Asymptotics of determinants for structured matrices

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Random Matrices and Related Topics in Jeju in May 2024

Toeplitz matrices and determinants

For $\phi \in L^1_{N\times N}$ defined on the unit circle $\mathbb{T}$ with Fourier coefficients

$$\phi_j = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}\,d\theta$$

define the Toeplitz and Hankel matrices by

$$T_n(\phi) = (\phi_{j-k})_{j,k=0}^{n-1} \quad \text{and} \quad H_n(\phi) = (\phi_{j+k+1})_{j,k=0}^{n-1}$$

which have the forms

$$\begin{pmatrix} \phi_0 & \phi_{-1} & \phi_{-2} & \cdots \\ \phi_1 & \phi_0 & \phi_{-1} & \cdots \\ \phi_2 & \phi_1 & \phi_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots \\ \phi_1 & \phi_2 & \phi_3 & \cdots \\ \phi_2 & \phi_3 & \phi_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We are interested in the behavior of

$$\det T_n(\phi) \quad \text{and} \quad \det\big(T_n(\phi) + H_n(\psi)\big)$$

as $n \to \infty$.

Toeplitz and Hankel operators

For $a \in L^\infty_{N\times n}$, define $T(a)$ and $H(a)$ on ℓ^2_N by

$$T(a) = (a_{j-k})_{j,k\geq 0} \quad \text{and} \quad H(a) = (a_{j+k+1})_{j,k\geq 0}.$$

Equivalently, we can view them as the operators

$$T(a)f = \left(\sum_{j=1}^{N} T(a_{kj})f_j\right)_{k=1}^{N} \quad \text{and} \quad H(a)f = \left(\sum_{j=1}^{N} H(a_{kj})f_j\right)_{k=1}^{N}$$

acting on the Hardy space $H^2_N = \{(f_1, \dots, f_N) : f_k \in H^2\}$, where

$$H^2 = \{f \in L^2(\mathbb{T}) : f_k = 0 \text{ for } k < 0\}$$

and

$$T(f) = PM_f \quad \text{and} \quad H(f) = PM_fJ$$

where P is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2, $M_fg = fg$, and $Jf(t) = \bar{t}f(\bar{t})$ ($t \in \mathbb{T}$) is the flip operator.

Szegő-Widom limit theorem

We say $a \in F$ if

$$\|a\|_F := \sum_{n=-\infty}^{\infty} |a_n| + \left(\sum_{n=-\infty}^{\infty} |n| \cdot |a_n|^2 \right)^{1/2} < \infty.$$

Theorem 1 (Szegő-Widom)

Let $\phi \in F_{N\times N}$ *be such that the determinant* $\det \phi(t)$ *does not vanish on all of* $\mathbb{T}$ *and has winding number zero. Then*

$$\lim_{n\to\infty} \frac{\det T_n(\phi)}{G[\phi]^n} = E(\phi) := \det T(\phi)T(\phi^{-1}),$$

where the right side is a well-defined operator determinant and

$$G[\phi] = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} (\log \det \phi)(e^{ix})\, dx \right) \tag{1}$$

in which $\log \det \phi$ *is continuous on* $\mathbb{T}$.

Comments on the Szegő-Widom limit theorem

- We can define $\det I + T$ when T is a trace class operator. Such T are compact with discrete eigenvalues λ_i that satisfy

$$\sum_{i=0}^{\infty} |\lambda_i| < \infty \quad \text{so that} \quad \det(I+T) = \prod_{i=0}^{\infty}(1+\lambda_i)$$

  is well defined.
- If both $T(\phi)$ and $T(\phi^{-1})$ are invertible, the proof is easier than in the general case where the stated assumption on $\det \phi(t)$ is equivalent to both $T(\phi)$ and $T(\phi^{-1})$ being Fredholm with index zero.
- The class F considered above can be replaced by more general classes, such as Krein algebras.
- In the scalar case $(N = 1)$ a multitude of different proofs of the classical Szegő Limit Theorem exist; including those that use Riemann-Hilbert problems (RHPs).

Borodin-Okounkov-Case-Geronimo formula

Requires an invertibility assumption on ϕ, so not as general but an extremely insightful and useful exact formula.

An invertible symbol ϕ has a Wiener-Hopf factorization

$$\phi = u_- u_+ = v_+ v_-$$

if the kth Fourier coefficients of $u_+, u_+^{-1}, v_+, v_+^{-1}$ vanish for $k > 0$ and $u_-, u_-^{-1}, v_-, v_-^{-1}$ vanish for $k < 0$.

