The random matrix Fyodorov-Hiary-Keating conjecture

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The Dyson Circular β-ensemble

The $C\beta E_n$ is a joint distribution on n points on the unit circle

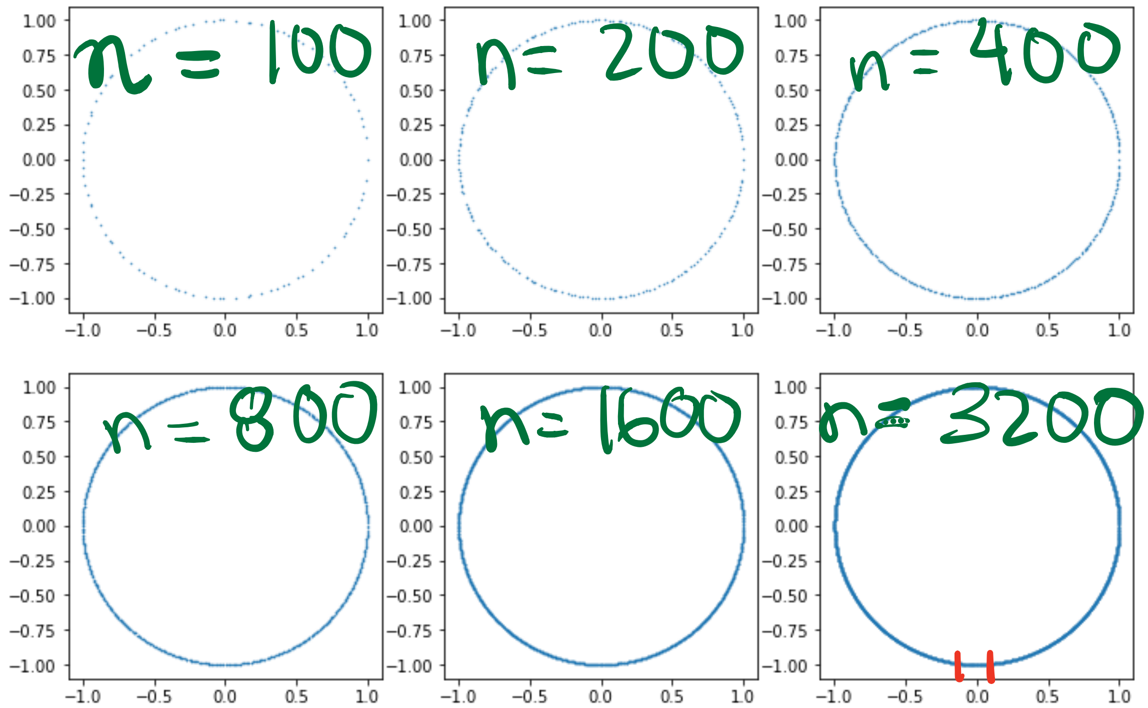
$$\frac{1}{Z_{n,\beta}} \prod_{1 \leq j < k \leq n} \left| e^{i\omega_j} - e^{i\omega_k} \right|^{\beta} d\omega_1 \, d\omega_2 \cdots d\omega_n$$

$\beta = 2 \leftrightarrow$ Eigenvalues of $\mathbb{U}_n$

Let X_n be the characteristic polynomial.

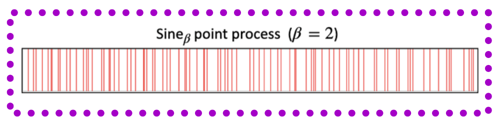
$$X_n(z) := \prod_{j=1}^{n} \left(1 - e^{i\omega_j} z\right)$$

The Sine process



$\beta = 2 \rightarrow \text{Sine}_2$

Dyson '1962 $\rightarrow$



Montgomery-Dyson Conjecture

Let $(\frac{1}{2}+i\gamma_j)$ be zeros of the Riemann Zeta fn.

