Analysis of singular subspaces under random perturbations

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Matrix denoising model

Example: In image processing, we only observe $\widetilde{S}$. What can be said about the true data matrix S?

Other applications: principal component analysis, matrix completion, community detection, etc.

Matrix denoising model

Signal: a real deterministic $N \times n$ matrix A with rank $r \ll \min\{N, n\}$.

Singular values: $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$.
SVD: $A = UDV^T = \sum_{i=1}^r \sigma_i u_i v_i^T$.

Model:

$$\widetilde{A} = \underbrace{A}_{\text{low-rank signal}} + \underbrace{E}_{\text{random noise}}$$

The (ordered) s.v.'s and s. vectors of $\widetilde{A}$ are $\widetilde{\sigma}_i$, $\widetilde{v}_i$ and $\widetilde{u}_i$.

▻ **Question**: How does the noise E affect the key parameters of A?

Key parameters: $\|A\|_{op} = \sigma_1$, top s.v.'s, top singular vectors, singular subspaces.

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Compare singular subspaces

For $1 \leqslant k \leqslant r$, denote

$$U_k := (u_1, \cdots, u_k) \quad \text{and} \quad \widetilde{U}_k := (\widetilde{u}_1, \cdots, \widetilde{u}_k).$$

Goal: Non-asymptotic bounds on

$$\min_{O \in \mathbb{O}^{k \times k}} \left\| U_k O - \widetilde{U}_k \right\|$$

for some kind of matrix norm $\| \cdot \|$ of interest.

- ▻ Unitarily invariant norms.
- ▻ $\| \cdot \|_{2,\infty}$: maximum row length.
- ▻ Linear and bilinear forms: $\|x^T(U_k O - \widetilde{U}_k)\|$ and $|x^T(U_k O - \widetilde{U}_k)y|$.

Extension:

$$\min_{O \in \mathbb{O}^{k \times k}} \left\| U_k O D_k - \widetilde{U}_k \widetilde{D}_k \right\|,$$

where $D_k = \text{diag}(\sigma_1, \cdots, \sigma_k)$.

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Deformation models: random matrix theory

Deformation models, asymptotical analysis, BBP phase transition.

▷ Extreme eigenvalues

- Convergent limit: Bai-Yao '12, Baik-Silverstein '06, Benaych-Georges- Nadakuditi '11 & '12, Capitaine-Donati-Martin '16, Ding '17, Knowles-Yin '13, Paul '07, etc.
- Fluctuation: Bai-Yao '08, Bao-Pan-Zhou '15, Benaych-Georges-Guionnet-Maida '11, Bloemendal-Knowles-Yau-Yin '16, Bloemendal-Virág '13 & '16, Capitaine-Donati-Martin-Féral '09, Cai-Han-Pan '20, Knowles-Yin '13, Renfrew-Soshnikov '12, etc.

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Unitarily invariant norms

Def: A norm $|||\cdot|||$ on $\mathbb{R}^{N\times n}$ is unitarily invariant if $|||A||| = |||UAV|||$ for all orthogonal matrices U, V.

Fact: $|||\cdot|||$ is unitarily invariant if and only if $|||A||| = f(\sigma_1, \dots, \sigma_{\min\{N,n\}})$ for all $A \in \mathbb{R}^{N\times n}$ for a symmetric gauge function f.

Examples

- Operator norm $\|A\|_{op}$ and Frobenius norm $\|A\|_F$.
- Schatten q-norm ($q \in [1,\infty]$): $\|A\|_q = \left(\sum_{i=1}^{\min\{N,n\}} \sigma_i^q\right)^{1/q}$.
- Ky Fan k-norm ($1 \leqslant k \leqslant \min\{N,n\}$): $\|A\|_{(k)} = \sum_{i=1}^{k} \sigma_i$.