Theorem 2 (Borodin-Okounkov-Case-Geronimo)

If the conditions of the Szegö-Widom theorem hold and if, in addition, $T(\phi)$ and $T(\phi^{-1})$ are invertible, then

$$\det T_n(\phi) = G(\phi)^n E(\phi) \cdot \det\left(I - H(z^{-n} v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+ z^{-n})\right),$$

where $\tilde{f}(z) = f(\bar{z})$.

Singular version of BOCG!?

Szegő-Widom theorem à la BOCG

Define $P_n x = (x_0, \ldots, x_{n-1}, 0, 0, \ldots)$ for $x \in \ell^2_N$ and let $Q = I - P_n$. Then

$$H(z^{-n} v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+ z^{-n}) = Q_n \, H(v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+) \, Q_n.$$

Since $Q_n x \to 0$ for each $x \in \ell^2_N$,

$$Q_n \, H(v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+) \, Q_n$$

tends to zero in the trace norm, and so

$$\det(I - Q_n \, H(v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+) \, Q_n) \to 1.$$

Thus, by the BOCG formula,

$$\begin{aligned} \det T_n(\phi) &= G(\phi)^n E(\phi) \cdot \det \left(I - H(z^{-n} v_- u_+^{-1}) H(\tilde{u}_-^{-1} \tilde{v}_+ z^{-n}) \right) \\ &\sim G(\phi)^n \, E(\phi) \end{aligned}$$

as $n \to \infty$.

Toeplitz plus Hankel matrices

For matrix symbols ϕ and ψ, consider

$$P_n T(\phi) + H(\psi) P_n \quad \text{or} \quad (\phi_{j-k} + \psi_{j+k+1}).$$

Their determinants arise naturally in applications:

- If $\psi = -\phi$ and if $\phi = g\tilde{g}$ for appropriately defined g, the determinant of $P_n T(\phi) + H(\psi) P_n$ computes the average of $\det g(U)$ over the subgroup of orthogonal matrices of size $2n$ with determinant 1.
- Other averages over the classical compact groups can be computed using different choices of ψ.

These averages are important in RMT and number theory[1]

[1] see, e.g., Baik, Rains: Algebraic aspects of increasing subsequences (2001); Conrey, Forrester, Snaith: Averages of ratios of characteristic polynomials for the compact classical groups (2005); Conrey, Snaith: Applications of the L-functions ratios conjectures (2007).

Asymptotics of Toeplitz plus Hankel determinants

This is somewhat similar to the case of Toeplitz determinants.

Theorem 3 (Basor-Ehrhard-V)

Suppose ϕ satisfies the conditions necessary for the BOCG identity and that $H(\psi)$ is a trace class operator. Then

$$\det(P_n T(\phi) + H(\psi) P_n) \sim G(\phi)^n E(\phi)\, \det(I + T(\phi)^{-1} H(\psi))$$

where the constants $G(\phi)$ and $E(\phi)$ are defined as before.

While the above result yields a general asymptotic formula, it is not in some sense complete. One would like a BOCG type formula and in the scalar case it would be useful to have more concrete descriptions for the constants.

Under additional assumptions, we obtain a BOCG type formula.

These include the following cases (but also other "compatible pairs"):

(I) $M(\phi) = T(\phi) + H(\phi)$,

(II) $M(\phi) = T(\phi) - H(\phi)$,

(III) $M(\phi) = T(\phi) - H(z^{-1}\phi)$,

Theorem 4 (Basor-Ehrhardt-V)

If $\phi = \phi_+\phi_-$ *and* ϕ *is even, that is,* $\phi(e^{i\theta}) = \phi(e^{-i\theta})$, *then*

$$\det P_n M(\phi) P_n = G(\phi)^n E \det(I + Q_n\, K\, Q_n),$$

where $E = \det T(\phi_+^{-1}) M(\phi) T(\phi_-^{-1})$ *and* $K = M(\phi_+^{-1}) T(\phi_+) - I$.

In each case (I)–(III), K can be expressed as a Hankel operator.

