Zeros of Bergman polynomials for Jordan domains with corners

Based on joint work with Erwin Miña-Díaz

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Random Matrices and Related Topics, Jeju, May 2024

Plan for the talk

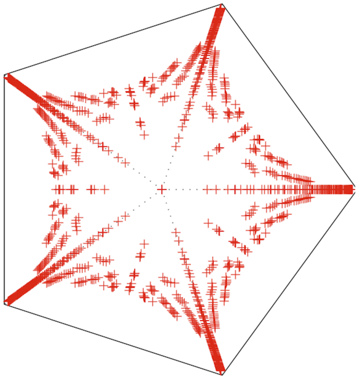

Fig. Zeros of Bergman polynomials for the pentagon (from [Saff-Stylianopoulos '08])

- Bergman polynomials for domains with piecewise analytic boundary
- The Triangle versus the Pentagon: dichotomy for zero distribution
- New asymptotic results for domains with "regular" corner singularities

Bergman orthogonal polynomials

Denote by $\{P_n\}_{n\geq 0}$ the **orthogonal polynomials** for a domain $D \subset \mathbb{C}$, obtained by applying the Gram-Schmidt process to $\{1, z, z^2, \ldots\}$ in the inner product

$$\langle f, g\rangle_D = \int_D f(z)\overline{g(z)}\mathrm{dA}(z),$$

where $\mathrm{dA}(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$.

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- The *polynomial* Bergman kernel $K_{n,D}(z,w) = \sum_{k=0}^{n-1} P_n(z)\overline{P_n(w)}$ governs the $\beta = 2$ Coulomb gas confined to D.
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- Relevant quantities are leading coefficients (rel. to partition function), strong asymptotics of P_n near $L = \partial D$.
- Our focus will be on Jordan domains D with piecewise analytic boundary and no cusps. Interested in **strong asymptotics** and **location of zeros**.

Carleman's theorem

Let D be a Jordan domain with *analytic* boundary L, and let Ω be the unbounded component of $\mathbb{C}\backslash L$. Let $\phi : \Omega \to \mathbb{C}\backslash\mathbb{D}$ be the exterior conformal map, $\phi(\infty) = \infty$, $\phi'(\infty) > 0$. Let $\Omega_\rho = \{z : |\phi(z)| > \rho\}$.

Theorem (Carleman, 1922). Under the above assymptions, there exists a number $\rho \in (0, 1)$ such that, as $n \to \infty$,

$$P_n(z) = \sqrt{n+1}\phi'(z)\phi(z)^n\Big(1 + \mathrm{O}(\rho^{2n})\Big), \qquad z \in \Omega_\rho.$$

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- Similar expansions hold in less regular domains and in weighted spaces $L^2(D, \omega\,\mathrm{dA})$ (see e.g. [Beckermann-Stylianopoulos '18], [Suetin '72], [Hedenmalm-W., '24]).

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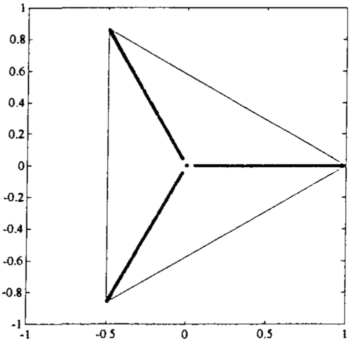
- Similar expansions hold in less regular domains and in weighted spaces $L^2(D, \omega\,\mathrm{d}A)$ (see e.g. [Beckermann-Stylianopoulos '18], [Suetin '72], [Hedenmalm-W., '24]).
- Typically **no information on interior** of D in lower regularity.

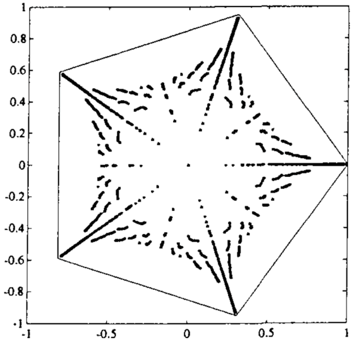
Orthogonal polynomials on regular N-gons

Regular 3–gon:



zeros of orthogonal polynomials up to degree 50.

Regular 5–gon:



zeros of orthogonal polynomials up to degree 50.

Conjecture (Eiermann-Stahl, '94): (A) For $N \in \{3, 4\}$, the zeros of P_n lie exactly on the segments connecting the vertices with the center.
(B) For all $N \geq 2$, the corners are the only points of L that attract zeros.

Distinsguishing $N = 3$ from $N = 5$

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Theorem (Levin-Saff-Stylianopoulos, '03, cf. Andrievskii-Blatt, '99)
Assume that $L = \partial D$ is a piecewise analytic Jordan curve. Then the following are equivalent:

(a) φ has a singularity on L

(b) $\nu_{P_n} \to \mu_D$ along some subsequence

(c) $\limsup_{n\to\infty} |P_n(z)|^{1/n} = 1$ on D

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- The interior map φ from the N-gon has singularity on L iff $N \geq 5$. Explains difference, and the Eiermann-Stahl conjecture (B) cannot be true.
- "Beware of predictions from plots" (Saff-Stylianopoulos, '08).
- Part (A) of the conjecture proven by Maymeskul-Saff using symmetry and reduction to orthogonality conditions on the segments.

