Asymptotics of Fredholm determinant associated with the Pearcey kernel

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Outline

- The Pearcey kernel
- Gap probability and Fredholm determinants
- Relation to the integrable systems: Lax pair, Hamiltonian
- Riemann-Hilbert analysis

Gaussian unitary ensemble with external source

- A deformed Gaussian unitary ensemble (GUE) [Brézin and Hikami, 1998]

$$\frac{1}{Z_n} e^{-n\mathrm{Tr}\left(\frac{M^2}{2} - AM\right)} dM, \tag{1}$$

  defined on the space of $n \times n$ Hermitian matrices, where Z_n is a normalization constant and A is a deterministic matrix which is diagonal with two eigenvalues a and $-a$ of equal multiplicity.

- The limiting mean density of eigenvalues is $v(x) = \frac{1}{\pi}|\operatorname{Im}\xi(x)|$, where $\xi(x)$ satisfies the Pastur's equation [Pastur, 1972]

$$\xi^3 - x\xi^2 - (a^2 - 1)\xi + xa^2 = 0.$$

- If $0 \le a < 1$, the eigenvalues accumulate on a single interval $[-x_1, x_1]$; while for $a > 1$, they accumulate on two disjoint ones: $[-x_1, -x_2] \cup [x_2, x_1]$.
- In the critical case $a = 1$, the gap between two disjoint interval closes at the origin and the limiting mean eigenvalue density exhibits a cusp-like singularity, i.e., the density vanishes like $|x|^{1/3}$ as $x \to 0$.

Non-intersecting Brownian paths

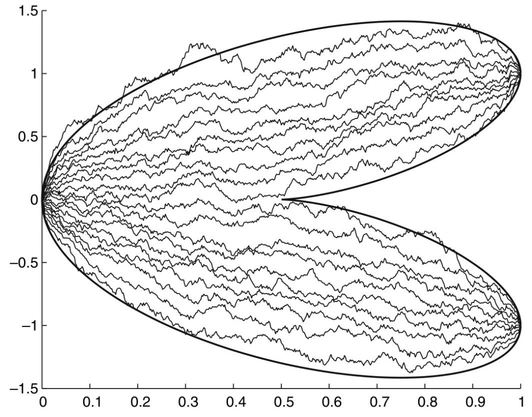


Figure: Non-intersecting Brownian paths that start at one point and end at two points.

[Bleher and Kuijlaars, 2007]

Limiting kernels

- The eigenvalues of the ensemble (1) are distributed according to a determinantal point process. [Brézin and Hikami, 1998], [Zinn-Justin, 1997]
- Inside the bulk: $v(x_0) > 0$

$$\lim_{n\to\infty} \frac{1}{nv(x_0)} K_n\left(x_0 + \frac{x}{nv(x_0)}, x_0 + \frac{y}{nv(x_0)}; a\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

- At the soft edge: $v(x) \sim \frac{c}{\pi}|x-x_1|^{1/2}$ as $x \to x_1$

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_n\left(x_1 + \frac{x}{(cn)^{2/3}}, x_1 + \frac{y}{(cn)^{2/3}}; a\right) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

- Upon letting $n \to \infty$ and $a \to 1$, the Pearcey kernel emerges near the origin: [Brézin and Hikami, 1998], [Tracy and Widom, 2006]

$$\lim_{n\to\infty} \frac{1}{n^{3/4}} K_n\left(\frac{x}{n^{3/4}}, \frac{y}{n^{3/4}}; 1 + \frac{\rho}{2\sqrt{n}}\right) = K^{\mathrm{Pe}}(x, y; \rho).$$

The Pearcey kernel

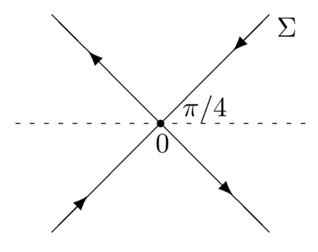
The Pearcey kernel K^{Pe} is defined as [Brézin and Hikami, 1998]

$$K^{\mathrm{Pe}}(x,y;\rho) = \int_0^\infty p(x+t)q(y+t)dt$$
$$= \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - \rho p(x)q(y)}{x-y}, \qquad (2)$$

where $\quad p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4}s^4 - \frac{\rho}{2}s^2 + isx} ds, \quad q(y) = \frac{1}{2\pi}\int_\Sigma e^{\frac{1}{4}s^4 + \frac{\rho}{2}s^2 + isy} ds$, and $\rho \in \mathbb{R}$.

