2D Coulomb Gases and Partition Functions

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Outline

2D Coulomb Gases

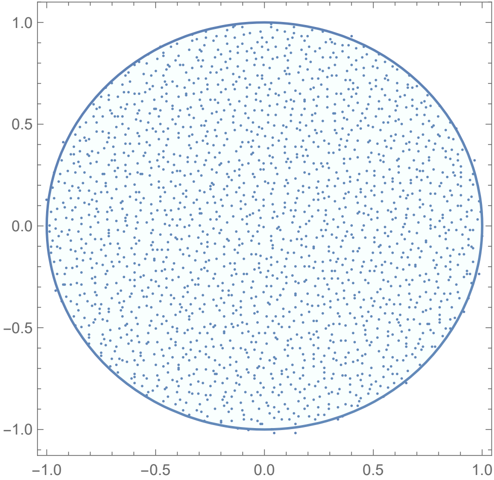
In the two-dimensional Coulomb gas model, we consider n particles as a system of point charges with the same sign located at points $\{z_j\}_{j=1}^n \in \mathbb{C}$, influenced by an external potential Q. We increase the number of point charges and the external potential such that in the scaling limit $(n \to \infty, N \to \infty$, while n/N is fixed), all the point charges are confined to a compact set in $\mathbb{C}$, which we call the **droplet** S_Q. The probability distribution is given by

$$\mathrm{d}\mathbf{P}_n = \frac{1}{\mathcal{Z}_n} \prod_{i<j} |z_i - z_j|^2 \cdot \exp\Big(- N \sum_{j=1}^{n} Q(z_j) \Big) \cdot \prod_{j=1}^{n} \mathrm{d}A(z_j),$$

where $\mathrm{d}A$ denotes the standard Lebesgue measure on the plane.
We focus on $n = N$.

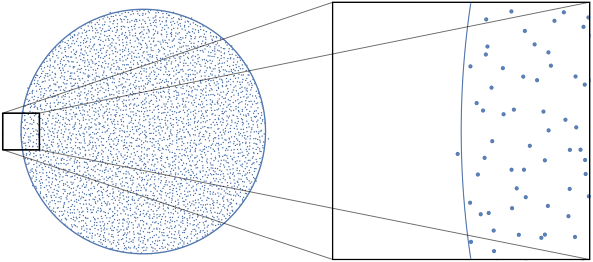
Ginibre Ensemble

$$Q(z) = |z|^2.$$



Ginibre, 1965, etc.

Statistical Behaviors



k-point correlation function

$$R_k(z_1, \ldots, z_k) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} \mathbf{P}_n \prod_{j=k+1}^{n} \mathrm{d}A(z_j),$$

where

$$\mathbf{P}_n = \frac{1}{\mathcal{Z}_n} \prod_{i<j} |z_i - z_j|^2 \cdot \exp\Big(-n \sum_{j=1}^{n} Q(z_j) \Big).$$

Correlation Function

$$R_k(z_1,\ldots,z_k)=\det\left[\mathbf{K}_n(z_i,z_j)\right]_{i,j=1}^{k},$$

where $\mathbf{K}_n(z,\zeta)$ is the correlation kernel given by

$$\mathbf{K}_n(z,\zeta):=\mathrm{e}^{-\frac{N}{2}Q(z)-\frac{N}{2}Q(\zeta)}\sum_{k=0}^{n-1}\frac{1}{h_k}p_k(z)\overline{p_k(\zeta)}.$$

Here $p_n(z)$ satisfies the orthogonality condition,

$$\int_{\mathbb{C}}p_n(z)\overline{p_m(z)}e^{-NQ(z)}\mathrm{d}A(z)=h_n\delta_{nm}\quad(n,m=0,1,2,\cdots).$$

A connection to orthogonal polynomials can be provided by *Heine's formula* i.e.,

$$p_n(z)=\mathbb{E}\prod_{j=1}^{n}(z-z_j).$$

Correlation Kernels

For general Q, when z_0 is in the bulk of the droplet, let $K_n(\xi, \eta)$ be the re-scaled correlation kernel, as $n \to \infty$, we have

$$\lim_{n\to\infty} c_n(\xi,\eta)K_n(\xi,\eta) = G(\xi,\eta) := \mathrm{e}^{\xi\overline{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)}.$$