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Principle angles between subspaces

Singular subspaces

$$\mathrm{U}_k = \mathrm{Span}\{u_1, \cdots, u_k\} \quad \text{and} \quad \widetilde{\mathrm{U}}_k = \mathrm{Span}\{\widetilde{u}_1, \cdots, \widetilde{u}_k\}.$$

The principal angles between U_k and $\widetilde{\mathrm{U}}_k$ are $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_k \leqslant \frac{\pi}{2}$, where the angles are defined recursively by

$$\cos(\theta_i) = \max_{u \in U_k, v \in \widetilde{U}_k} u^T v = u_i^T v_i, \qquad \|u\| = \|v\| = 1$$

subject to $u^T u_j = 0,\ v^T v_j = 0, \quad j = 1, \dots, i-1.$

Denote

$$\sin \angle(\mathrm{U}_k, \widetilde{\mathrm{U}}_k) = \mathrm{diag}(\sin\theta_1, \cdots, \sin\theta_k),$$
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Fact: The SVD $U_k^T \widetilde{U}_k = O_1 \cos\angle(\mathrm{U}_k, \widetilde{\mathrm{U}}_k) O_2^T$.

For unitarily invariant $\|\cdot\|$, take $O := O_1 O_2^T$,

$$\min_{O \in \mathbb{O}^{k\times k}} \| U_k O - \widetilde{U}_k \| \leqslant \sqrt{2} \| \sin\angle(\mathrm{U}_k, \widetilde{\mathrm{U}}_k) \|.$$

Classical bounds:

P_W: the orthogonal projection onto a subspace W.

▻ Wedin sine theorem (1972): for any unitarily invariant $\|\cdot\|$,

$$\| \sin\angle(\mathrm{U}_k, \widetilde{\mathrm{U}}_k) \| \leqslant \frac{\max\{ \| P_{U_k^\perp} E P_{\widetilde{V}_k} \|, \| P_{V_k^\perp} E^T P_{\widetilde{U}_k} \| \}}{\sigma_k - \widetilde{\sigma}_{k+1}}.$$

Hermitian case: Davis-Kahan sine theorem (1970).

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Low-rank signal + random noise

Classical bounds wasteful when A is low-rank and E is random.

E.g. E of size $n \times n$, entries i.i.d. $\mathcal{N}(0,1)$. Spectral gap $\delta_1 := \sigma_1 - \sigma_2$.

- Wedin: $\sin \angle(u_1, \widetilde{u}_1) \leqslant 2\frac{\|E\|_{op}}{\delta_1} \lesssim \frac{\sqrt{n}}{\delta_1}$ w.h.p.

▷ Vu (2010): E i.i.d. Rademacher matrix

$$\sin^2 \angle(u_1, \widetilde{u}_1) \lesssim \frac{\sqrt{r}}{\delta_1} + \frac{\|E\|_{op}}{\sigma_1} + \frac{\|E\|_{op}}{\delta_1} \cdot \frac{\|E\|_{op}}{\sigma_1} \quad \text{w.h.p.}$$

Key observation: When A is low rank, the true dimension is r, the rank of A, rather than its size N or n.

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Previous results

▷ O'Rourke-Vu-W. (2013): If E has good concentration properties,

$$\sin \angle(u_1, \tilde{u}_1) \lesssim \underbrace{\frac{C(r)}{\delta_1}}_{\text{Wedin}} + \underbrace{\frac{\|E\|_{op}}{\sigma_1}}_{SNR} + \underbrace{\frac{\|E\|_{op}}{\delta_1} \cdot \frac{\|E\|_{op}}{\sigma_1}}_{?} \quad \text{w.h.p.}$$

Similar results for $\| \sin \angle(U_k, \tilde{U}_k)\|_{op}$.