Finite sections of functions of Toeplitz operators

Theorem 5 (Basor-Ehrhardt-V)

If $\phi \in F$, f is analytic on a region containing $\sigma(T(\phi))$, and both $T(f(\phi))$ and $T(f(\phi)^{-1})$ are invertible, then

$$\det P_n f(T(\phi))P_n \sim G(f(\phi))^n \det f(T(\phi))T((f(\phi)^{-1}).$$

Proof. To show first that $f(T(\phi)) - T(f(\phi))$ is trace class, note

$$T(\phi)T(\phi^{-1}) = I - H(\phi)H(\widetilde{\phi}^{-1})$$

is the identity plus a trace class operator, so that $T(\phi)^{-1} - T(\phi^{-1})$ is trace class. Write

$$f(T(\phi)) - T(f(\phi)) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)\,(z - T(\phi))^{-1}\,dz - \frac{1}{2\pi i}T(\int_{\mathcal{C}} f(z)\,(z-\phi)^{-1}\,dz)$$

where $\mathcal{C}$ is any contour in $\mathbb{C} \setminus \sigma(T(\phi))$, and use an approximation argument to show that $f(T(\phi)) - T(f(\phi))$ is also trace class.

Proof cont. Since $f(T(\phi)) - T(f(\phi)) = T$ for some $T \in S_1$, it follows that

$$f(T(\phi))T(f(\phi))^{-1} - I = T' \implies f(T(\phi))T(f(\phi))^{-1} = I + T'$$

for some $T' \in S_1$. Thus, since $T(f(\phi))^{-1} - T((f(\phi)^{-1})$ is also trace class, it follows that $f(T(\phi))\, T((f(\phi))^{-1}) = I + S$ for some $S \in S_1$, so

$$\det f(T(\phi))T((f(\phi))^{-1})$$

is well defined.

Now write $f(\phi) = (f(\phi))_+(f(\phi))_-$ where $(f(\phi))_\pm$ are the Wiener-Hopf factors of $f(\phi)$ guaranteed to exist by the invertibility of $T(f(\phi)^{-1})$.

Notice

$$P_n f(T((\phi))P_n = P_n T(f(\phi))_+)T(f(\phi))_+^{-1})f(T(\phi))T(f(\phi)_-^{-1})T(f(\phi)_-)P_n.$$

Since the outside operators are lower and upper triangular, the RHS equals

$$P_n T(f(\phi))_+)P_n T(f(\phi))_+^{-1})f(T(\phi))T(f(\phi)_-^{-1})P_n T(f(\phi)_-)P_n.$$

Taking determinants shows that $\det P_n f(T((\phi))P_n$ equals

$$\underbrace{(\det(f(\phi)_+)_0)^n\,(\det(f(\phi)_-)_0)^n}_{=G(f(\phi))^n}\ \underbrace{\det P_n T(f(\phi))_+^{-1})f(T(\phi))T(f(\phi)_-^{-1})P_n}_{\to \det f(T(\phi))T((f(\phi)^{-1})}.$$

Example 6

If $f(z) = e^z$, then

$$\det P_n e^{T(\phi)} P_n \sim G(e^{\phi})^n \det e^{T(\phi)} T(e^{-\phi}) \quad \text{as } n \to \infty.$$

In the scalar case, the constant can be computed

$$\det e^{T(\phi)} T(e^{-\phi}) = \exp\Big(\frac{1}{2} \sum_{k=1}^{\infty} k \phi_k \phi_{-k}\Big).$$

Symbols with singularities

When $N = 1$, the asymptotics are known for symbols possessing zeros, (integrable) singularities, jumps, and nonzero winding:

- Ehrhardt 2001 using operator theory
- Deift, Its, and Krasovsky 2011 using RHPs

We are interested in the jump type singularities. Let $N = 1$ and

$$\phi(t) = \phi_0(t) \prod_{k=1}^{R} u_{\beta_k,\tau_k}(t) \tag{2}$$

where ϕ_0 is smooth with no zeros and no winding and

$$u_{\beta,\tau}(t) = (-t/\tau)^{\beta} = \exp(i\beta \arg(-t/\tau)) = e^{i\beta(\theta-\theta_0-\pi)}, \qquad t = e^{i\theta} \tag{3}$$

with $\tau = e^{i\theta_0}$ and $|\arg(\,\cdot\,)| < \pi$.

Notice that

$$u_{\beta,\tau}(\tau \pm 0) = \exp(\mp i\beta\pi)$$

has one jump at τ.

Theorem 7 (Ehrhardt 2001; Deift, Its, Krasovsky 2011)

Suppose that

$$|\operatorname{Re}\beta_k - \operatorname{Re}\beta_j| < 1 \quad \text{and} \quad \beta_k \notin \mathbb{Z}\setminus\{0\}$$

for $1 \leq j,k \leq R$. *Then*

$$D_n[\phi] \sim G(\phi_0)^n\, n^{\Omega} E, \qquad \textit{as } n \to \infty, \tag{4}$$

where $E \neq 0$ *can be described and* $\Omega = -\sum_{k=1}^{R} \beta_k^2$.