Ameur, Hedenmalm, and Makarov (2011)
When z_0 is on the boundary of the droplet, let $K_n(\xi, \eta)$ be the re-scaled correlation kernel, as $n \to \infty$, we have

$$\lim_{n\to\infty} c_n(\xi,\eta)K_n(\xi,\eta) = G(\xi,\eta)\mathrm{erfc}(\xi+\overline{\eta}),\ \mathrm{erfc}(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty \mathrm{e}^{-t^2/2}\mathrm{d}t.$$

Hedenmalm and Wennman (2021)

Partition Functions: Predictions

Given the external potential Q, the partition function:

$$Z_n^Q := \int_{\mathbb{C}^n} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} e^{-n\,Q(z_j)}\,\mathrm{d}A(z_j).$$

It is conjectured that if the droplet is *connected*, as $n \to \infty$, the partition function Z_n^Q has the asymptotic expansion of the form

$$\log Z_n^Q = C_1 n^2 + C_2 n \log n + C_3 n + C_4 \log n + C_5 + o(1).$$

Partition Functions: Recent Progress

$$C_1 = -I_Q[\sigma_Q],$$

where σ_Q is the unique minimizer (a.k.a. the equilibrium measure) of the weighted logarithmic energy $I_Q[\mu]$,

$$I_Q[\mu] := \int_{\mathbb{C}^2} \log \frac{1}{|z-w|} \, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} Q \, d\mu.$$

Hedenmalm-Makarov (2013)

$$C_2 = \frac{1}{2}.$$

Sandier-Serfaty (2015)

$$C_3 = \frac{\log(2\pi)}{2} - 1 - \frac{1}{2} \int_{\mathbb{C}} \log(\Delta Q) \, d\sigma_Q.$$

Leblé-Serfaty (2017), Bauerschmidt-Bourgade-Nikula-Yau (2019)

Partition Functions: Predictions

$$C_4 = \frac{6-\chi}{12},$$

where χ is the Euler characteristic of the droplet.
Jancovici-Manificat-Pisani (1994), Telléz-Forrester (1999)

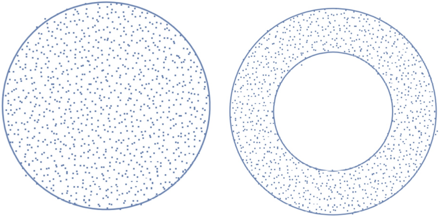
$$C_5 = \frac{\log(2\pi)}{2} + \chi\,\zeta'(-1) + F_Q,$$

where ζ is the Riemann zeta function, and F_Q can be expressed in terms of the zeta-regularized determinant of the exterior droplet,

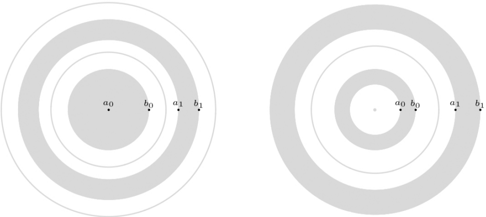
$$F_Q = -\frac{1}{2}\log\det\nolimits_\zeta(\Delta_{\mathbb{C}\setminus S_Q}).$$

Zabrodin-Wiegmann (2006)

If Q is radially symmetry: $Q(z) = Q(|z|)$.
For subharmonic potential $\Delta Q(z) > 0$:
Byun-Kang-Seo (2023)



For models with spectral gaps:
Ameur-Charlier-Cronvall (2023)



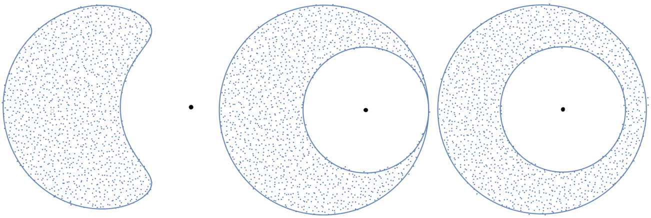
A Conditional Ginibre Ensemble

We consider the external potential,

$$Q(z) = |z|^2 + 2c \log \frac{1}{|z-a|},$$

where $c > 0$ and $a > 0$. The probability distribution is given by

$$\frac{1}{Z_n(a,c)} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} |z_j - a|^{2nc} \mathrm{e}^{-n|z_j|^2} \mathrm{d}A(z_j).$$



$a > a_{cri}$, $\quad a \sim a_{cri} := \sqrt{c+1} - \sqrt{c}$, $\quad a < a_{cri}$