▷ O'Rourke-Vu-W. (2023): E i.i.d. $\mathcal{N}(0,1)$ entries. For any $k \in [r]$, if

$$\sigma_k \geqslant 2\|E\|_{op}, \quad \delta_k = \sigma_k - \sigma_{k+1} \geqslant c_1 r^2 \sqrt{\log(N+n)},$$

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then

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New results

▷ W. (2024): E i.i.d. $\mathcal{N}(0,1)$ entries. Assume

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$k_0 = \min\{k, r-k\}$. For any unitarily invariant $|\!|\!| \cdot |\!|\!|$, w.h.p.

$$|\!|\!| \sin \angle(U_k, \widetilde{U}_k) |\!|\!| \leqslant c_3\sqrt{k \cdot k_0}\frac{\sqrt{r+\log(N+n)}}{\delta_k} + 2\frac{|\!|\!| P_{U^\perp} E P_{\widetilde{V}_k} |\!|\!| + |\!|\!| P_{V^\perp} E^T P_{\widetilde{U}_k} |\!|\!|}{\sigma_k}.$$

Compare with Wedin's bound: from $|\!|\!| P_{U_k^\perp} E P_{\widetilde{V}_k} |\!|\!| = |\!|\!| (P_{U_{k+1,r}} + P_{U^\perp}) E P_{\widetilde{V}_k} |\!|\!|$,

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$$\begin{aligned} &\|\!|\sin\angle(U_k,\widetilde{U}_k)|\!\| \\ &\leqslant c_3\sqrt{k\cdot k_0}\frac{\sqrt{r+\log(N+n)}}{\delta_k} + 2\frac{\|\!|P_{U^\perp}EP_{\widetilde{V}_k}|\!\| + \|\!|P_{V^\perp}E^TP_{\widetilde{U}_k}|\!\|}{\sigma_k}. \end{aligned}$$

<u>Compare with Wedin's bound</u>: from $\|\!|P_{U_k^\perp}EP_{\widetilde{V}_k}|\!\| = \|\!|(P_{U_{k+1,r}} + P_{U^\perp})EP_{\widetilde{V}_k}|\!\|$,

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Specially, assume $\sigma_k \geqslant 2\|E\|_{op}, \quad \delta_k \geqslant c_1 r\sqrt{r + \log(N+n)}$.

- (Operator norm)

$$\|\sin\angle(U_k, \widetilde{U}_k)\|_{op} \lesssim \sqrt{k}\frac{\sqrt{r + \log(N+n)}}{\delta_k}\mathbf{1}_{\{k\neq r\}} + \frac{\|E\|_{op}}{\sigma_k}.$$

- (Frobenius norm)

$$\|\sin\angle(U_k, \widetilde{U}_k)\|_{F} \lesssim \sqrt{k}\frac{\sqrt{r + \log(N+n)}}{\delta_k}\mathbf{1}_{\{k\neq r\}} + \sqrt{k}\frac{\|E\|_{op}}{\sigma_k}.$$

Recent literatures

▷ ℓ_2 analysis: $\min_{s\in\{\pm1\}} \|\tilde{u}_k - su_k\|_2$ or $\|\sin\angle(U_k, \tilde{U}_k)\|_*$ with $* \in \{op, F\}$.

Deterministic setting

- Yu-Wang-Samworth '15, Wu-Lei '13, Fan-Wang-Zhong '17, Cai-Zhang '18, Zhang-Cai-Wu '22, Luo-Han-Zhang '21.

"Random noise" setting

- Wang '12: non-asymptotic dist. of s. vectors when entries of E i.i.d. $\mathcal{N}(0,1)$.
- Allez-Bouchaud '13: eigenvector dynamics (symmetric case).
- Benaych-Georges-Enriquez-Michaïl '18: perturbative expansion of the coordinates of the e.vectors.
- Zhong '17: bound on $\sin\angle(u_1, \tilde{u}_1)$ for E sub-gaussian (symmetric case).