(Montgomery Conjecture) '73

For nice enough test functions h

$$\left(\frac{2\pi}{T\log T}\right)^{k-1} \times \sum_{\substack{(\gamma_j)_1^k \\ \in [0,T]}} h\left((\gamma_1-\gamma_j)_{j=2}^{k} \times \frac{(\log T)}{2\pi}\right) \xrightarrow[T\to\infty]{} \mathbb{E}\, h(p_2,\dots,p_k)$$

Palm-Process of sine-2

Verified for special test functions

Montgomery (k=2)

Rudnick-Sarnak general k

Value Distribution of ξ

Theorem (Selberg 1946) For any open $E \subseteq \mathbb{R}^2$,

$$\frac{1}{T}\left|\left\{T\le t\le 2T,\ \frac{\log \xi(1/2+it)}{\sqrt{1/2\log\log T}}\in E\right\}\right| \xrightarrow[T\to\infty]{} \iint_E \frac{e^{-(x^2+y^2)/2}}{2\pi}\,dx\,dy$$

RMT

CβE

Theorem (Keating-Snaith '00)

$$\mathbb{P}\left(\frac{\log X_n(1)}{\sqrt{1/\beta\log n}}\in E\right) \xrightarrow[n\to\infty]{} \iint_E \frac{e^{-(x^2+y^2)/2}}{2\pi}\,dx\,dy$$

$\left[X_n(z) := \prod_{j=1}^{n}\left(1-e^{i\omega_j}z\right)\right]$

How strong is the analogy?

Value Distribution of ξ

What about the correlations?

Fixed nonequal $\omega_j \in [0, 2\pi]$

Theorem Hughes-Keating O'Connell '01

$$\mathbb{P}\left(\bigcap_{j=1}^{K} \left\{ \frac{\log X_n(\omega_j)}{\sqrt{1/\beta \, \log n}} \in E_j \right\} \right) \xrightarrow[n\to\infty]{} \prod_{j=1}^{K} \iint_{E_j} \frac{e^{-(x^2+y^2)/2}}{2\pi} \, dx\, dy$$

The Correlation Structure is invisible

The Fyodorov-Hiary-Keating Conjecture

RMT-Part

FHK ('12) Conjecture: (for $\beta = 2$)

(3/4 accounts for nontrivial correlations)

$$\left(\overbrace{\max_{|z|=1} \log |X_n(z)|}^{M_n} - \sqrt{\frac{2}{\beta}}\left(\overbrace{\log n - \frac{3}{4}\log\log n}^{m_n}\right)\right) \xrightarrow[n\to\infty]{d} G_1 * G_2$$

the convolution of two independent Gumbels

- G_1 is universal, in the class of log-correlated fields
- G_2 being Gumbel is much less universal

The Fyodorov-Hiary-Keating Conjecture ζ part

FHK ('12) Conjecture

$$\left(\max_{|u|\le 1} \log\left|\zeta\left(\tfrac12 + i(\hat{T}+u)\right)\right| - \sqrt{\frac{2}{\beta}}\left(\log n - \frac{3}{4}\log\log n\right)\right) \xrightarrow[n\to\infty]{d} \; ?$$

$\hat{T} = \mathrm{Unif}(e^n, e^{2n})$

Partial progress:

Najnudel '18, Bourgade '18, Arguin-Bourgade-Radziwiłł '20

Convergence in law of the maximum

Existing work:

- $(\beta=2)$ $\dfrac{M_n - m_n}{\log n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$ '15 Arguin, Belius, Bourgade
- $(\beta=2)$ $\dfrac{M_n - m_n}{\log\log n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$ '16 P'-Zeitouni
- (all $\beta>0$) $\{M_n - m_n\}$ tight '16 Chhaibi-Madaule -Najnudel

closest technically to this work

Convergence in law of the maximum

Theorem 'P-Zeitouni '22.

$$M_n - m_n \xrightarrow[n\to\infty]{(d)} \text{Gumbel}\left(\tfrac{1}{\sqrt{2}\beta}\right) * \mathcal{Q}(\beta) + c$$

where • $\mathcal{Q}(\beta)^*$ is the mass of a critical chaos

*(elaboration to follow)

• $c = c(\beta)$ is a constant.