Balogh, Bertola, Lee, and McLaughlin (2015)

Main Results

Theorem (Byun-Seo-Yang '24)

As $n \to \infty$, we have

$$\log Z_n(a,c) = -I_Q[\sigma_Q]n^2 + \frac{1}{2} n \log n + \Big(\frac{\log(2\pi)}{2} - 1\Big) n + \frac{6-\chi}{12}\log n + \frac{\log(2\pi)}{2} + \chi\,\zeta'(-1) + \mathcal{F}(a,c) + \mathcal{E}_n,$$

where

$$I_Q[\mu_Q] = \begin{cases} \frac{3}{4} + \frac{3c}{2} + \frac{c^2}{2}\log c - \frac{(c+1)^2}{2}\log(c+1) - ca^2, & a < a_{cri}, \\ \frac{3}{8} + \frac{a^2}{8} + \frac{3}{8a^2q^4} - \frac{5}{8q^2} + \Big(\frac{3}{4} + \frac{a^2}{8}\Big)a^2q^2 - \frac{3a^4q^4}{8} & \\ \quad + \log(2aq) + 2c\log(2aq^2) + \log\frac{(1+a^2q^2-2a^2q^4)^{c^2}}{(1+a^2q^2)^{(c+1)^2}}, & a > a_{cri}, \end{cases}$$

$q = q(a)$ is given explicitly,

Main Results

Theorem (To be continued)

$$\chi = \begin{cases} 0, & a < a_{cri}, \\ 1, & a > a_{cri}, \end{cases}$$

$$\mathcal{F}(a,c) = \begin{cases} \frac{1}{12}\log\left(\frac{c}{1+c}\right), & a < a_{cri}, \\ \frac{1}{24}\log\left(\frac{(1+a^2q^2-2a^2q^4)^4}{(1+a^2q^2)^4(1-q^2)^3(1-a^4q^6)}\right), & a > a_{cri}, \end{cases}$$

and

$$\mathcal{E}_n = \begin{cases} \sum_{k=1}^{M}\left(\frac{B_{2k}}{2k(2k-1)}\frac{1}{n^{2k-1}} + \frac{B_{2k+2}}{4k(k+1)}\left(\frac{1}{(c+1)^{2k}} - \frac{1}{c^{2k}}\right)\frac{1}{n^{2k}}\right) \\ \quad + O(\frac{1}{n^{2M+1}}), & a < a_{cri}, \\ O\left(\frac{1}{n}\right), & a > a_{cri}, \end{cases}$$

for any $M > 0$, where $\{B_k\}$ are the Bernoulli numbers.

Theorem (Byun-Seo-Yang '24)

For a fixed $c > 0$, let

$$a := a_{cri} - \frac{(\sqrt{c+1}-\sqrt{c})^{1/3}\mathbf{s}}{2(c^2+c)^{1/6}n^{2/3}} + O\Big(\frac{1}{n^{4/3}}\Big).$$

Then as $n \to \infty$, we have

$$\begin{aligned}\log Z_n(a,c) = &-\Big(\frac{3}{4}+\frac{3c}{2}+\frac{c^2}{2}\log c - \frac{(c+1)^2}{2}\log(c+1) - ca^2\Big)n^2 \\ &+ \frac{1}{2}n\log n + \Big(\frac{\log(2\pi)}{2}-1\Big)n + \frac{1}{2}\log n \\ &+ \frac{\log(2\pi)}{2} + \frac{1}{12}\log\Big(\frac{c}{1+c}\Big) + \log F_{\mathrm{TW}}(c^{-2/3}\mathbf{s}) + O\Big(\frac{1}{n^{2/3}}\Big),\end{aligned}$$

where $F_{\mathrm{TW}}(t) := \exp\big(-\int_t^\infty (x-t)\mathbf{q}(x)^2\mathrm{d}x\big)$ is the Tracy-Widom distribution.