Why does the interior map φ matter?

The polynomials are dense in $A^2(D)$, so the Bergman kernel is

$$K_D(z,w) = \frac{\varphi'(z)\overline{\varphi'(w)}}{(1-\varphi(z)\overline{\varphi(w)})^2} = \sum_{k\geq 0} \overline{P_k(w)} P_k(z).$$

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Lemma (Walsh, '69). Let $F(z)$ be an analytic function on D with a Fourier series expansion

$$F(z) = \sum_{n\geq 0} \alpha_n P_n(z).$$

Then, if ρ is maximal the maximal number such that $F(z)$ is analytic on the domain D_ρ interior to $|\phi(z)| = \rho$ then $\limsup |\alpha_n|^{1/n} = \frac{1}{\rho}$.

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- Akin to radius of convergence, but requires only modest a priori knowledge of the polynomials $P_n(z)$.
- If φ has a singularity on L, then $\limsup_{n\geq 0} |P_n(w)|^{1/n} = 1$ on D, restricting the domain of validity of $P_n(z) \sim \sqrt{n+1}\phi'(z)\phi(z)^n$.

Domains with analytic corner singularities

Q. What about domains D for which φ lacks singularities on L?

Definition. We let $\mathcal{A}_1$ be the class of Jordan domains for which any interior conformal map φ extends analytically through L, and has zeros $z_1, \ldots, z_q$ on L.

- If $D \in \mathcal{A}_1$, then any corner has aperture π/m, $m \in \{2, 3, 4, \ldots\}$ (the order of φ at the corner). This condition is *not sufficient*.
- Membership in $\mathcal{A}_1$ can equivalently be characterized by invariance of $2n_j - 1$-fold Schwarz reflection around the corner of angle π/n_j.

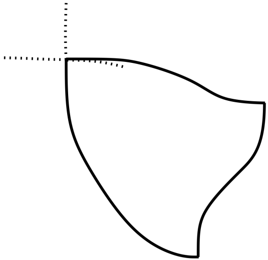

Illustration of a domain with non-analytic corner singularity of aperture $\pi/2$.

The main result

Theorem (Miña-Díaz-W., '24). Let $D \in \mathcal{A}_1$. There exists an open set Ω^* containing $L \backslash \{z_1, \dots, z_q\}$, and a univalent function Φ on Ω^* with $\Phi|_\Omega = \phi$, such that the strong asymptotics

$$P_n(z) = \sqrt{n+1}\Phi'(z)\Phi(z)^n \left(1 + \mathrm{O}\left(\frac{\log n}{n}\right)\right)$$

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Asymptotics in the **interior of** D is the most challenging part. Comes from an approximate *integral representation*: for $D \in \mathcal{A}_1$, we have

$$P_n(z) \sim \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \int_{|w|=1} \frac{w^n}{h(w) - \varphi(z)} \mathrm{d}w, \qquad z \in D,$$

where $h = \varphi \circ \phi^{-1}$.

The domain D and the maximal set Ω^*

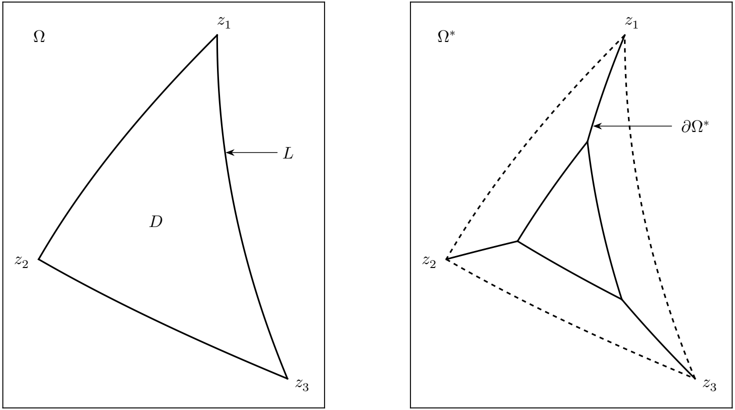

Figure: Illustration of the curve $L = \partial D$ and sets Ω and Ω^*, for an admissible domain D with three corners z_1, z_2, and z_3.

The domain D_1

The function $h(w)$ is a homeomorphism of $\mathbb{T}$, and is *analytic* in a maximal annulus $\{\mu < |w| < \frac{1}{\mu}\}$, $\mu < 1$. For $z \in D$, consider the solutions $\{w_1, \ldots, w_k\}$ in $\{\mu < |z| < 1\}$ to $h(w) = \varphi(z)$.

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- For $z \in D_1$, we can thus apply the residue theorem,

$$P_n(z) \sim \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \int_{|w|=1} \frac{w^n}{h(w) - \varphi(z)} \mathrm{d}w$$
$$= (n+1)\phi_1'(z)\phi_1^n(z)(1 + \mathrm{O}(\rho^n))$$

  for some $\rho < 1$.