The Killip-Nenciu '04 representation

Let $\{\gamma_k\}_0^\infty$ be independent, complex, "Verblunsky coefficients" rotationally invariant, and have $|\gamma_k|^2 \overset{\text{law}}{=} \mathrm{Beta}(1, \beta(k+1)/2)$

Let $\log \Phi_{k+1}^*(e^{i\theta}) \overset{\log}{=} \Phi_k^*(e^{i\theta}) + \log\left(1-\gamma_k e^{i\,\mathrm{Im}\log \Phi_n^*(e^{i\theta})}\right)$

$\Phi_0^*(\theta)=1.$

$$\log X_n(e^{i\theta}) = \log \Phi_{n-1}^*(e^{i\theta}) + \log\left(1+e^{i\alpha + \mathrm{Im}\log \Phi_{n-1}^*(e^{i\theta})}\right)$$

Then with Z_j iid $\mathcal{CN}(0,1)$, $Y_j(\theta) = \mathrm{Im}\log \Phi_j^*(e^{i\theta})$

$$\boxed{\log X_n(e^{i\theta}) \approx \log \Phi_{n-1}^*(e^{i\theta}) \approx i(n+1)\theta + \sum_{j=1}^n \sqrt{\frac{4}{\beta}}\,\frac{Z_j}{\sqrt{j}}\, e^{i[Y_j(\theta)-Y_j(0)]}}$$

The derivative martingale

Let $\quad \mathcal{D}_n = \frac{1}{2\pi n}\int_0^{2\pi} \left|\Phi_n^*(e^{i\theta})\right|^{\sqrt{2\beta}}\left(\sqrt{2}\log n - \sqrt{\beta}\log \Phi_n^*(e^{i\theta}) + O_n(1)\right).$

"the derivative martingale".

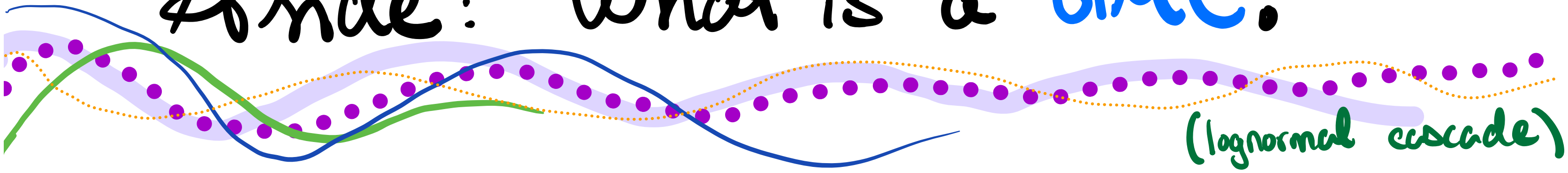
(deterministic bounded sequence)

Theorem (P'-Zeitouni '22) $\mathcal{D}_{2^n}$ converges almost surely to a finite, a.s. positive random variable $\mathcal{D}_\infty$