- Proved by Basor 1978 under the assumption
  (a) $\operatorname{Re}\beta_k = 0$ for all $1 \leq k \leq R$.
- Basor 1979 and Böttcher 1982 under the weaker assumption
  (b) $|\operatorname{Re}\beta_k| < 1/2$ for all $1 \leq k \leq R$.
- If $|\operatorname{Re}\beta_k - \operatorname{Re}\beta_j| = 1$ is attained for at least some j,k, then the Fisher-Hartwig asymptotics breaks down and a generalized asymptotic formula was proved by Deift, Its, Krasovsky 2011 using RHPs.

Symbol classes

To specify the smoothness condition, introduce two classes of functions, which generalize $C^{1+\varepsilon}(\mathbb{T})$ of differentiable functions with a Hölder-Lipschitz continuous derivative of order $0 < \varepsilon < 1$.

Definition 8

Denote by $PC^{1+\varepsilon}(\mathbb{T};\Gamma)$ the set of functions $a \in PC(\mathbb{T};\Gamma)$ that are is continuously differentiable on $\mathbb{T} \setminus \Gamma$ with a derivative satisfying a Hölder condition of order $\varepsilon > 0$ on each arc (τ_k, τ_{k+1}).
Furthermore, let

$$C^{1+\varepsilon}_{\mathrm{pw}}(\mathbb{T};\Gamma) = PC^{1+\varepsilon}(\mathbb{T};\Gamma) \cap C(\mathbb{T}), \tag{5}$$

(the continuous functions with a piecewise Hölder derivative).

Both $PC^{1+\varepsilon}(\mathbb{T};\Gamma)$ and $C^{1+\varepsilon}_{\mathrm{pw}}(\mathbb{T};\Gamma)$ are Banach algebras with

$$\|a\| = \|a\|_\infty + \sum_{k=1}^{R} \sup_{\theta_k < x < y < \theta_{k+1}} \frac{|a'(e^{ix}) - a'(e^{iy})|}{|x-y|^{\varepsilon}},$$

where $\theta_{R+1} = \theta_1 + 2\pi$.

Main theorem

Given a matrix $B \in \mathbb{C}^{N\times N}$, define the matrix analog of $u_{\beta,\tau}$ by

$$u_{B,\tau}(t) = (-t/\tau)^B = \exp(iB\arg(-t/\tau)), \qquad t \in \mathbb{T}. \tag{6}$$

Notice that

$$u_{B,\tau}(\tau \pm 0) = e^{\mp\pi i B}$$

has a single jump at τ.

Theorem 9 (Basor-Ehrhardt-V arXiv:2307.00825)

Let $\phi \in PC^{1+\varepsilon}_{N\times N}(\mathbb{T};\Gamma)$ and suppose that $T(\phi)$ is Fredholm of index zero.
(i) *Then ϕ admits a unique representation of the form*

$$\phi(t) = \phi_0(t)\phi_1(t)\cdots\phi_R(t) \tag{7}$$

where $\phi_0 \in C^{1+\varepsilon}_{\mathrm{pw}}(\mathbb{T};\Gamma)^{N\times N}$ is an invertible function with $\mathrm{wind}(\det\phi_0) = 0$ *and*

$$\phi_k(t) = u_{B_k,\tau_k}(t), \qquad 1 \le k \le R,$$

with the matrices $B_k \in \mathbb{C}^{N\times N}$ having the property that the real parts of all their eigenvalues $\beta_k^{(1)},\ldots,\beta_k^{(N)}$ are contained in $I = (-1/2, 1/2)$.

(ii) Moreover,

$$\lim_{n\to\infty} \frac{\det T_n(\phi)}{G^n n^{\Omega}} = E \tag{8}$$

where

$$G = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} (\log\det\phi_0)(e^{ix})\,dx\right), \tag{9}$$

$$\Omega = -\sum_{k=1}^{R}\sum_{j=1}^{N} (\beta_k^{(j)})^2, \tag{10}$$

$$E = \prod_{k=1}^{R}\prod_{j=1}^{N} G(1+\beta_k^{(j)})\,G(1-\beta_k^{(j)}) \tag{11}$$

$$\times \det\Big(T(\phi)T(\phi_R)^{-1}\cdots T(\phi_1)^{-1}T(\phi_1^{-1})^{-1}\cdots T(\phi_R^{-1})^{-1}T(\phi^{-1})\Big),$$

where the Barnes G-function is defined by

$$G(1{+}z) = (2\pi)^{z/2}e^{-(z+1)z/2-\gamma_E z^2/2}\prod_{k=1}^{\infty}\left(\left(1+\frac{z}{k}\right)^k e^{-z+z^2/(2k)}\right) \tag{12}$$

with γ_E being Euler's constant.