Recent literatures

▻ Entrywise or $\ell_\infty/\ell_{2,\infty}$ analysis: $\|\widetilde{u}_k - s u_k\|_\infty$ or $\|\widetilde{U}_k - U_k O\|_{2,\infty}$

- Eldridge-Belkin-Wang '13: general bounds, symmetric case, use Neumann trick to expand the perturbed e.vectors.
- Abbe-Fan-Wang-Zhong '20: symmetric case, A finite rank and incoherent. Large eigengaps. General assumption on random noise E.
- Zhong-Boumal '18: phase synchronization model, Gaussian noise.
- Cape-Tang-Priebe '19: general bounds under $2 \to \infty$ norm.
- Chen-Fan-Ma-Wang '19: ℓ_∞ norm errors of the leading e.vector of asymmetric probability transition matrix.
- Lei '19: $\ell_{2,\infty}$ eigenspace perturbation bounds for symmetric random matrices, general noise E.
- Cai-Li-Chi-Poor-Chen '21: matrix dimensions are highly unbalanced.
- Abbe-Fan-Wang '20: $\ell_{2,p}$ analysis of eigenspaces.
- Agterberg, Lubbers, and Priebe 22': the noise matrix allowed to have dependence within rows and heteroskedasticity.

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- Eldridge-Belkin-Wang '13: general bounds, symmetric case, use Neumann trick to expand the perturbed e.vectors.
- Abbe-Fan-Wang-Zhong '20: symmetric case, A finite rank and incoherent. Large eigengaps. General assumption on random noise E.
- Zhong-Boumal '18: phase synchronization model, Gaussian noise.
- Cape-Tang-Priebe '19: general bounds under $2 \to \infty$ norm.
- Chen-Fan-Ma-Wang '19: ℓ_∞ norm errors of the leading e.vector of asymmetric probability transition matrix.
- Lei '19: $\ell_{2,\infty}$ eigenspace perturbation bounds for symmetric random matrices, general noise E.
- Cai-Li-Chi-Poor-Chen '21: matrix dimensions are highly unbalanced.
- Abbe-Fan-Wang '20: $\ell_{2,p}$ analysis of eigenspaces.
- Agterberg, Lubbers, and Priebe 22': the noise matrix allowed to have dependence within rows and heteroskedasticity.

Recent literatures

▻ Entrywise or $\ell_\infty/\ell_{2,\infty}$ analysis: $\|\widetilde{u}_k - su_k\|_\infty$ or $\|\widetilde{U}_k - U_k O\|_{2,\infty}$.

- Yan-Chen-Fan '21: $\ell_{2,\infty}$ bound for the full singular vector matrices, distributional characterization, general noise E.
- Bhardwaj-Vu '23: a stochastic version of Wedin for general noise E.
- Yan-Wainwright '24: improve Yan-Chen-Fan '21 for E sub-Gaussian entries.

▻ Linear or bilinear forms: $\|x^T(\widetilde{u}_k - su_k)\|$ or $|x^T(\widetilde{U}_k - U_k O)y|$.

- Koltchinskii-Xia '16: concentration bounds for bilinear forms of singular subspaces under Gaussian noise.
- Koltchinskii- Lounici '16: concentration and asymptotic distributions of bilinear forms of the spectral projectors for principle components under Gaussian noise in a general Hilbert space.
- Li-Cai-Poor-Chen '21: improve Koltchinskii-Xia 16', small eigen-gap assumption.
- Cheng-Wei-Chen '21: heteroscedastic noise, small eigen-gap.
- Agterberg '23: distributional characterization and statistical inference for $a^T\widetilde{u}_k$, small eigen-gap.

New results: ℓ_∞ and $\ell_{2,\infty}$ bounds

Observe: from the SVD $U_k^T \widetilde{U}_k = O_1 \cos\angle(\mathrm{U}_k, \widetilde{\mathrm{U}}_k) O_2^T$, take $O = O_1 O_2^T$,

$$\min_{O \in \mathbb{O}^{k\times k}} \|\widetilde{U}_k - U_k O\|_{2,\infty} \leqslant \|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} + \|U_k\|_{2,\infty} \|\sin\angle(U_k, \widetilde{U}_k)\|_{op}^2.$$

▻ W. (2024): Assume $\sigma_k \geqslant 2\|E\|_{op}$ and $\delta_k \gtrsim r\sqrt{r + \log(N+n)}$. w.h.p.

$$\|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} \lesssim \sqrt{k}\frac{\sqrt{r+\log(N+n)}}{\delta_k}\|U\|_{2,\infty} + \sqrt{k}\frac{\sqrt{r\log(N+n)}}{\sigma_k}(1 + \|U\|_{2,\infty}).$$

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New results: linear and bilinear forms

▻ W. (2024): Let x, y be unit vectors.