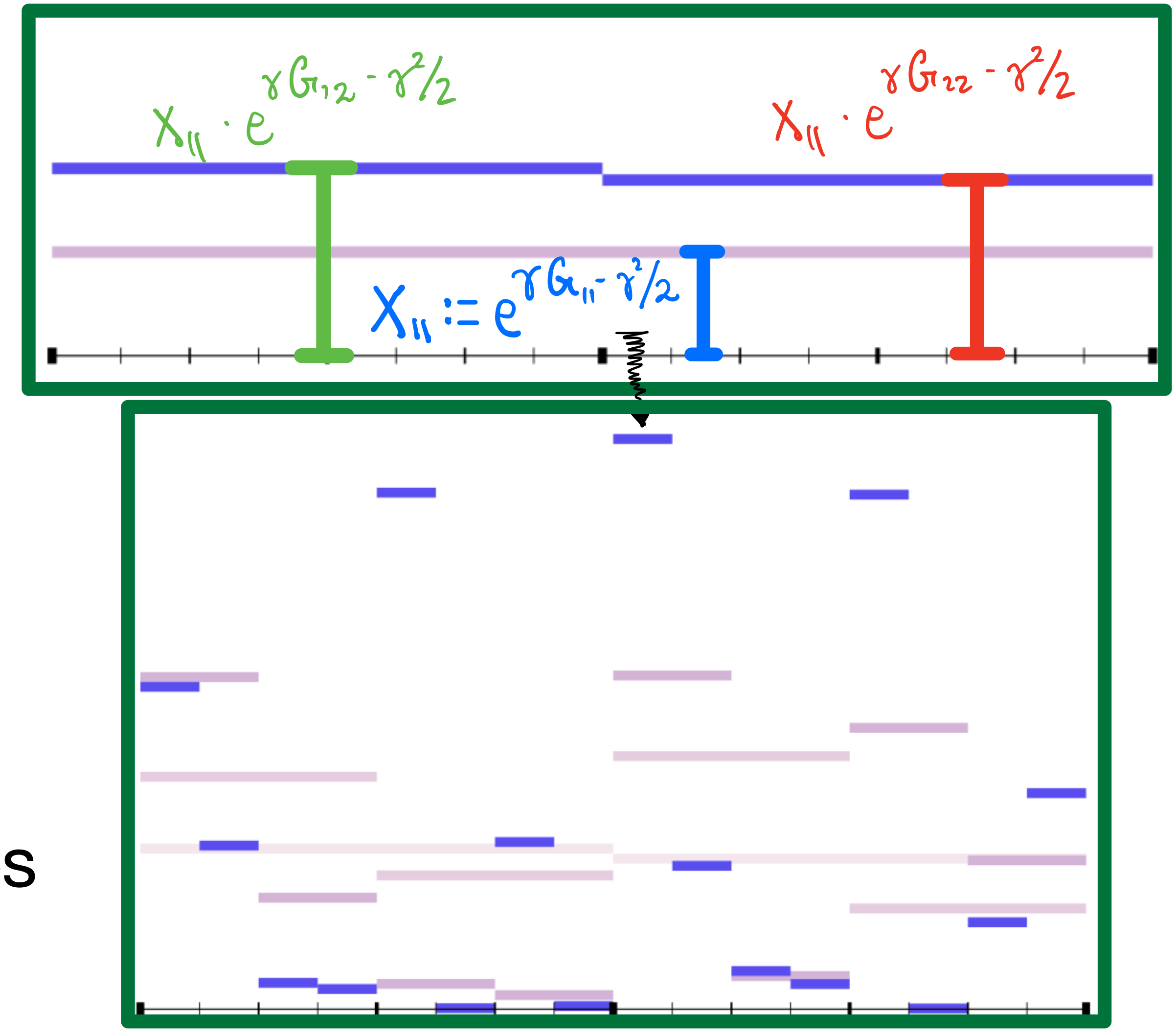
this is the shift $\mathcal{D}(\beta)$

(in fact $\frac{1}{2\pi n}\left|\Phi_n^*(e^{i\theta})\right|^{\sqrt{2\beta}}\left(\sqrt{2}\log n - \sqrt{\beta}\log \Phi_n^*(e^{i\theta})\right)d\theta$ converges to an a.s. nonatomic random measure, a multiplicative chaos $M_\infty(d\theta)$)

Aside: what is a GMC?



Gaussian
Multiplicative
Chaos



A random measure built by randomly rescaling a density at different scales

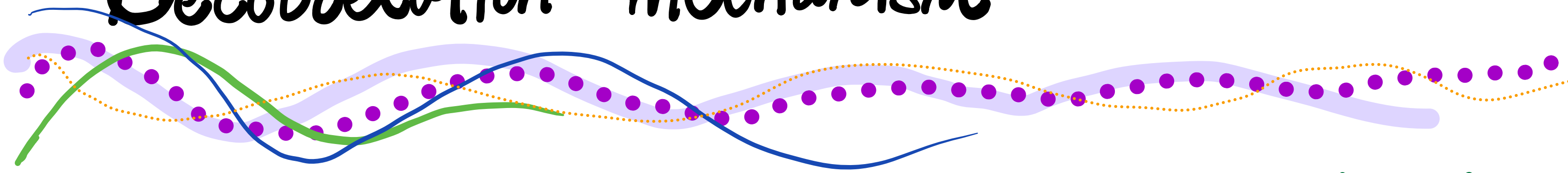
Decorrelation mechanism

To develop the ideas,

$$\log \Phi_{n-1}^*(e^{i\theta}) \approx i(n+1)\theta + \sum_{j=1}^{n} \sqrt{\frac{4}{\beta}}\, \frac{Z_j}{\sqrt{j}}\, e^{i \operatorname{Im} \log \Phi_j^*(e^{i\theta})}$$

