is symmetric and smooth around $s = 0$, as $r \to 1$,

$$\mathcal{P}_r(h) = h(0) + \frac{\mathcal{I}(Th)}{2} y + \frac{1}{4}\left\{\mathcal{I}(Th) - \frac{h''(0)}{2}\right\} y^2 + \frac{1}{8}\left\{\frac{3}{2}\mathcal{I}(Th) - h''(0) - \frac{1}{2}\mathcal{I}(T^2h)\right\} y^3 + O(y^4)$$

$$\mathcal{Q}_r(h) = 2h(0)y^{-1} - h(0) + \frac{1}{4}h''(0)y + \left(\frac{1}{4}\mathcal{I}(T^2h) + \frac{1}{8}h''(0)\right) y^2 + \left(\frac{1}{4}\mathcal{I}(T^2h) + \frac{1}{16}h''(0) - \frac{1}{64}h^{(4)}(0)\right) y^3 + \left(\frac{1}{4}\mathcal{I}(T^2h) - \frac{1}{16}\mathcal{I}(T^3h) + \frac{1}{32}h''(0) - \frac{3}{128}h^{(4)}(0)\right) y^4 + O(y^5),$$

where $y = 1 - r^2$ and $\mathcal{I}(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)ds,\ Th(s) = \frac{h(s)-h(0)}{1-\cos s}$.

Boundary behavior of the 1-intensity

Using the asymptotics before, we obtain the following asymptotic behavior of the directional density.

Theorem (S.)

Let $F_\varphi = F(\cdot + \varphi)$. Suppose $dF_\varphi(s) = f_\varphi(s)ds$ and $f_\varphi(s)$ is smooth around $s = 0$.

(i) When $f_\varphi(0) > 0$,

$$\rho_1(re^{i\varphi}) = \frac{1}{\pi(1-r^2)^2} - \frac{1}{4f_\varphi(0)^2}\left\{f'_\varphi(0)^2 + \mathcal{I}(T\widehat{f_\varphi})^2\right\} + O((1-r^2)).$$

(ii) When $f_\varphi(0) = 0$,

$$\rho_1(re^{i\varphi}) = \frac{1}{\pi(1-r^2)} \frac{f''_\varphi(0)}{2\mathcal{I}(T\widehat{f_\varphi})} + O(1).$$

Here

$$T\widehat{f_\varphi}(s) = \frac{f_\varphi(s) + f_\varphi(-s) - 2f_\varphi(0)}{2(1-\cos s)}, \quad \mathcal{I}(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)ds.$$

Boundary behavior of the 1-intensity

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$$\rho_1(re^{i\varphi}) = \frac{1}{\pi(1-r^2)}\frac{f''_\varphi(0)}{2\mathcal{I}(T\widehat{f_\varphi})} + O(1).$$

(iii) When $f_\varphi(0) = f''_\varphi(0) = 0$,

$$\rho_1(re^{i\varphi}) = \frac{1}{\pi}\frac{\mathcal{I}(T^2\widehat{f_\varphi})^2 - \mathcal{I}(T^2\widehat{f_\varphi}\cos s)^2 - \mathcal{I}(T^2\check{f_\varphi}\sin s)^2}{4\mathcal{I}(T\widehat{f_\varphi})^2} + O(1-r^2).$$

Boundary behavior of the 1-intensity

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Analytic continuation

- From the previous theorem (iii), if the spectral density $f(s) = 0$ on a closed arc I, then the number of zeros of X_Ξ near I is of $O(1)$.
- This suggests that $X_\Xi(z)$ could be analytically continued across I.

Theorem (S.)

Suppose F is absolutely continuous with $dF(t) = f(t)dt$. Then $I := (\mathrm{supp}\, f)^c$ is the regular set of X_Ξ a.s., i.e., $X_\Xi(z)$ can be analytically continued across I a.s. In particular, if Ξ is purely non-deterministic, then the circle of convergence is the natural boundary for X_Ξ.

Remark. The Gaussian process $\Xi = \{\xi_k\}_{k \in \mathbb{Z}}$ is *purely non-deterministic* iff the spectral measure $dF(s)$ is absolutely continuous and its density $f(s)$ satisfies $\int_{-\pi}^{\pi} \log f(s) ds > -\infty$.

Summary and future works

Summary:

- We considered the zeros of Gaussian power series with dependent Gaussian coefficients.
- More dependence on the coefficients, less zeros there are.
- Spectral measures play a crucial role for the asymptotic behavior of the density of zeros as $r \to 1$.

Future directions:

- Analyze the case where the spectral measure is *singular*.
- What is the variance behavior? CLT?

Thank you for your attention!

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Thank you for your attention!

Reference

Noda, K., Shirai, T. Expected Number of Zeros of Random Power Series with Finitely Dependent Gaussian Coefficients. J. Theor. Probab. 36, 1534–1554 (2023).

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