Asymptotic expansion of the hard-to-soft edge transition

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Introduction

The Wishart-Laguerre ensemble

The Wishart-Laguerre ensemble $\boldsymbol{X}$ is a complex random matrix defined by

$$\boldsymbol{X} = (X_1 + \mathrm{i}X_2)(X_1 + \mathrm{i}X_2)^*,$$

where X_1 and X_2 are two $n \times (n + \nu)$, $\nu \geq 0$, random matrices, whose element is chosen to be an independent normal random variable.

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Question: Where are the eigenvalues of $\boldsymbol{X}$?

Marchenko-Pastur law

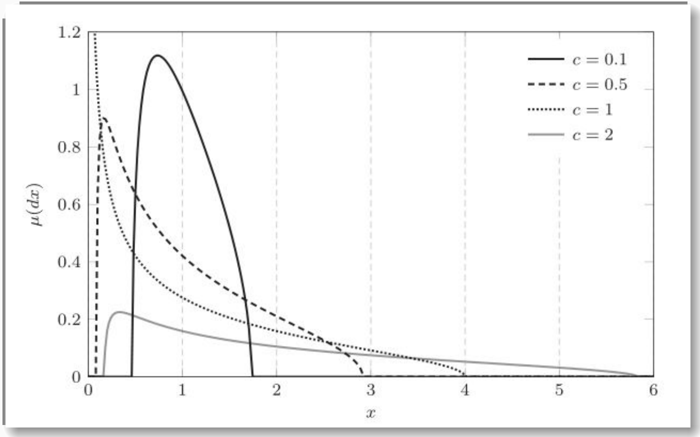
As $n \to \infty$, it is well known that the empirical spectral measure of $\boldsymbol{X}$ converges weakly to the Marchenko-Pastur (MP) distribution μ given by

$$\mu(\mathrm{d}x) = \frac{\sqrt{(x - x_-)(x_+ - x)}}{2\pi c x}\,\mathrm{d}x, \qquad x \in (x_-, x_+),$$

where $c = \lim n/(n + \nu) \in (0, 1]$ and $x_\pm = (1 \pm \sqrt{c})^2$.

[Marchenko-Pastur, '67]

Marchenko-Pastur law



Marchenko-Pastur distribution for different values of c. Source: [Couillet-Liao, '22]

Determinantal point process

Joint probability density function for the eigenvalues of $\boldsymbol{X}$:

$$\frac{1}{C_{n,\nu}} \prod_{l=1}^{n} x_l^{\nu} e^{-x_l} \prod_{1 \le j < k \le n} (x_j - x_k)^2 = \det \left[K_n^{\nu}(x_j, x_k) \right]_{j,k=1,\ldots,n}$$

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This is a determinantal point process (DPP) with the correlation kernel $K_n^{\nu}(x, y)$ expressed in terms of classical Laguerre polynomials. All the information of DPP is contained in the correlation kernel.

Microscopic limits of the correlation kernel

Microscopic limits $\Rightarrow$ local fluctuation

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The local limits and the associated DPPs are **universal** in RMT and beyond.

Ulam's problem

Let S_n be the set of all permutations on $\{1, 2, \ldots, n\}$. Given $\sigma \in S_n$,

$$L_n(\sigma) := \text{length of longest increasing subsequence in } \sigma,$$

i.e., the maximum of all k such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ with $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k)$.

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Example

Let $\sigma = 4635217$. Since both 457 and 357 are the longest increasing subsequences of length 3, it follows that

$$L_7(\sigma) = 3.$$

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- $\mathbb{E}(L_n) \sim c\sqrt{n}$

  [Hammersley, '72]

- $\mathbb{E}(L_n) \sim 2\sqrt{n}$

  [Vershik-Kerov, '77; Logan-Shepp, '77]

- $\lim_{n\to\infty} \mathbb{P}\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq t\right) = F(t)$, where $F(t)$ is the Tracy-Widom distribution.

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The connections with other problems such as last passage percolation and RMT can be found in the book of Romik (2015).