$i = \sqrt{-1}$

Decorrelation mechanism



To develop the ideas,

Gaussian Random Wave
$\left(\exp\left(\sqrt{4/\beta}\, G_n(\theta) + i(n+1)\theta \right) \right)$
G_n a GAF

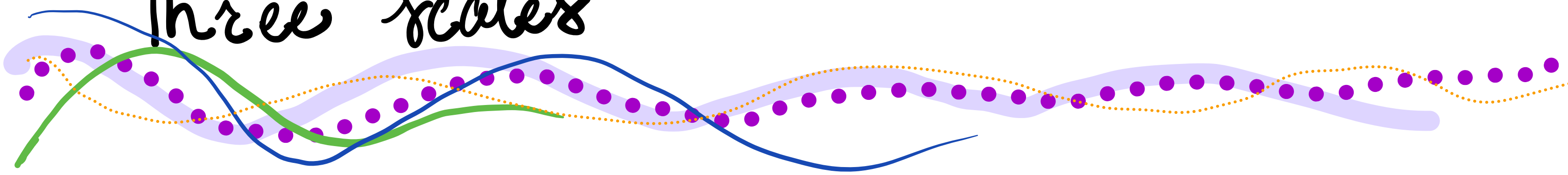
$$\log \Phi^*_{n-1}(e^{i\theta}) \approx i(n+1)\theta + \underbrace{\sum_{j=1}^{n} \sqrt{\frac{4}{\beta}}\, \frac{Z_j}{\sqrt{j}}\, e^{i(j+1)\theta}}_{\sqrt{4/\beta}\, G_n}$$