The functions p and q satisfy

$$p'''(x) = xp(x) + \rho p'(x) \quad \text{and} \quad q'''(y) = -yq(y) + \rho q'(y).$$



The Pearcey kernel

The Pearcey kernel appears in various models:

- large complex correlated Wishart matrices; [Hachem, Hardy and Najim, 2016]
- a two-matrix model with special quartic potential; [Geudens and Zhang, 2015]
- general complex Hermitian Wigner-type matrices at the cusps; [Erdős, Krüger and Schröder, 2020]
- the non-intersecting Brownian motions at the critical time; [Bleher and Kuijlaars, 2007], [Adler, Orantin and van Moerbeke, 2010]
- a combinatorial model on random partitions. [Okounkov and Reshetikhin, 2007]

Gap probability: Tracy-Widom distribution as an example

- Gap probability: $E(A)$ = Prob(there is no eigenvalues in the interval A).
- Let λ_n be the largest eigenvalue of a GUE matrix M_n. Then, we have

$$\text{Prob}\left[M_n \text{ has no eigenvalue in } (2\sqrt{n} + \frac{s}{n^{1/6}}, +\infty)\right]$$
$$= \text{Prob}\left[\frac{\lambda_n - 2\sqrt{n}}{n^{-1/6}} \le s\right] \longrightarrow \det[I - \mathcal{K}^{\text{Ai}}_{(s,\infty)}],$$

where $\mathcal{K}^{\text{Ai}}_{(s,\infty)}$ is the trace-class operator acting on $L^2(s,\infty)$ with the Airy kernel $K^{\text{Ai}}(x,y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y}$.

- Tracy-Widom distribution: [Tracy and Widom, 1994]

$$F_{\text{TW}}(s) := \det[I - \mathcal{K}^{\text{Ai}}_{(s,\infty)}] = \exp\left(-\int_s^\infty (x-s) y_{HM}^2(x)dx\right).$$

Tracy-Widom distribution

- $y_{HM}(x)$ is the Hastings-McLeod solution to the Painlevé II equation

$$y''(x) = xy(x) + 2y^3(x), \quad \text{with}$$

$$y(x) \sim \mathrm{Ai}(x),\ x \to +\infty,\ y(x) \sim \sqrt{\frac{-x}{2}}\left(1 + \frac{1}{8x^3} + O(x^{-6})\right),\ x \to -\infty.$$

- We may also rewrite $F_{\mathrm{TW}}(s)$ as [Forrester and Witte, 2001]

$$F_{\mathrm{TW}}(s) = \exp\left(-\int_s^\infty H(x)dx\right),$$

where $H(x)$ is related to the Hamiltonian for the Painlevé II equation.

- Large gap asymptotics: [Deift, Its and Krasovsky, 2008]

[Baik, Buckingham and DiFranco, 2008]

$$\ln\det[I - \mathcal{K}^{\mathrm{Ai}}_{(s,\infty)}] = \frac{s^3}{12} - \frac{1}{8}\ln(-s) + \frac{1}{24}\ln 2 + \zeta'(-1) + O(|s|^{-\frac{3}{2}}),$$

as $s \to -\infty$, where $\zeta(s)$ is the Riemann zeta-function.

Deformed Tracy-Widom distribution

- Thinning is a classical operation in the theory of point processes: *remove* each particle with probability $1-\gamma$. If one thins a determinantal point process, the resulting process is still determinantal.
- The deformed Tracy-Widom distribution: [Bohigas et al., 2009]

$$\det[I-\gamma\mathcal{K}^{\text{Ai}}_{(s,\infty)}] = \exp\left(-\int_s^\infty (x-s)y_{AS}^2(x)dx\right), \qquad \gamma\in(0,1).$$

- $y_{AS}(x)$ is the Ablowitz-Segur solution to the Painlevé II equation satisfying

$$y_{AS}(x)\sim\sqrt{\gamma}\,\text{Ai}(x), \qquad 0<\gamma<1, \quad \text{as } x\to+\infty.$$