If the description of E is of no interest, the main result can be simplified, an the product representation (7) is not needed.

Corollary 10

Let $\phi \in PC^{1+\varepsilon}(\mathbb{T};\Gamma)^{N\times N}$ and and suppose that $T(\phi)$ is Fredholm of index zero.
Then

$$\lim_{n\to\infty} \frac{\det T_n(\phi)}{G^n n^{\Omega}} = E$$

where

$$G = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} (\log c)(e^{ix})\,dx\right), \tag{13}$$

$$\Omega = -\sum_{k=1}^{R} \operatorname{trace}\left((L_k)^2\right), \tag{14}$$

$$L_k = \frac{1}{2\pi i}\log\left(\phi(\tau_k+0)^{-1}\phi(\tau_k-0)\right) \quad \text{and} \quad c(t) = \frac{\det\phi(t)}{\prod_{k=1}^{R} u_{\beta_k,\tau_k}(t)}$$

Moreover, $E \neq 0$ if and only if $T(\phi)$ and $T(\phi^{-1})$ are invertible.

Remark. Our results only give the actual asymptotic behavior of $\det T_n(\phi)$ when E is nonzero (which is not easy to determine).

Comments on the main theorem

- The part involving the Barnes G-function in the constant E is always nonzero.
- The second part of the constant E is a well-defined operator determinant, i.e., it is the determinant of an operator of the form identity plus a trace class operator.
- In particular, the Toeplitz operators $T(\phi_k)$ and $T(\phi_k^{-1})$, $1 \leq k \leq R$, appearing therein are invertible.
- Note that $T(\phi_0)$ and $T(\phi_0^{-1})$ do not occur in the product. In fact, $T(\phi_0)$, $T(\phi_0^{-1})$, $T(\phi)$, or $T(\phi^{-1})$ need not be invertible. Our assumptions only imply that these four operators are Fredholm of index zero (equivalent to invertibility if $N = 1$).
- It is possible the operator-determinant (and hence E) is zero, namely when $T(\phi)$ or $T(\phi^{-1})$ is not invertible.
- If ϕ has no jumps ($R = 0, \Gamma = \emptyset$), we get the Szegő-Widom limit theorem. No other general explicit expression is known for the operator determinant in the constant E in the block case ($N \geq 2$). For certain very special classes, E can be computed, e.g. the work of A. Its et al. below (using RHPs).

Approach

Based in part on the localization or separation theorem of Basor 1979 which states that when the symbols ϕ and ψ do not have common singularities and satisfy certain invertibility and smoothness criteria off the singularites, then

$$\lim_{n\to\infty} \frac{D_n[\phi\psi]}{D_n[\phi]D_n[\psi]} = \det\left(T^{-1}(\phi)T(\phi\psi)T^{-1}(\psi)\right)\left(T^{-1}(\tilde{\phi})T(\widetilde{\phi\psi})T^{-1}(\tilde{\psi})\right).$$

Thus, our strategy is

(1) Compute $\lim_{n\to\infty} D_n[f]$ for "canonical symbols" f.

(2) Prove a localization theorem that pieces together the asymptotics from (1).

Approach cont.

- It may seem Basor's original idea should easily transform to the matrix-valued symbol case.
- However, her localization theorem requires at each step that certain semi-infinite Toeplitz operators be invertible.
- In the scalar case this is not an issue because if two invertible Toeplitz operators have bounded symbols that have disjoint singularities, then the Toeplitz operator with the product symbol is also invertible.
- However in the block case, one can only say that the resulting operator is Fredholm of index zero.
- Thus a new version of the localization theorem needs to be proved that does not require the same invertibility conditions.

Application: Entanglement entropy

Entanglement entropy of various quantum spin chain models, such as the XX, XY and Ising chains, can be computed using the Szegő-Widom limit theorem or determinants involving Toeplitz matrices generated by 2×2 matrix-valued symbols that possess jump discontinuities.

The use of the Szegő-Widom limit theorem still requires the computation of the constant in the expansion that is known only in rare cases, such as the work of Its, Mezzadri and Mo, who computed the von Neumann entropy of entanglement of the ground state of a wide family of one-dimensional quantum spin chain models (incl. the XX and the XY models).

The basic idea of how Toeplitz determinants enter the study of entanglement is as follows.