The Tracy-Widom distribution

Definition:

$$F(t) = \det(I - \mathcal{K}_{\mathrm{Ai}}),$$

where $\mathcal{K}_{\mathrm{Ai}}$ is the integral operator acting on $L^2(t, \infty)$ with the Airy kernel

$$K^{\mathrm{Ai}}(x, y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y},$$

i.e.,

$$F(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_t^{\infty} \cdots \int_t^{\infty} \det(K^{\mathrm{Ai}}(\xi_i, \xi_j))_{i,j=1}^n \, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n.$$

It describes the limiting distributions of extreme eigenvalues for many random matrix ensembles.

Hammersley's Poissonization of L_n

A key role is played by the following Poissonization of L_n:

$$\mathbb{P}\left(L_{N_r} \le l\right) := P(r; l) = e^{-r} \sum_{n=0}^{\infty} \mathbb{P}(L_n \le l) \frac{r^n}{n!},$$

where $N_r \in \{0, 1, 2, \ldots\}$ is a random variable with a Poisson distribution of intensity $r > 0$.

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where $N_r \in \{0, 1, 2, \ldots\}$ is a random variable with a Poisson distribution of intensity $r > 0$.

Combinatorial interpretation: the cumulative distribution of the length of longest up/right path from $(0, 0)$ to $(1, 1)$ with nodes chosen randomly according to N_r.

Hammersley's Poissonization of L_n

To answer Ulam's question, one needs to show

$$\lim_{r\to\infty} \mathbb{P}\left(\frac{L_{N_r} - 2\sqrt{r}}{r^{1/6}} \le t\right) = F(t).$$

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This, together with Johansson's de-Poissonization lemma, will lead to the distribution of L_n.

[Johansson, '98]

Hammersley's Poissonization of L_n

There are several representations of $P(r; l)$:

- a Toeplitz determinant of modified Bessel functions,

  [Gessel, '90]
- Fredholm determinats of various (discrete) integral operators,

  [Baik-Deift-Johansson, '00; Borodin-Okounkov-Olshanski, '00; Johansson, '01; ...]
- a unitary group integral.

  [Rains, '98]

Asymptotics of $P(r; l)$

The fluctuation of $P(r; l)$ around $2\sqrt{r}$ can be proved by

- the Toeplitz determinant representation + Riemann-Hilbert approach,

  [Baik-Deift-Johansson, '99]

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- the Toeplitz determinant representation + Riemann-Hilbert approach,

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- the Fredholm determinat representation

$$P(r; l) = E_2^{\mathrm{hard}}(4r; l),$$

where

$$E_2^{\mathrm{hard}}(s; \nu) := \det(I - K_\nu^{\mathrm{Bes}})\Big|_{L^2(0,s)}$$

with

$$K_\nu^{\mathrm{Bes}}(x, y) := \frac{J_\nu(\sqrt{x})\sqrt{y}J_\nu'(\sqrt{y}) - \sqrt{x}J_\nu'(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)}$$

being the Bessel kernel.

[Borodin-Forrester, '03; Forrester-Hughes, '94]

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being the Bessel kernel.

- The distribution of $P(r; l)$ for large r is equivalent to the hard-to-soft edge transition

$$\lim_{\nu\to\infty} E_2^{\mathrm{hard}}\left(\left(\nu - t(\nu/2)^{1/3}\right)^2; \nu\right) = F(t).$$

[Borodin-Forrester, '03]

Today's topic

Can we improve the distribution formula of L_n by establishing the first few finite-size correction terms or the asymptotic expansion? This is also known as **edgeworth expansions** in the literature.

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Relevant works:

- finite size corrections relating to distributions of the length of longest increasing subsequences,

  [Forrester-Mays, '23; Bornemann, '23]
- finite size corrections relating to the Laguerre ensemble,

  [Perret-Schehr, '16; Forrester-Trinh, '19]
- finite size corrections relating to the elliptic Ginibre matrices,

  [Byun-Lee, '23]
- $\cdots$

Main results

An expansion from the Bessel kernel to the Airy kernel

Theorem [Yao-Z., arXiv:2309.06733]