- Large gap asymptotics: [Bothner and Buckingham, 2018]

$$\ln\det[I-\gamma\mathcal{K}^{\text{Ai}}_{(s,\infty)}] = -\frac{2v}{3\pi}(-s)^{\frac{3}{2}}+\frac{v^2}{4\pi^2}\ln(8(-s)^{\frac{3}{2}})+\ln\left(G(1+\frac{iv}{2\pi})G(1-\frac{iv}{2\pi})\right)+O(|s|^{-1})$$

as $s\to-\infty$, where $v=-\ln(1-\gamma)\geq 0$ and $G(z)$ is the Barnes G-function.

Gap probabilities

- For the classical sine and Bessel kernels, the associated gap probabilities have been well understood, including the integral representations, large gap asymptotics, and the corresponding deformed cases.

  Baik, Basor, Bothner, Buckingham, Budylin, Buslaev, Charlier, Claeys, Deift, DiFranco, Its, Krasovsky, Tracy, Widom, Zhou...

- Painlevé kernels arise to describe various phase transitions in the critical situations under a double scaling limit. The corresponding Fredholm determinants are also studied. [D., Xu and Zhang, 2018], [Xu and D., 2019]

- Riemann-Hilbert problems (RHP) is a powerful method to derive the asymptotics.

Our target

In this talk, we consider

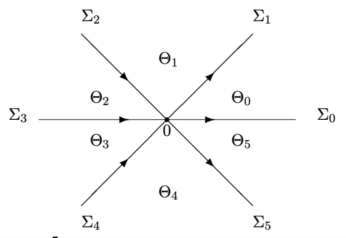
$$F(s;\gamma,\rho) := \ln\det\left(I-\gamma\mathcal{K}^{\mathrm{Pe}}_{s,\rho}\right), \qquad 0\le\gamma\le 1,$$

where $\mathcal{K}^{\mathrm{Pe}}_{s,\rho}$ is the trace class operator acting on $L^2(-s,s)$ with the Pearcey kernel.

RHP associated with the Pearcey kernel

A 3×3 matrix-valued function $\Psi(z) = \Psi(z;\rho)$ satisfies: [Bleher and Kuijlaars, 2007]

(1) $\Psi(z)$ is analytic in $\mathbb{C} \setminus \{\cup_{j=0}^{5}\Sigma_j\}$, where $\Sigma_{0,3} = \mathbb{R}_\pm, \Sigma_{1,4} = e^{\frac{\pi i}{4}}\mathbb{R}_\pm, \Sigma_{2,5} = e^{-\frac{\pi i}{4}}\mathbb{R}_\pm$.



(2) $\Psi_+(z) = \Psi_-(z)J_\Psi(z), \quad z \in \cup_{j=0}^{5}\Sigma_j$, where the jump matrices are

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ z \in \Sigma_0; \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ z \in \Sigma_1; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \ z \in \Sigma_2;$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ z \in \Sigma_3; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \ z \in \Sigma_4; \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \ z \in \Sigma_5.$$

RHP associated with the Pearcey kernel

(3) As $z \to \infty$ and $\pm \operatorname{Im} z > 0$, we have

$$\Psi(z) = \sqrt{\frac{2\pi}{3}} e^{\frac{\rho^2}{6}} i\Psi_0 \left(I + \frac{\Psi_{-1}}{z} + \mathcal{O}(z^{-2}) \right) \operatorname{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}} \right) L_{\pm} e^{\Theta(z)},$$

where $L_{\pm}$ are the constant matrices

$$L_+ = \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix}, \qquad L_- = \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}, \qquad \omega = e^{\frac{2\pi i}{3}},$$

and

$$\Theta(z) = \Theta(z;\rho) = \begin{cases} \operatorname{diag}(\theta_1(z;\rho), \theta_2(z;\rho), \theta_3(z;\rho)), & \operatorname{Im} z > 0, \\ \operatorname{diag}(\theta_2(z;\rho), \theta_1(z;\rho), \theta_3(z;\rho)), & \operatorname{Im} z < 0, \end{cases}$$

with $\theta_k(z;\rho) = \frac{3}{4}\omega^{2k} z^{\frac{4}{3}} + \frac{\rho}{2}\omega^k z^{\frac{2}{3}}, \quad k = 1, 2, 3.$

(4) $\Psi(z)$ is bounded near the origin.