Consider the Hamiltonian

$$H_\alpha = -\frac{\alpha}{2} \sum_{0 \le j \le k \le M-1} \left((A_{jk} + \gamma B_{jk}) \sigma_j^x \sigma_k^x \prod_{l=j+1}^{k-1} \sigma_l^z \right.$$
$$\left. + (A_{jk} - \gamma B_{jk}) \sigma_j^y \sigma_k^y \prod_{l=j+1}^{k-1} \sigma_l^z \right) - \sum_{j=0}^{M-1} \sigma_j^z, \quad (15)$$

where $\sigma_j^x, \sigma_j^y, \sigma_j^z$ stand for the Pauli matrices which describe spin operators on the jth lattice site of a chain with M sites, A is symmetric, B is antisymmetric, and both are translation-invariant.

If we divide the system into two subchains, denoting the part containing the first L spins by A and the second part containing the remaining $M - L$ spins by B with $1 \ll L \ll M$, then the von Neumann entropy $S(\rho_A)$ is given by

$$S(\rho_A) = -\operatorname{trace} \rho_A \log \rho_A, \quad (16)$$

where $\rho_A = \operatorname{trace}_B \rho_{AB}$ and $\rho_{AB} = |\Psi_g\rangle\langle\Psi_g|$.

Its, Mezzadri, Mo used the following

$$S(\rho_A) = \lim_{\epsilon \to 0} \frac{1}{4\pi i} \int_{\Gamma(\epsilon)} e(1+\epsilon, \lambda) \frac{d \log D_L(\lambda)}{d\lambda} d\lambda, \tag{17}$$

where $\Gamma(\epsilon)$ is the contour depicted in Figure 1 and oriented counterclockwise,

$$e(x, y) = -\frac{x+y}{2} \log\left(\frac{x+y}{2}\right) - \frac{x-y}{2} \log\left(\frac{x-y}{2}\right),$$

and $D_L(\lambda)$ is the Toeplitz determinant of some symbol ϕ depending on the model.

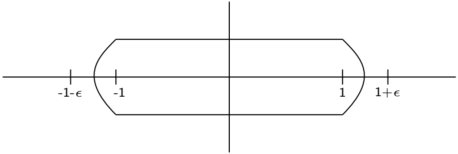

Figure: The contour $\Gamma(\epsilon)$ of the integral in (17).

Previously, the entropy of the XY model and its generalization was computed using the Szegő-Widom theorem when ϕ is nice enough.

However, in critical cases, such as when $\alpha = 1$, ϕ has jumps and the Szegő-Widom limit theorem no longer applies.

This motivates the study of the asymptotics of Toeplitz determinants with piecewise continuous matrix-valued symbols.

Consider the symbol

$$\phi(e^{i\theta}) = \begin{pmatrix} i\lambda & g(\theta) \\ -g(\theta)^{-1} & i\lambda \end{pmatrix}$$

where

$$g(\theta) = \frac{\alpha\cos\theta - 1 - i\gamma\alpha\sin\theta}{|\alpha\cos\theta - 1 - i\gamma\alpha\sin\theta|}.$$

When $\alpha = 1$ this symbol has a jump at $\theta = 0$.

The "jump ratio matrix" is

$$\frac{1}{\lambda^2-1}\begin{pmatrix} \lambda^2+1 & 2\lambda \\ 2\lambda & \lambda^2+1 \end{pmatrix}.$$

The eigenvalues of the above are

$$\frac{\lambda+1}{\lambda-1} \quad \text{and} \quad \frac{\lambda-1}{\lambda+1}.$$

Thus when λ is real and in $(-1,1)$ the corresponding βs have real parts $\pm 1/2$, which is not covered by our theorem, although it is possible the results still hold.

At $\lambda = \pm 1$, the symbol is not invertible and thus also not covered by our theorem. For other values of λ the asymptotics of the determinants are covered by our results. Note that $\det \phi = 1-\lambda^2$ is a constant, and hence the I-winding number of ϕ is zero.