- Linear forms: w.h.p.

$$\left\|x^T(\widetilde{U}_k - P_{U_k}\widetilde{U}_k)\right\| \lesssim \sqrt{k}\frac{\sqrt{r+\log(N+n)}}{\delta_k}\|x^T U\| + \sqrt{k}\frac{\sqrt{r\log(N+n)}}{\sigma_k}\left(1+\|x^T U\|\right).$$

- Bilinear forms: w.h.p.

$$\left|x^T(\widetilde{U}_k - P_{U_k}\widetilde{U}_k)y\right| \lesssim \frac{\sqrt{r+\log(N+n)}}{\delta_k}\|x^T U\|\sqrt{\|y\|_0} + \sqrt{r\log(N+n)}\left(1+\|x^T U\|\right)\left|\sum_{i=1}^{k}\frac{y_i}{\sigma_i}\right|.$$

In particular, together with the ℓ_2 bound,

$$\min_{O\in\mathbb{O}^{k\times k}}\|\widetilde{U}_k - U_k O\|_{\max} \lesssim \frac{\sqrt{r+\log(N+n)}}{\delta_k}\|U\|_{2,\infty} + \frac{\sqrt{r\log(N+n)}}{\sigma_k}(1+\|U\|_{2,\infty}) + \frac{\|E\|_{op}^2}{\sigma_k^2}\|U_k\|_{2,\infty}$$

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$$\left\|x^T(\widetilde{U}_k - P_{U_k}\widetilde{U}_k)\right\| \lesssim \sqrt{k}\frac{\sqrt{r+\log(N+n)}}{\delta_k}\|x^TU\| + \sqrt{k}\frac{\sqrt{r\log(N+n)}}{\sigma_k}\left(1+\|x^TU\|\right).$$

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New results: weighted singular subspace perturbation

Proposition:

$$\min_{O\in\mathbb{O}^{r\times r}} \|\widetilde{U}_r\widetilde{D}_r - UO\widetilde{D}_r\|_{2,\infty} \leqslant \|\widetilde{U}_r\widetilde{D}_r - P_U\widetilde{U}_r\widetilde{D}_r\|_{2,\infty} + \|U\|_{2,\infty}\|\sin\angle(U,\widetilde{U}_r)\|_{op}\|E\|_{op}.$$

▻ W. (2024): Assume $\sigma_r \geqslant 2\|E\|_{op}$. w.h.p.

$$\min_{O\in\mathbb{O}^{r\times r}} \|\widetilde{U}_r\widetilde{D}_r - UO\widetilde{D}_r\|_{2,\infty} \lesssim r\sqrt{\log(N+n)}(1+\|U\|_{2,\infty}) + \|U\|_{2,\infty}\frac{\|E\|_{op}^2}{\sigma_r}.$$

Applications in analyzing the spectral algorithm for the Gaussian Mixture Model and submatrix localization problem.

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Sketch of proof

Linearization: $\widetilde{\mathcal{A}} = \mathcal{A} + \mathcal{E}$, where

$$\mathcal{A} := \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{E} := \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix}.$$

Assume

$$\mathcal{A} = \mathcal{U}\mathcal{D}\mathcal{U}^T \quad \text{and} \quad \widetilde{\mathcal{A}} = \widetilde{\mathcal{U}}\widetilde{\mathcal{D}}\widetilde{\mathcal{U}}^T.$$

Non-zero e.v.'s of $\mathcal{A}$: $\lambda_j = \sigma_j$ and $\lambda_{j+r} = -\sigma_j$ for $1 \leqslant j \leqslant r$.