The Pearcey kernel

- The above RH problem has a unique solution. For $z \in \Theta_1$, we have

[Bleher and Kuijlaars, 2007]

$$\Psi(z) = \begin{pmatrix} \mathcal{P}_0(z) & \mathcal{P}_1(z) & \mathcal{P}_4(z) \\ \mathcal{P}_0'(z) & \mathcal{P}_1'(z) & \mathcal{P}_4'(z) \\ \mathcal{P}_0''(z) & \mathcal{P}_1''(z) & \mathcal{P}_4''(z) \end{pmatrix}$$

with $\mathcal{P}_j(z) = \mathcal{P}_j(z;\rho) = \int_{\Gamma_j} e^{-\frac{1}{4}t^4 - \frac{\rho}{2}t^2 + itz} dt$, $j = 0, 1, \ldots, 5$, where

$$\Gamma_0 = (-\infty, +\infty),\ \Gamma_1 = (i\infty, 0] \cup [0, \infty),\ \Gamma_2 = (i\infty, 0] \cup [0, -\infty),$$
$$\Gamma_3 = (-i\infty, 0] \cup [0, -\infty), \Gamma_4 = (-i\infty, 0] \cup [0, \infty),\ \Gamma_5 = (-i\infty, i\infty).$$

- Let $\widetilde{\Psi}$ be the analytic extension of the restriction of Ψ on the region Θ_1 to the whole complex plane. The Pearcey kernel then admits the following equivalent representation:

$$\gamma K^{\mathrm{Pe}}(x, y; \rho) = \frac{\mathbf{f}(x)^t \mathbf{h}(y)}{x - y},$$

where $\mathbf{f}(x) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} := \widetilde{\Psi}(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\quad \mathbf{h}(y) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} := \frac{\gamma}{2\pi i} \widetilde{\Psi}(y)^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Integrable kernel

- If K is integrable and $(I-\mathcal{K})^{-1}$ exists, then the kernel $R(x,y)$ for the resolvent operator $R=(I-\mathcal{K})^{-1}-I$ is also integrable:

  [Its, Izergin, Korepin and Slavnov 1990]

$$R(x,y)=\frac{\mathbf{F}^t(x)\mathbf{H}(y)}{x-y}$$

  with $\mathbf{F}(x)=\Big((I-\mathcal{K})^{-1}\mathbf{f}\Big)(x)$ and $\mathbf{H}(x)=\Big((I-\mathcal{K}^t)^{-1}\mathbf{h}\Big)(x)$.

- Riemann-Hilbert problem associated with integrable kernels:
  - (a) $Y(z)$ is analytic for $z\in\mathbb{C}\backslash\Gamma$, where Γ is an interval in $\mathbb{R}$;
  - (b) For $x\in\Gamma$, $Y_+(x)=Y_-(x)\Big(I-2\pi i\,\mathbf{f}(x)\mathbf{h}^t(x)\Big)$;
  - (c) As $z\to\infty$, $Y(z)=I+\frac{Y_{-1}}{z}+\mathcal{O}(z^{-2})$;
  - (d) $Y(z)$ satisfies certain conditions at the endpoints of Γ.

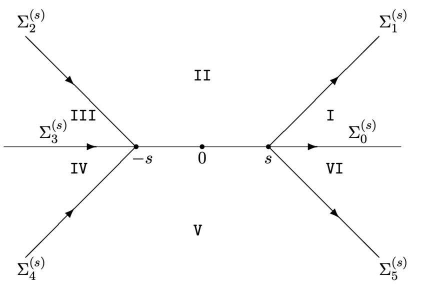
Lemma (Deift, Its and Zhou, 1997)

The unique solution of the above RH problem is: $\quad Y(z)=I-\displaystyle\int_\Gamma\frac{\mathbf{F}(x)\mathbf{h}^t(x)}{x-z}dx.$