Theorem 11 (Basor-Ehrhardt-V)

Suppose that $\lambda \notin [-1,1]$. Then

$$D_n[\phi] \sim (1-\lambda^2)^n \; n^{\Omega} \; E$$

where $\Omega = -2\beta^2$, E is given in (11)*, and $\beta = \frac{1}{2\pi i}\log(\frac{\lambda+1}{\lambda-1})$ with the appropriately chosen logarithms.*

Another example where the jump discontinuities occur can be found in the works of F. Ares, J.G Esteve, F. Falceto, A.R. de Queiroz where similar entanglement problems related to spin chain models (such as fermionic and Kitaev chains) are studied (non-rigorously). There the symbol in question is of the form

$$\phi(e^{i\theta}) = \lambda I - \frac{1}{\Lambda(\theta)} M(\theta), \tag{18}$$

where

$$M(\theta) = \begin{pmatrix} h + 2\cos\theta & G(\theta) \\ -G(\theta) & -h - 2\cos\theta \end{pmatrix}, \ \Lambda(\theta) = \sqrt{(h + 2\cos\theta)^2 + |G(\theta)|^2}$$

$$G(\theta) = \begin{cases} -i(\pi + \theta), & -\pi \leq \theta < -\theta_0 \\ -i\theta, & -\theta_0 < \theta < \theta_0 \\ i(\pi - \theta), & \theta_0 < \theta \leq \pi \end{cases}$$

Again we can use our results to compute the asymptotics of $D_n(\phi)$.

Special Session on Random Matrix Theory and Free Probability

Organized by Haakan Hedenmalm, Jani Virtanen, and hopefully someone from free probability

IWOTA 2024 in Canterbury, England
August 12-16, 2024

`https://blogs.kent.ac.uk/iwota2024`

The topics include applications of operator theory, complex analysis, and probability to random matrix theory, mathematical physics, and hopefully connections and an exchange of ideas between these areas and free probability.

For further info and to participate,
contact Jani Virtanen at jani.virtanen@helsinki.fi.

Preliminary results: pure jumps

For pure jumps, via Jordan normal forms, diagonal forms, and upper triangular forms, it can be shown the matrix case completely reduces to the scalar case, for which these results are known.

Proposition 12

Assume that the eigenvalues $\beta^{(1)}, \ldots, \beta^{(N)}$ of an $N \times N$ matrix B have real parts in $I = (-1/2, 1/2)$. Then the operators $T(u_{B,\tau})$ and $T(\tilde{u}_{B,\tau})$ are invertible on $(\ell^2)^N$ and the sequence $T_n(u_{B,\tau})$ is stable. Furthermore,

$$\det T_n(u_{B,\tau}) = E n^{\Omega}(1 + o(1)), \qquad \text{as } n \to \infty$$

with

$$\Omega = -\sum_{k=1}^{N} (\beta^{(k)})^2, \qquad E = \prod_{k=1}^{N} G(1 + \beta^{(k)}) G(1 - \beta^{(k)}),$$

where $G(z)$ stands for the Barnes G-function (12).

Localization result

Theorem 13

Assume $\phi = \phi_0\phi_1\cdots\phi_R$ *where* $\phi_k = u_{B_k,\tau_k}$, $1 \le k \le R$, *and such that the eigenvalues of* $B_k \in \mathbb{C}^{N\times N}$ *have real parts in the interval* $I = (-1/2, 1/2)$. *Moreover, suppose also that* $\phi_0 \in C^{1+\varepsilon}_{\mathrm{pw}}(\mathbb{T};\Gamma)^{N\times N}$ *is such that both* $T(\phi_0)$ *and* $T(\tilde{\phi}_0)$ *are invertible on* $(\ell^2)^N$. *Then*

$$\lim_{n\to\infty} \frac{\det T_n(\phi)}{G[\phi_0]^n \prod_{k=1}^{R} \det T_n(\phi_k)} = E \tag{19}$$

where $E = E_1E_2E_3$ *and*

$$\begin{aligned} E_1 &= \det T(\phi)T(\phi_R)^{-1}\cdots T(\phi_0)^{-1}, \\ E_2 &= \det T(\tilde{\phi})T(\tilde{\phi}_R)^{-1}\cdots T(\tilde{\phi}_0)^{-1}, \\ E_3 &= \det T(\phi_0)T(\phi_0^{-1}), \end{aligned}$$

and $G[\phi_0]$ *is defined in* (1).

Proof of the localization result

It can be shown $H(a)H(\tilde{b})$ and $H(\tilde{a})H(b)$ are trace class when $a = \phi_0 \cdots \phi_{k-1}$ and $b = \phi_k$, $1 \le k \le R$ (see Lemma 4.2).