Corresponding e.vectors: $\mathbf{u}_j := \frac{1}{\sqrt{2}} \begin{pmatrix} u_j \\ v_j \end{pmatrix}$ and $\mathbf{u}_{j+r} := \frac{1}{\sqrt{2}} \begin{pmatrix} u_j \\ -v_j \end{pmatrix}$.

Similar notations $\widetilde{\lambda}_j$, $\tilde{\mathbf{u}}_j$ for $\widetilde{\mathcal{A}}$.

Sketch of proof

Consider the Green function $G(z) := (zI - \mathcal{E})^{-1}$.

Key ingredient: [Isotropic local law]

For $z \in \mathbb{C}$ with $|z| \geqslant 2(\sqrt{N} + \sqrt{n})$,

$$|\langle x, (G(z) - \Phi(z))\, y\rangle| \lesssim \frac{\sqrt{\log(N+n)}}{|z|^2} \quad \text{w.h.p.}$$

Here

$$\Phi(z) := \begin{pmatrix} \frac{1}{\phi_1(z)} I_N & 0 \\ 0 & \frac{1}{\phi_2(z)} I_n \end{pmatrix},$$

where

$$\phi_1(z) := z - \sum_{t \in [N+1, N+n]} G_{tt}(z), \quad \phi_2(z) := z - \sum_{s \in [1, N]} G_{ss}(z).$$

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Sketch of proof

By the isotropic local law:

- Denote $\mathrm{D} = \{z \in \mathbb{C} : 2(\sqrt{N} + \sqrt{n}) \leqslant |z| \leqslant 2n^3\}$.

$$\max_{z \in \mathrm{D}} |z|^2 \left\| \mathcal{U}^T \left(G(z) - \Phi(z) \right) \mathcal{U} \right\|_{op} \lesssim \sqrt{r + \log(N+n)} \quad \text{w.h.p.}$$

- Singular value location:

$$|\phi_1(\tilde{\sigma}_j)\phi_2(\tilde{\sigma}_j) - \sigma_j^2| \lesssim r\sqrt{r + \log(N+n)}\sigma_j \quad \text{w.h.p.}$$

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Sketch of proof

Start with the decomposition

$$\tilde{\mathbf{u}}_1 = (\mathbf{u}_1\mathbf{u}_1^T)\tilde{\mathbf{u}}_1 + \mathcal{P}_1\tilde{\mathbf{u}}_1 + \mathcal{Q}\tilde{\mathbf{u}}_1,$$

where $\mathcal{P}_1 = \mathcal{U}_1\mathcal{U}_1^T$ and $\mathcal{U}_1$ is the matrix of eigenvectors of $\mathcal{A}$ excluding $\mathbf{u}_1$. $\mathcal{Q}$ is the orthogonal projection matrix onto the null space of $\mathcal{A}$.

ℓ_2 bounds:

Take the ℓ_2 norm on both sides $1 = \cos^2\angle(\mathbf{u}_1, \tilde{\mathbf{u}}_1) + \|\mathcal{P}_1\tilde{\mathbf{u}}_1\|^2 + \|\mathcal{Q}\tilde{\mathbf{u}}_1\|^2$. Hence,

$$\sin^2\angle(\mathbf{u}_1, \tilde{\mathbf{u}}_1) = \|\mathcal{P}_1\tilde{\mathbf{u}}_1\|^2 + \|\mathcal{Q}\tilde{\mathbf{u}}_1\|^2 \leqslant \|\mathcal{U}_1^T\tilde{\mathbf{u}}_1\|^2 + \|\mathcal{Q}\tilde{\mathbf{u}}_1\|^2.$$

Note $\|\mathcal{U}_1^T\tilde{\mathbf{u}}_1\|^2 = \sum_{j=2}^{2r} |\mathbf{u}_j^T\tilde{\mathbf{u}}_1|^2$.