RHP associated with the Fredholm determinant

Define $\Phi(z) = \Phi(z;s)$ as follows:

$$\Phi(z) = \frac{\Psi_0^{-1}}{\sqrt{\frac{2\pi}{3}e^{\frac{\rho^2}{6}}i}} \begin{cases} Y(z)\Psi(z), & z \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{VI}, \\ Y(z)\widetilde{\Psi}(z), & z \in \mathrm{II}, \\ Y(z)\widetilde{\Psi}(z)\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \mathrm{V}, \end{cases}$$



RHP associated with the Fredholm determinant

The function $\Phi(z) = \Phi(z; s)$ has the following properties:

(1) $\Phi(z)$ is analytic in $\mathbb{C} \setminus \{\cup_{j=0}^{5} \Sigma_j^{(s)} \cup [-s, s]\}$

(2) $\Phi_+(z) = \Phi_-(z) J_\Phi(z)$, $z \in \cup_{j=0}^{5} \Sigma_j^{(s)} \cup (-s, s)$, where the jump matrices are

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ z \in \Sigma_0^{(s)}; \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ z \in \Sigma_1^{(s)}; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \ z \in \Sigma_2^{(s)};$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ z \in \Sigma_3^{(s)}; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \ z \in \Sigma_4^{(s)}; \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \ z \in \Sigma_5^{(s)};$$

$$\begin{pmatrix} 1 & 1-\gamma & 1-\gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in (-s, s).$$

RHP associated with the Fredholm determinant

(3) As $z \to \infty$ and $\pm \operatorname{Im} z > 0$, we have

$$\Phi(z) = \left(I + \frac{\Phi_{-1}}{z} + \mathcal{O}(z^{-2})\right) \operatorname{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}\right) L_{\pm} e^{\Theta(z)},$$

where $\Phi_{-1} = \Psi_{-1} + \Psi_0^{-1} Y_{-1} \Psi_0$.

(4) As $z \to s$, we have

$$\Phi(z) = \widehat{\Phi}_1(z) \begin{pmatrix} 1 & -\frac{\gamma}{2\pi i} \ln(z-s) & -\frac{\gamma}{2\pi i} \ln(z-s) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{cases} I, & z \in \mathtt{II}, \\ \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \mathtt{V}, \end{cases}$$

where $\widehat{\Phi}_1(z)$ is analytic at $z = s$ satisfying the following expansion

$$\widehat{\Phi}_1(z) = \Phi_0^{(0)}(s) \left(I + \Phi_1^{(0)}(s)(z-s) + \mathcal{O}((z-s)^2)\right), \qquad z \to s.$$

(5) As $z \to -s$, the behavior of $\Phi(z)$ is determined by the following symmetric relation

$$\Phi(z) = -\operatorname{diag}(1, -1, 1)\Phi(-z) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Differential identity

Proposition

We have

$$\frac{d}{ds}F(s;\gamma,\rho) = \frac{d}{ds}\ln\det\left(I-\gamma\mathcal{K}^{\mathrm{Pe}}_{s,\rho}\right)$$
$$= -\frac{\gamma}{\pi i}\left[\left(\Phi_1^{(0)}(s)\right)_{21} + \left(\Phi_1^{(0)}(s)\right)_{31}\right]. \qquad (3)$$

Proof.

First,

$$\frac{d}{ds}\ln\det(I-\gamma\mathcal{K}^{\mathrm{Pe}}_{s,\rho}) = -\mathrm{tr}\left((I-\gamma\mathcal{K}^{\mathrm{Pe}}_{s,\rho})^{-1}\gamma\frac{d}{ds}\mathcal{K}^{\mathrm{Pe}}_{s,\rho}\right) = -R(s,s) - R(-s,-s).$$

From the lemma of Deift et al., we also get

$$\mathbf{F}(x) = Y_+(x)\mathbf{f}(x), \quad \mathbf{H}(x) = Y_+(x)^{-t}\mathbf{h}(x), \qquad x \in (-s,s).$$