Let

$$\hat{K}_\nu^{\mathrm{Bes}}(x,y) := \sqrt{\phi_\nu'(x)\phi_\nu'(y)}K_\nu^{\mathrm{Bes}}(\phi_\nu(x),\phi_\nu(y))$$

with $\phi_\nu(t) = \nu^2(1-h_\nu t)^2$ and $h_\nu = 2^{-\frac{1}{3}}\nu^{-\frac{2}{3}}$ be the symmetrically transformed Bessel kernel. For any $\mathfrak{m} \in \mathbb{N}$, we have, as $h_\nu \to 0^+$,

$$\hat{K}_\nu^{\mathrm{Bes}}(x,y) = K^{\mathrm{Ai}}(x,y) + \sum_{j=1}^{\mathfrak{m}} K_j(x,y)h_\nu^j + h_\nu^{\mathfrak{m}+1}\cdot\mathcal{O}\left(e^{-(x+y)}\right),$$

uniformly valid for $t_0 \le x, y < h_\nu^{-1}$ with t_0 being any fixed real number. Preserving uniformity, the expansion can be repeatedly differentiated w.r.t. the variable x and y.

An expansion from the Bessel kernel to the Airy kernel

Theorem [Yao-Z., arXiv:2309.06733]

Here, K^{Ai} is the Airy kernel and

$$K_j(x,y) = \sum_{\kappa,\lambda \in \{0,1\}} p_{j,\kappa\lambda}(x,y)\mathrm{Ai}^{(\kappa)}(x)\mathrm{Ai}^{(\lambda)}(y)$$

with $p_{j,\kappa\lambda}(x,y)$ being polynomials in x and y. Moreover, we have

$$\begin{aligned} K_1(x,y) = \frac{1}{10}\big(&-3(x^2+xy+y^2)\mathrm{Ai}(x)\mathrm{Ai}(y) + 2(\mathrm{Ai}(x)\mathrm{Ai}'(y) + \mathrm{Ai}'(x)\mathrm{Ai}(y)) \\ &+ 3(x+y)\mathrm{Ai}'(x)\mathrm{Ai}'(y)\big). \end{aligned}$$

Remarks

The above theorem was stated by Bornemann (arXiv:2301.02022) under the condition that $0 \leq \mathfrak{m} \leq \mathfrak{m}_* = 100$, and it was conjectured therein to be true without such a restriction.

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One can lift the expansion of the kernel to the associated Fredholm determinants, as rigorously shown by Bornemann.

Asymptotic expansion of the hard-to-soft edge transition

Corollary [Yao-Z., arXiv:2309.06733]

With E_2^{hard} defined as before, we have, for any $\mathfrak{m} \in \mathbb{N}$, as $h_\nu \to 0^+$,

$$E_2^{\mathrm{hard}}(\phi_\nu(t);\nu) = F(t) + \sum_{j=1}^{\mathfrak{m}} F_j(t) h_\nu^j + h_\nu^{\mathfrak{m}+1} \cdot \mathcal{O}\left(e^{-3t/2}\right),$$

uniformly valid for $t_0 \le t < h_\nu^{-1}$ with t_0 being any fixed real number. Preserving uniformity, the expansion can be repeatedly differentiated w.r.t. the variable t. Here, F denotes the Tracy-Widom distribution and F_j are certain smooth functions.

Remarks

It can be shown that

$$F_1(t) = \frac{3t^2}{10}F'(t) - \frac{1}{5}F''(t),$$

$$F_2(t) = \left(\frac{2}{175} + \frac{32t^2}{175}\right)F'(t) + \left(-\frac{16t}{175} + \frac{9t^4}{200}\right)F''(t) - \frac{3t^2}{50}F'''(t) + \frac{1}{50}F^{(4)}(t).$$

[Bornemann, '23]

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[Bornemann, '23]

The above expansion serves as a preparatory step in establishing the expansion of L_n.