Application I: Characteristic Polynomials of the Ginibre Ensemble

For the Ginibre ensemble $\mathbf{G}_n$, the probability density is given by

$$\frac{1}{Z_n^{Gin}} \prod_{i<j} |z_i - z_j|^2 \exp\Big(- n \sum_{j=1}^{n} |z_j|^2 \Big), \quad Z_n^{Gin} = n! \frac{G(n+1)}{n^{n(n+1)/2}}.$$

Characteristic polynomials of the Ginibre ensemble $\mathbf{G}_n$ are given by

$$\begin{aligned} \mathbb{E}\big|\mathbf{G}_n - z\big|^{2\gamma} &= \mathbb{E} \prod_{j=1}^{n} |z - z_j|^{2\gamma} \\ &= \frac{1}{Z_n^{Gin}} \int_{\mathbb{C}^n} \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{n} |z - z_j|^{2\gamma} \mathrm{e}^{-n|z_j|^2} \mathrm{d}A(z_j) \\ &= \frac{Z_n(|z|, \gamma)}{Z_n^{Gin}} \end{aligned}$$

Application I: Characteristic polynomials of $\mathbf{G}_n$

Let $\gamma = nc$.

Theorem (Byun-Seo-Yang '24)

For fix $c > 0$, as $n \to \infty$, we have

$$\mathbb{E}\Big|\det(\mathbf{G}_n - z)\Big|^{2cn} = n^{\frac{1-\chi}{12}}\, e^{(\chi-1)\zeta'(-1)}\, \mathcal{G}(|z|)\exp\Big(\mathcal{H}(|z|)n^2 + \widetilde{\mathcal{E}}_n\Big),$$

$$\mathcal{G}(z) = \begin{cases} \Big(\frac{c}{1+c}\Big)^{\frac{1}{12}}, & |z| < a_{cri}, \\ \Big(\frac{F(z)^4(F(z)^4+z^2F(z)^2-2z^2)^4}{(F(z)^2+z^2)^4(F(z)-1)^3(F(z)^6-z^4)}\Big)^{\frac{1}{24}}, & |z| > a_{cri}, \end{cases}$$

$$\mathcal{H}(z) = \begin{cases} c\,z^2 - \frac{3c}{2} + \frac{(c+1)^2}{2}\log(c+1) - \frac{c^2}{2}\log c, & |z| < a_{cri}, \\ \frac{3}{8} - \frac{z^2}{8} - \frac{3F(z)^4}{8z^2} + \frac{5F(z)^2}{8} - \Big(\frac{3}{4} + \frac{z^2}{8}\Big)\frac{z^2}{F(z)^2} + \frac{3}{8}\frac{z^4}{F(z)^4} & \\ \quad + \log\Big(\frac{F(z)^{2c^2-1}(F(z)^2+z^2)^{(c+1)^2}}{(2z)^{2c+1}(F(z)^4+z^2F(z)^2-2z^2)^{c^2}}\Big), & |z| > a_{cri}. \end{cases}$$

Fixed $\gamma > 0$

The bulk case $|z| < 1$,

$$\mathbb{E}\left|\det(\mathbf{G}_n - z)\right|^{2\gamma} = n^{\frac{\gamma^2}{2}} \frac{\exp\left(\gamma n |z|^2\right)}{\exp(\gamma n)} \frac{(2\pi)^{\gamma/2}}{G(\gamma+1)}(1 + o(1)).$$

Webb-Wong (2019)
The edge case $|z| \sim 1$,

$$\mathbb{E}\left|\det(\mathbf{G}_n - z)\right|^{2\gamma} = n^{\frac{\gamma^2}{2}} \frac{\exp\left(\gamma n |z|^2\right)}{\exp(\gamma n)} \frac{(2\pi)^{\gamma/2}}{G(\gamma+1)} F_\gamma(\sqrt{n}(1 - |z|^2))(1 + o(1)),$$

where F_γ is related to Painlevé IV. **Deaño-Simm (2022)**
Unified Formula:

$$\mathbb{E}\left|\det(\mathbf{G}_n - z)\right|^{2\gamma} = n^{-\gamma n} \exp\left(\gamma n \, |z|^2\right) \frac{G(\gamma + n + 1)}{G(\gamma+1)G(n+1)}(1 + o(1)).$$

Application II: LUE

We consider the Laguerre unitary ensemble (LUE). The probability distribution of eigenvalues $\{\lambda_j\}_{j=1}^n$ is proportional to

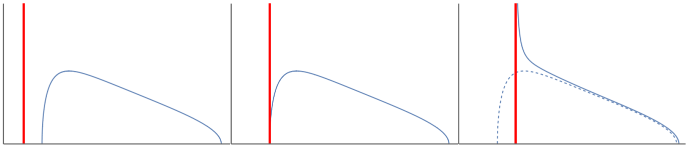
$$\prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_{j=1}^{n} \lambda_j^{\alpha n} e^{-n \sum_{j=1}^n \lambda_j}, \qquad (\lambda_n > \cdots > \lambda_1 > 0),$$