- Bound $|\mathbf{u}_2^T\tilde{\mathbf{u}}_1|^2$: from $(\mathcal{E} + \mathcal{A})\tilde{\mathbf{u}}_1 = \tilde{\sigma}_1\tilde{\mathbf{u}}_1$,

$$\tilde{\mathbf{u}}_1 = (\tilde{\sigma}_1 I - \mathcal{E})^{-1}\mathcal{A}\tilde{\mathbf{u}}_1 = \Phi(\tilde{\sigma}_1)\mathcal{A}\tilde{\mathbf{u}}_1 + (G(\tilde{\sigma}_1) - \Phi(\tilde{\sigma}_1))\,\mathcal{A}\tilde{\mathbf{u}}_1.$$

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Next, $\mathbf{u}_2^T\tilde{\mathbf{u}}_1 = \mathbf{u}_2^T\Phi(\tilde{\sigma}_1)\underbrace{\mathcal{A}\tilde{\mathbf{u}}_1}_{\sigma_2\mathbf{u}_2(\mathbf{u}_2^T\tilde{\mathbf{u}}_1)+\sum_{i\neq 2}\lambda_i\mathbf{u}_i(\mathbf{u}_i^T\tilde{\mathbf{u}}_1)} + \underbrace{\mathbf{u}_2^T\left(G(\tilde{\sigma}_1)-\Phi(\tilde{\sigma}_1)\right)\mathcal{U}}_{\text{controlled by local law}}\cdot\mathcal{D}\mathcal{U}^T\tilde{\mathbf{u}}_1.$

Gather $\mathbf{u}_2^T\tilde{\mathbf{u}}_1$ terms:

$$\left(1-\sigma_2\frac{1}{2}\left(\phi_1(\tilde{\sigma}_1)^{-1}+\phi_2(\tilde{\sigma}_1)^{-1}\right)\right)\mathbf{u}_2^T\tilde{\mathbf{u}}_1 \approx \left(1-\frac{\sigma_2}{\sigma_1}\right)\mathbf{u}_2^T\tilde{\mathbf{u}}_1.$$

ℓ_∞ bounds: taking the ℓ_∞ norm on both sides:

$$\|\tilde{\mathbf{u}}_1-(\mathbf{u}_1\mathbf{u}_1^T)\tilde{\mathbf{u}}_1\|_\infty \leqslant \|\mathcal{P}_1\tilde{\mathbf{u}}_1\|_\infty + \|\mathcal{Q}\tilde{\mathbf{u}}_1\|_\infty \leqslant \|\mathcal{U}\|_{2,\infty}\|\mathcal{U}_1^T\tilde{\mathbf{u}}_1\| + \|\mathcal{Q}\tilde{\mathbf{u}}_1\|_\infty.$$

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$$\mathcal{Q}\tilde{\mathbf{u}}_1 = \mathcal{Q}\left(G(\tilde{\sigma}_1)-\Phi(\tilde{\sigma}_1)\right)\mathcal{A}\tilde{\mathbf{u}}_1$$

by the isotropic local law.

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Next, $\mathbf{u}_2^T\tilde{\mathbf{u}}_1 = \mathbf{u}_2^T\Phi(\tilde{\sigma}_1)\underbrace{\mathcal{A}\tilde{\mathbf{u}}_1}_{\sigma_2\mathbf{u}_2(\mathbf{u}_2^T\tilde{\mathbf{u}}_1)+\sum_{i\neq 2}\lambda_i\mathbf{u}_i(\mathbf{u}_i^T\tilde{\mathbf{u}}_1)} + \underbrace{\mathbf{u}_2^T\left(G(\tilde{\sigma}_1)-\Phi(\tilde{\sigma}_1)\right)\mathcal{U}}_{\text{controlled by local law}}\cdot\mathcal{D}\mathcal{U}^T\tilde{\mathbf{u}}_1.$

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THANK YOU!

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