Then, a straightforward computation yields (3). □

Lax Pair

Proposition

From the RHP for Φ, we have

$$\frac{\partial}{\partial z}\Phi(z;s) = L(z;s)\Phi(z;s), \quad \frac{\partial}{\partial s}\Phi(z;s) = U(z;s)\Phi(z;s), \tag{4}$$

where

$$\begin{aligned} L(z;s) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} + A_0(s) + \frac{A_1(s)}{z-s} + \frac{A_2(s)}{z+s}, \\ U(z;s) &= -\frac{A_1(s)}{z-s} + \frac{A_2(s)}{z+s} \end{aligned}$$

with $A_0(s) = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{2}p_0(s) & 0 & 1 \\ 0 & \sqrt{2}q_0(s) & 0 \end{pmatrix}$, $A_1(s) = \begin{pmatrix} q_1(s) \\ q_2(s) \\ q_3(s) \end{pmatrix}\begin{pmatrix} p_1(s) & p_2(s) & p_3(s) \end{pmatrix}$, *and* $A_2(s)$ *being related to* $A_1(s)$ *through the following relation*

$$A_2(s) = \mathrm{diag}(1,-1,1)A_1(s)\,\mathrm{diag}(1,-1,1). \tag{5}$$

Lax Pair

Proof.

All jumps in the RH problem for Φ are independent of z and s. This implies

$$L(z;s) := \frac{\partial}{\partial z}\Phi(z;s)\cdot\Phi(z;s)^{-1}, \qquad U(z;s) := \frac{\partial}{\partial s}\Phi(z;s)\cdot\Phi(z;s)^{-1}$$

are analytic in the complex z plane except for possible isolated singularities at $z=\pm s$ and $z=\infty$.

As $z\to\infty$,
$$L(z;s) = \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} z + \begin{pmatrix}0&1&0\\ \frac{\rho}{3}&0&1\\0&\frac{\rho}{3}&0\end{pmatrix} + \left[\Phi_{-1}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\right] + \mathcal{O}(z^{-1}).$$

As $z\to s$,
$$L(z;s) \sim -\frac{1}{z-s}\frac{\gamma}{2\pi i}\Phi_0^{(0)}(s)\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}\Phi_0^{(0)}(s)^{-1}.$$

By setting $\begin{pmatrix}q_1(s)\\q_2(s)\\q_3(s)\end{pmatrix} = \Phi_0^{(0)}(s)\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}p_1(s)\\p_2(s)\\p_3(s)\end{pmatrix} = -\frac{\gamma}{2\pi i}\Phi_0^{(0)}(s)^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix}$, we obtain the expression of $A_1(s)$. □

The Hamiltonian system

- From the general theory of Jimbo-Miwa-Ueno, the Hamiltonian $H(s) = H(p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3; s)$ is given by

$$H(s) = -\frac{\gamma}{2\pi i}\operatorname{Tr}\left(\Phi_1^{(0)}(s)\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right) = -\frac{\gamma}{2\pi i}\left[\left(\Phi_1^{(0)}(s)\right)_{21} + \left(\Phi_1^{(0)}(s)\right)_{31}\right].$$

- We have

$$\begin{aligned} H(s) = {} & \sqrt{2}p_0(s)p_2(s)q_1(s) + \sqrt{2}p_3(s)q_0(s)q_2(s) + p_1(s)q_2(s) + p_2(s)q_3(s) \\ & + sp_3(s)q_1(s) + \frac{1}{2s}\left(p_1(s)q_1(s) - p_2(s)q_2(s) + p_3(s)q_3(s)\right)^2, \end{aligned} \tag{6}$$

and

$$q_k'(s) = \frac{\partial H}{\partial p_k}, \qquad p_k'(s) = -\frac{\partial H}{\partial q_k}, \qquad k = 0, 1, 2, 3.$$

- The above Hamiltonian is equivalent to that used in Brézin and Hikami (PRE, 1998) under a simple rescalling.

Integral representation

Theorem

With the Hamiltonian $H(s)$ given in (6), we have

$$F(s;\gamma,\rho) = 2\int_0^s H(\tau)d\tau, \qquad \gamma \in [0,1], \quad s \in (0,+\infty). \tag{7}$$