Thus, Widom's formula $T(ab) = T(a)T(b) + H(a)H(\tilde{b})$ shows that

$$K_1 = T(\phi) - T(\phi_0)T(\phi_1)\cdots T(\phi_R) \tag{20}$$

$$K_2 = T(\tilde{\phi}) - T(\tilde{\phi}_0)T(\tilde{\phi}_1)\cdots T(\tilde{\phi}_R) \tag{21}$$

are trace class. Define

$$P_n(f_0, f_1, \ldots) = (f_0, f_1, \ldots, f_{n-1}, 0, 0, \ldots)$$
$$W_n(f_0, f_1, \ldots) = (f_{n-1}, f_{n-2}, \ldots, f_0, 0, 0, \ldots).$$

Using another formula of Widom

$$T_n(ab) = T_n(a)T_n(b) + P_n H(a)H(\tilde{b})P_n + W_n H(\tilde{a})H(b)W_n$$

and induction, we can show that

$$T_n(\phi) = T_n(\phi_0)T_n(\phi_1)\cdots T_n(\phi_R) + P_n K_1 P_n + W_n K_2 W_n + C_n \tag{22}$$

where C_n tends to zero in the trace norm.

Because $T(\phi_0)$ and $T(\tilde{\phi}_0)$ are invertible, we know that $T_n(\phi_0)$ and $T_n(\tilde{\phi}_0)$ are stable. Also, by Proposition 12, $T_n(\phi_k)$ and $T_n(\tilde{\phi}_k)$ are stable. Thus, for $j = 0, \ldots, R$,

$$T_n(\phi_j)^{-1} \to T(\phi_j) \quad \text{and} \quad T_n(\tilde{\phi}_j)^{-1} \to T(\tilde{\phi}_j)$$

strongly. Put

$$A_n = T_n(\phi) T_n(\phi_R)^{-1} \cdots T_n(\phi_0)^{-1}.$$

Using the second Widom's formula, (20), (21), and (22), we get

$$\begin{aligned} A_n = P_n &+ P_n K_1 P_n T_n(\phi_R)^{-1} \cdots T_n(\phi_0)^{-1} P_n \\ &+ W_n K_2 P_n T_n(\widetilde{\phi}_R)^{-1} \cdots T_n(\widetilde{\phi}_0)^{-1} W_n + C_n'. \end{aligned}$$

Using the strong convergence of the inverses (and their adjoints),

$$A_n = P_n + P_n L_1 P_n + W_n L_2 W_n + C_n$$

with certain $C_n \to 0$ in trace norm and

$$\begin{aligned} L_1 &= K_1 T(\phi_R)^{-1} \cdots T(\phi_0)^{-1} = T(\phi) T(\phi_R)^{-1} \cdots T(\phi_0)^{-1} - I, \\ L_2 &= K_2 T(\tilde{\phi}_R)^{-1} \cdots T(\tilde{\phi}_0)^{-1} = T(\tilde{\phi}) T(\tilde{\phi}_R)^{-1} \cdots T(\tilde{\phi}_0)^{-1} - I, \end{aligned}$$

which are both trace class operators.

Taking the determinant of A_n and passing to the limit gives the left side of the following, while using Lemma 3.5, it follows that the limit equals the product of two well-defined operator determinants,

$$\lim_{n\to\infty} \frac{\det T_n(\phi)}{\prod_{k=0}^{R} \det T_n(\phi_k)} = \lim_{n\to\infty} \det A_n = \det(I + L_1)\det(I + L_2).$$

Using the previous expressions for L_1 and L_2, we get

$$\lim_{n\to\infty} \frac{\det T_n(\phi)}{\prod_{k=0}^{R} \det T_n(\phi_k)} = E_1 E_2$$

where

$$E_1 = \det T(\phi)T(\phi_R)^{-1} \cdots T(\phi_0)^{-1},$$
$$E_2 = \det T(\tilde{\phi})T(\tilde{\phi}_R)^{-1} \cdots T(\tilde{\phi}_0)^{-1}.$$

It remains to apply the Szegő-Widom limit theorem to $\det T_n(\phi_0)$.

Removing the invertibility assumption

It remains to remove the extra assumption that $T(\phi_0)$ and $T(\phi_0^{-1})$ are invertible from the previous result and replace it by a weaker condition on ϕ_0.

The proof of the Szegő-Widom limit theorem given by Widom 1976 gets around this problem by using his elegant perturbation result:

- If $T(\phi)$ is a Fredholm operator with index zero one can find a matrix Laurent polynomial q such that $T(\phi + \lambda q)$ is invertible whenever $0 < |\lambda| < \varepsilon$.

A similar argument may also work in our case, but it seems that one would need a result to simultaneously perturb $T(\phi)$ and $T(\widetilde{\phi})$ with the same q to make them both invertible. Such a result has not yet been established.

Our approach to remove the invertibility condition is rather different and technical.

Open problem. Precise smoothness on ϕ_0 that is required for the asymptotics. This seems open even when $N = 1$.