About the proofs

The proof of Bornemann

Recall that

$$K_\nu^{\mathrm{Bes}}(x, y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J'_\nu(\sqrt{y}) - \sqrt{x}J'_\nu(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)}.$$

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It is natural to use Olver's expansion of Bessel functions of large order in the transition region:

$$J_\nu(\nu+\tau\nu^{1/3}) \sim \frac{2^{1/3}}{\nu^{1/3}}\mathrm{Ai}(-2^{1/3}\tau)\sum_{k=0}^{\infty}\frac{A_k(\tau)}{\nu^{2k/3}} + \frac{2^{1/3}}{\nu^{1/3}}\mathrm{Ai}'(-2^{1/3}\tau)\sum_{k=0}^{\infty}\frac{B_k(\tau)}{\nu^{2k/3}},$$

for $|\arg\nu| \le \pi/2-\delta$ and fixed τ. Here, $A_k(\tau)$ and $B_k(\tau)$ are are certain rational polynomials of increasing degree.

[Olver, '52]

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for $|\arg\nu| \le \pi/2 - \delta$ and fixed τ. Here, $A_k(\tau)$ and $B_k(\tau)$ are are certain rational polynomials of increasing degree.

[Olver, '52]

It is **difficult** to check the divisibility by $x - y$ for all the coefficients.

A Riemann-Hilbert characterization of the Bessel kernel

The Bessel kernel admits the following representation:

$$K_\nu^{\mathrm{Bes}}(x,y) = \frac{1}{2\pi\mathrm{i}(x-y)} \begin{pmatrix} -e^{-\frac{\pi\mathrm{i}\nu}{2}} & e^{\frac{\pi\mathrm{i}\nu}{2}} \end{pmatrix} \Psi_+(y)^{-1}\Psi_+(x) \begin{pmatrix} e^{\frac{\pi\mathrm{i}\nu}{2}} \\ e^{-\frac{\pi\mathrm{i}\nu}{2}} \end{pmatrix}.$$

where

$$\Psi(z) = \sqrt{\pi}e^{-\frac{\pi\mathrm{i}}{4}} \begin{pmatrix} I_\nu\left((-z)^{\frac{1}{2}}\right) & -\frac{\mathrm{i}}{\pi}K_\nu\left((-z)^{\frac{1}{2}}\right) \\ (-z)^{\frac{1}{2}} I_\nu'\left((-z)^{\frac{1}{2}}\right) & -\frac{\mathrm{i}}{\pi}(-z)^{\frac{1}{2}} K_\nu'\left((-z)^{\frac{1}{2}}\right) \end{pmatrix} \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-\pi\mathrm{i}\nu} & 1 \end{pmatrix}, & \arg z \in (0, \frac{\pi}{3}), \\ I, & \arg z \in (\frac{\pi}{3}, \frac{5\pi}{3}), \\ \begin{pmatrix} 1 & 0 \\ e^{\pi\mathrm{i}\nu} & 1 \end{pmatrix}, & \arg z \in (\frac{5\pi}{3}, 2\pi), \end{cases}$$

with I_ν and K_ν being the modified Bessel functions of order ν.

Ideas of the proof

The matrix-valued function Ψ is solves a Riemann-Hilbert problem uniquely.

- Asymptotic analysis of Ψ for large $\nu \longrightarrow$ kernel expansion.
- This direct approach provides a better understanding of the coefficients in the expansion.

Ideas of the proof

Let $\mathrm{Ai}(x)$ be the classical Airy function. It is readily seen from the differential equation

$$\frac{\mathrm{d}^2\mathrm{Ai}(z)}{\mathrm{d}z^2} = z\mathrm{Ai}(z)$$

that

$$\mathrm{Ai}^{(m)}(x) = P_m(x)\mathrm{Ai}(x) + Q_m(x)\mathrm{Ai}'(x),$$

where $P_m(x)$ and $Q_m(x)$ are polynomials with $P_0(x) = 1$ and $Q_0(x) = 0$. They satisfy the recurrence relations

$$P_{n+1}(x) = P_n'(x) + xQ_n(x), \qquad Q_{n+1}(x) = Q_n'(x) + P_n(x).$$

Ideas of the proof

Lemma

With polynomials P_m and Q_m defined through the m-the derivative of $\mathrm{Ai}(x)$, we have

$$\sum_{j=0}^{N} \frac{1}{j!(N-j)!} \left(P_j(x)Q_{N+1-j}(x) - Q_j(x)P_{N+1-j}(x)\right) = 0, \qquad N \geq 1.$$

Thanks for your attention!

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Questions?