$H(s)$ satisfies the following asymptotic behaviors: as $s \to 0^+$,

$$H(s) = \mathcal{O}(1),$$

and, as $s \to +\infty$,

$$H(s) = \begin{cases} -\frac{3s^{5/3}}{2^{11/3}} + \frac{\rho s}{4} - \frac{\rho^2 s^{1/3}}{3\cdot 2^{7/3}} - \frac{1}{9s} + \mathcal{O}(s^{-5/3}), & \gamma = 1, \\ \sqrt{3}\beta i s^{1/3} - \frac{\beta i \rho}{\sqrt{3}\, s^{1/3}} - \frac{4\beta^2}{3s} - \frac{2\sqrt{3}\beta i}{9s}\cos(2\vartheta(s)) + \mathcal{O}(s^{-5/3}), & \gamma \in [0,1), \end{cases}$$

where $\quad \beta := \frac{1}{2\pi i}\ln(1-\gamma) \in i\mathbb{R}_+, \ \gamma \in [0,1),$

and $\quad \vartheta(s) := \vartheta(s;\beta) = -\frac{3\sqrt{3}}{8}s^{\frac{4}{3}} + \frac{\sqrt{3}\rho}{4}s^{\frac{2}{3}} + \arg\Gamma(1-\beta) - \beta i\left(\frac{4}{3}\ln s + \ln\left(\frac{9}{2}\right)\right).$

Large gap asymptotics

Theorem

$$F(s;1,\rho) = -\frac{9s^{\frac{8}{3}}}{2^{\frac{17}{3}}} + \frac{\rho s^2}{4} - \frac{\rho^2 s^{\frac{4}{3}}}{2^{\frac{10}{3}}} - \frac{2}{9}\ln s + \frac{\rho^4}{216} + C + \mathcal{O}(s^{-\frac{2}{3}}), \qquad s \to +\infty,$$

uniformly for ρ in any compact subset of $\mathbb{R}$, where C is an undetermined constant independent of ρ and s.

Theorem

$$F(s;\gamma,\rho) = \frac{3\sqrt{3}\beta i}{2} s^{\frac{4}{3}} - \sqrt{3}\rho\beta i s^{\frac{2}{3}} - \frac{8\beta^2}{3}\ln s$$
$$- 2\beta^2 \ln\left(\frac{9}{2}\right) + 2\ln\left(G(1+\beta)G(1-\beta)\right) + \mathcal{O}(s^{-\frac{2}{3}}), \qquad s \to +\infty,$$

uniformly for γ and ρ in any compact subset of $[0,1)$ and $\mathbb{R}$, respectively, where $\beta = \frac{1}{2\pi i}\ln(1-\gamma)$ and $G(z)$ is the Barnes G-function.

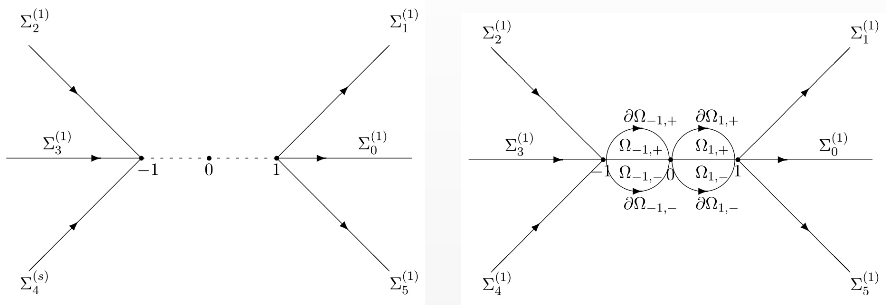


Figure: Contour in the RH analysis: $\gamma = 1$ (left); $0 < \gamma < 1$ (right)

Global parametrix: $\gamma = 1$

(1) $N(z)$ is analytic in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$.

(2) N satisfies the jump condition

$$N_+(x) = N_-(x) \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & x > 1, \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & x < -1. \end{cases}$$

(3) As $z \to \infty$, we have

$$N(z) = \left(I + O(z^{-1})\right) \operatorname{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}\right) L_{\pm}.$$

Global parametrix: $0 < \gamma < 1$

(1) $N(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(2) $N_+(z) = N_-(z) J_N(z), \ z \in \mathbb{R}$, where

$$J_N(z) = \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & z \in (-\infty, -1), \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (1, +\infty), \end{cases} \qquad J_N(z) = \begin{cases} \begin{pmatrix} 0 & 0 & 1-\gamma \\ 0 & 1 & 0 \\ \frac{1}{\gamma-1} & 0 & 0 \end{pmatrix}, & z \in (-1, 0), \\ \begin{pmatrix} 0 & 1-\gamma & 0 \\ \frac{1}{\gamma-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (0, 1). \end{cases}$$

(3) As $z \to \infty$, we have

$$N(z) = \left(I + O(z^{-1})\right) \operatorname{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}\right) L_{\pm}.$$

Local parametrix

- When $\gamma = 1$, local parametrices near $z = \pm 1$ are given in terms of the Bessel parametrix.
- When $0 \leq \gamma < 1$,
  - local parametrices near $z = \pm 1$ are given in terms of the confluent hypergeometric parametrix.
  - local parametrix near $z = 0$ is given in terms of the Pearcey parametrix.