Where $\alpha \geq 0$. The density of the equilibrium measure is known as the Marchenko-Pastur distribution

$$\frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi x} \cdot \mathbf{1}_{[\lambda_-, \lambda_+]}(x), \qquad \lambda_\pm := (\sqrt{\alpha + 1} \pm 1)^2.$$

Application II: Large Deviation of the smallest eigenvalue of the LUE

We focus on the statistics of the smallest eigenvalue λ_1: $\mathbb{P}[\lambda_1 > t]$.



$$t < \lambda_-, \qquad t = \lambda_-, \qquad t > \lambda_-$$

For a fixed $c > 0$, we have

$$\mathbb{P}\left[\lambda_1 > \frac{x^2}{c}\right] = \mathrm{e}^{-ncx^2} \frac{Z_n(x,c)}{Z_n(0,c)}\Big|_{n \to n/c}, \quad c = 1/\alpha.$$

Nishigaki-Kamenev 2002, Forrester-Rains 2009

Application II: LUE

Theorem (Byun-Seo-Yang '24)

As $n \to \infty$, we have the following:

- *If $t < \lambda_-$, we have*

$$\log \mathbb{P}\big[\lambda_1 > t\big] = O(N^{-\infty}).$$

- *If $t > \lambda_-$, we have*

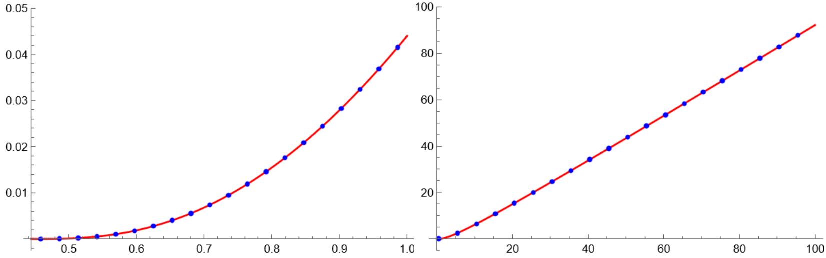
$$\log \mathbb{P}\big[\lambda_1 > t\big] = -\Phi(t;\alpha)n^2 - \frac{1}{12}\log(\alpha n) + \zeta'(-1) + \Psi(t;\alpha) + O\Big(\frac{1}{n}\Big),$$

where $\Phi(t;\alpha)$ and $\Psi(t;\alpha)$ are given explicitly.

$$\Phi(t;\alpha) \sim \frac{\sqrt{\alpha+1}}{12\lambda_-^2}(t-\lambda_-)^3, \quad t \to \lambda_-,$$

$$\Phi(t;\alpha) \sim t - \alpha \log t, \quad t \to \infty.$$

Leading term matches the Katzav-Castillo Formula (2010).

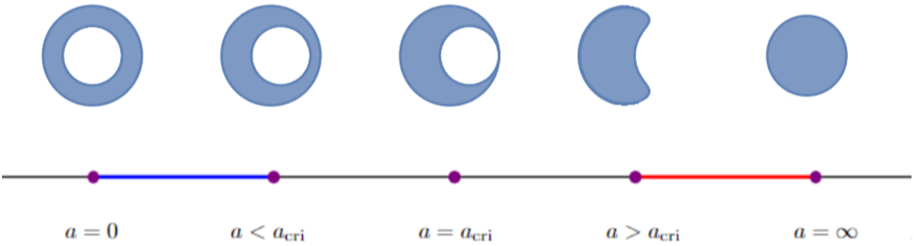


Strategy of the Proof

- Deformation of the partition function
- Fine asymptotic behavior of orthogonal polynomials via Riemann-Hilbert problems.

$$\log Z_n(a,c) = \log Z_n(0,c) + \int_0^a \frac{d}{dt} \log Z_n(t,c)\,dt$$

$$= \log Z_n^{\mathrm{Gin}} + (2c \log a) n^2 - \int_a^\infty \left(\frac{d}{dt} \log Z_n(t,c) - \frac{2cn^2}{t} \right) dt.$$



Thank You For Your Attention!