$i = \sqrt{-1}$

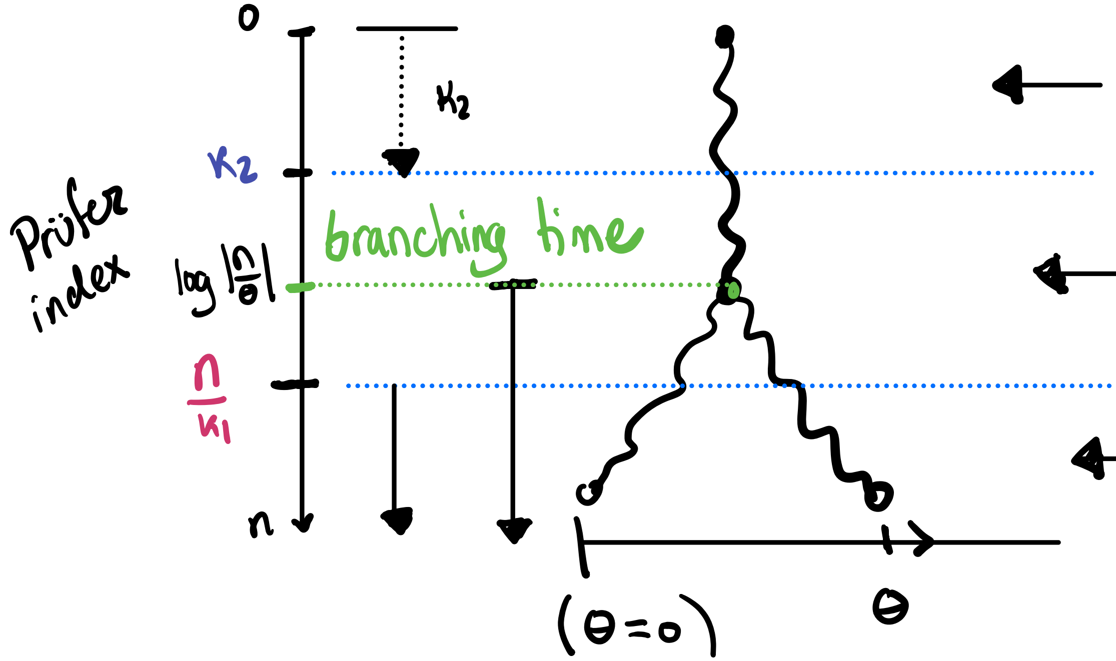
G_n is a Gaussian log-correlated field

Qualitatively, $G_n \approx \log \Phi^*_{n-1}(e^{i\theta})$ except for fine properties $(\theta \asymp 1/n)$.

Three scales



Fix K_1, K_2 large $\{\log \Phi_k^*(e^{i\theta})\}_{k=0}^{n}$



non-universal initial part ($Q_\infty(\beta)$)

≈ Gaussian random wave

non-universal stochastic ξ-function regime

Decoration

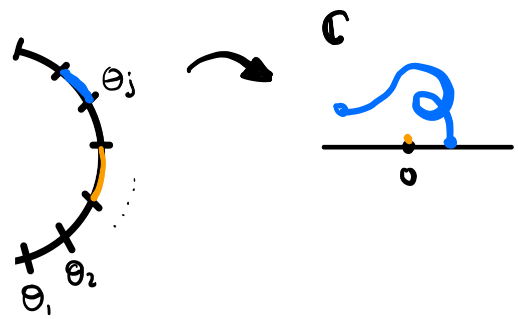
Landscape Convergence

Break $[0,2\pi]$ into intervals of length κ_1/n.

$I_1, I_2, \ldots, I_{n/\kappa_1}$ w/ left endpoints $\theta_1, \ldots, \theta_{n/\kappa_1}$

Define the PP on $\mathbb{R} \times C([0,\kappa_1],\mathbb{C})$

$$\mathcal{E}x_{\kappa_1,n} = \sum_{j=1}^{n/\kappa_1} \delta_{\left(\theta_j,\ \underbrace{\left(\frac{\Phi_n^*(\exp(i(\theta_j + t/n)))}{\exp(\sqrt{2/\beta}\, m_n)} \,:\, t \in [0,\kappa_1]\right)}_{\text{is usually } \approx 0}\right)}$$



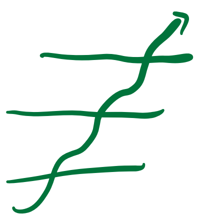
Theorem P'-Zeitouni '22 $\lim_{\kappa_1 \to \infty} \lim_{n \to \infty} d\left(\mathcal{E}x_{\kappa_1,n},\ \text{Poisson}\left(M_\infty(d\theta) \times \mathfrak{P}_{\kappa_1}(df)\right)\right) = 0$

- $\mathfrak{P}_{\kappa_1}(df)$ is the law of a randomly scaled stochastic-ζ function on a wide interval $[0,\kappa_1]$

The Stochastic-ξ function

Theorem Valko-Virag '20

$$\exp\left(\log \Phi^{*}_{e^t n/k_1}(e^{i\theta n})\right) \xrightarrow[n\to\infty]{d} \exp\left(i\mathcal{U} + \varphi_t(\theta)\right)$$



$\mathcal{U} \sim \mathrm{Unif}([0,2\pi])$ and

$$d\varphi_t(\theta) = i e^t \theta + \sqrt{\tfrac{4}{\beta}}\, dZ_t\, e^{i \operatorname{Im} \varphi_t(\theta)} \qquad t \in [0, \log k_1]$$

$\uparrow$ CBM

Zeros of $\sin\left(\mathcal{U} + \operatorname{Im} \varphi_{\log k_1}(\theta)\right)$ follow the sine-process

Conclusion

- For the RMT-FHK conjecture, the law of $D(\beta)$ is open
- The law of Stochastic-ζ around a high point is open
- Arguin-Bourgade-Radziwiłł - tightness of recentered maximum of ζ. Exact law is at least as hard as Montgomery conjecture.

Thanks!