Another differential identity

Proposition

We have the following differential identities with respect to the parameter γ:

$$\frac{\partial}{\partial\gamma}\left(\sum_{k=0}^{3} p_k(s)q_k'(s) - H(s)\right) = \frac{d}{ds}\left(\sum_{k=0}^{3} p_k(s)\frac{\partial}{\partial\gamma}q_k(s)\right). \tag{8}$$

Moreover, H is related to the action differential $\sum_{k=0}^{3} p_k q_k' - H$ by

$$\sum_{k=0}^{3} p_k(s)q_k'(s) - H(s) = H(s) + \frac{1}{4}\frac{d}{ds}\left(2p_0(s)q_0(s) + p_2(s)q_2(s) + 2p_3(s)q_3(s) - 3sH(s)\right). \tag{9}$$

Proof.

Differentiating $H(p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3; s)$ about γ, we get

$$\frac{\partial}{\partial\gamma}H(s) = \sum_{k=0}^{3}\left(\frac{\partial H}{\partial p_k}\frac{\partial}{\partial\gamma}p_k(s) + \frac{\partial H}{\partial q_k}\frac{\partial}{\partial\gamma}q_k(s)\right) = \sum_{k=0}^{3}\left(q_k'(s)\frac{\partial}{\partial\gamma}p_k(s) - p_k'(s)\frac{\partial}{\partial\gamma}q_k(s)\right).$$

□

Large gap asymptotics: constant determination for $\gamma < 1$

First, we obtain from (9) that

$$\int_0^s H(\tau)d\tau = \int_0^s \left(\sum_{k=0}^{3} p_k(\tau)q_k'(\tau) - H(\tau)\right)d\tau - \frac{1}{4}\Big[2p_0(\tau)q_0(\tau) + p_2(\tau)q_2(\tau) + 2p_3(\tau)q_3(\tau) - 3\tau H(\tau)\Big]_{\tau=0}^{s}.$$

By integrating both sides of (8) with respect to s, it follows that

$$\frac{\partial}{\partial\gamma}\int_0^s \left(\sum_{k=0}^{3} p_k(\tau)q_k'(\tau) - H(\tau)\right)d\tau = \sum_{k=0}^{3} p_k(s)\frac{\partial}{\partial\beta}q_k(s) - \sum_{k=0}^{3} p_k(0)\frac{\partial}{\partial\beta}q_k(0). \tag{10}$$

Note that $\sum_{k=0}^{3} p_k(s)q_k'(s) - H(s) \equiv 0$, when $\gamma = 0$. Integrating both sides of (10) with respect to γ, we obtain

$$\int_0^s \left(\sum_{k=0}^{3} p_k(\tau)q_k'(\tau) - H(\tau)\right)d\tau = \int_0^\gamma \left(\sum_{k=0}^{3} p_k(s)\frac{\partial}{\partial\beta'}q_k(s)\right)d\gamma'. \tag{11}$$

Summary

- We obtained an integral formula for the Fredholm determinant associated with the Pearcey kernel, which involves the associated Hamiltonian and holds for all $\gamma \in [0, 1]$.
- The large gap asymptotics for the Fredholm determinant are derived as $s \to +\infty$, which is not uniformly valid for $\gamma \in [0, 1]$. The exponent of the leading term drops by half, that is, $s^{\frac{8}{3}}$ in the unthinned ($\gamma = 1$) case reduces to $s^{\frac{4}{3}}$ in the thinned ($0 < \gamma < 1$) case.
- The constant term is given explicitly in terms of the Barnes G-function when $\gamma \in [0, 1)$. However, the constant terms is still unknown when $\gamma = 1$.

Thank You!

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Thank You!