- Any point sufficiently close to L but *away from the corners* is in D_1, and we have $\phi_1 = \phi$ there.

Broad proof outlines

- Sharp **exterior asymptotics** due to Beckermann-Stylinaopoulos. A small improvement near ∂L gives

$$P_n(z) = \sqrt{n+1}\phi'(z)\phi(z)^n\Big(1 + \mathrm{O}\Big(\frac{|\phi(z)|^2}{n(|\phi(z)|-1)^2}\Big)\Big), \quad z \in \Omega$$

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- The integral expansion on D_1 gives the **interior asymptotics**

$$P_n(z) = \sqrt{n+1}\phi'(z)\phi(z)^n\Big(1 + \mathrm{O}\Big(\frac{1}{n(|1-\phi(z)|)^B}\Big)\Big)$$

valid for $z \in D$ near L but bounded away from the corners.

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Q. How to bridge the gap in the asymptotic formulas across L?

Inspiration from the Phragmén-Lindelöf principle

Suppose that f is holomorphic on the strip $S = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < 1\}$, with $|f(z)| \leq 1$ on ∂S. If moreover $|f(z)| \leq \exp(e^{\frac{\pi\alpha}{2}|z|})$ for some $\alpha < 1$, then we have $|f(z)| \leq 1$ throughout S.

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Recall the proof: Introduce the auxiliary function

$$h_n(z) = \exp\big(-\varepsilon_n(e^{\frac{\pi\beta z}{2}} + e^{-\frac{\pi\beta z}{2}})\big),$$

where ε_n decays to 0 (e.g., $\frac{1}{n}$), and $\beta \in (\alpha, 1)$. We have $|h(z)| \leq 1$ on S.

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- By the a priori bound,

$$|g(z)h_n(z)| \leq \exp\left(e^{\pi\alpha|z|/2} - \cos\left(\frac{\pi\beta}{2}\right)\varepsilon_n e^{\pi\beta|x|/2}\right).$$

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- For large $R > 0$, the right-hand side is ≤ 1 on $S \cap \{|\mathrm{Re}(z)| = R\}$. By the *maximum principle* $|g(z)h_n(z)| \leq 1$ on the strip S.
- When $n \to \infty$, $h_n(z) \to 1$ locally uniformly, so $|g(z)| \leq 1$ on S.

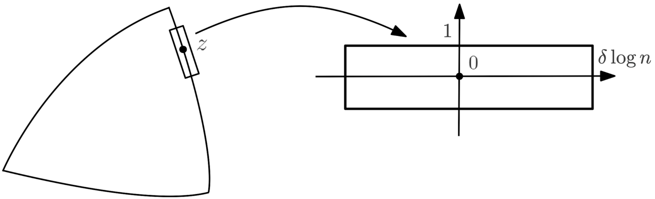
Gluing interior and exterior asymptotics

Let

$$A_n(z) = \frac{P_n(z)}{\sqrt{n+1}\phi'(z)\phi(z)^n} - 1.$$

We want to show that $A_n(z) = \mathrm{O}(\frac{(\log n)^c}{n})$.

- Interior/exterior asymptotics: The upper bound holds when $\mathrm{d}(z, L) \geq \frac{1}{\log n}$ and z is away from the corners.
- A priori bound of the form $|A_n(z)| \leq C^n$ is relatively easy.



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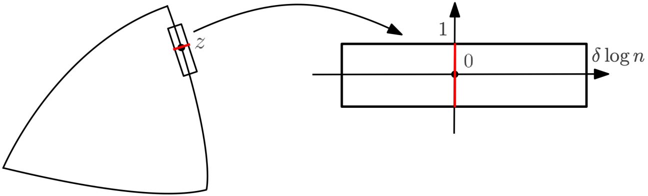

T_n rectangle around z as above, of "width" δ and shrinking "height" $\frac{1}{\log n}$.

Conformal map $f_n : T_n \to S_n$, where $S_n = \{|\mathrm{Re}\, w| \leq \delta \log n,\ |\mathrm{Im}\, w| < 1\}$.
We put

$$g_n(z) = A_n \circ f_n^{-1}(z).$$

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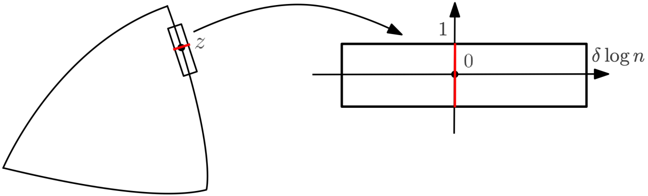

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- Phragmén-Lindelöf argument: $|g_n(z)h_n(z)| \le C\frac{(\log n)^c}{n}$ on S_n.
- We conclude that $|g_n(it)| \le C\frac{(\log n)^c}{n}$ for $|t| \le 1$, which proves that A_n satisfies the desired bound on a normal line through z of length $2/\log n$. This proves the claim.

Thank you for your attention!