Gaussian fluctuations for spin systems and point processes: near-optimal rates via quantitative Marcinkiewicz's theorem

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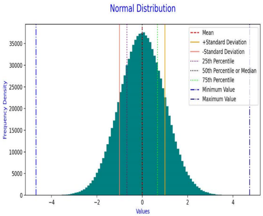
Gaussian Fluctuations



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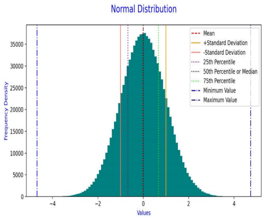

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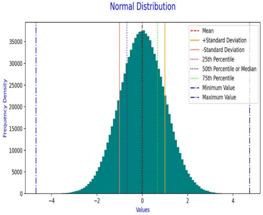

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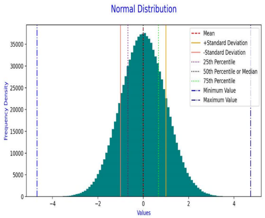

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- Most approaches to Gaussian fluctuations involve
  - approximation via an appropriate *low-complexity* function of roughly independent (or weakly dependent) random variables
  - These random components should individually have small *influence*.
- From classical CLT for sums of weakly dependent random variables to so-called *invariance principles*, e.g. for low degree polynomials of independent random bits, and a lot in between ...

Marcinkiewicz's Theorem

Let X be a real random variable. Set $\Psi_X(u) := \mathbb{E}[e^{iuX}]$.

Theorem (Marcinkiewicz, 1939)

If Ψ_X is an entire characteristic function that is of the form $\exp(P(u))$ for some polynomial P, then X has to be a Gaussian.

Marcinkiewicz's Theorem

Observe that

- $\Psi = \exp(f)$ for some entire function f :: Ψ has no zeros on the whole of $\mathbb{C}$.
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Question

What are the optimal conditions under which a Marcinkiewicz-type theorem holds?

- Let $M_f(r) := \max\{|f(z)| : |z| = r\}$. Then $\lim_{r\to\infty} \frac{\log^+ M_f(r)}{r} > 0$ does not fall within the ambit of Marcinkiewicz's Theorem, e.g. it is realised in the case of the Poisson distribution.

- In 1960, Linnik conjectured that a Marcinkiewicz type theorem holds if

$$\lim_{r\to\infty} \frac{\log^+ M_f(r)}{r} = 0$$

where
- Ostrovskii confirmed that Linnik's conjecture is true, as soon as

$$\limsup_{r\to\infty} \frac{\log^+ M_f(r)}{r} = 0,$$

using ideas from Wiman-Valiron theory.

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- Natural hypothesis : '$\Psi_X = \exp(f)$ for some entire function f' replaced by 'Ψ_X has no zeros on a disk $\overline{\mathbb{D}(0,r)}$'.
- Natural conclusion : exact Gaussianity replaced by distance of X from a Gaussian, depending on the size of $\overline{\mathbb{D}(0,r)}$ and the growth rate of $\log M_f \simeq \log\log \Psi_X$ on $\overline{\mathbb{D}(0,r)}$.

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Our Quantitative Marcinkiewicz Theorem is the following :

Theorem (Dinh, G., Tran, Tran)

Suppose there is a number $r > 0$ *such that*

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Then, for some universal constant $A > 0$ we have

$$\sup_{x\in\mathbb{R}} |F_{\bar{X}}(x) - \phi(x)| \leq 2|\sigma - 1| + A \cdot \frac{1 + \log^+ \log \max_{|u|=r} |\Psi(u)|}{r}.$$

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Then the sequence $\{X_n\}$ satisfies the CLT, i.e., $\hat{X}_n \to N(0,1)$ as $n \to \infty$.

Indeed, we have for some universal constant $A > 0$

$$\sup_{x\in\mathbb{R}} |F_{\hat{X}_n}(x) - \phi(x)| \le A \cdot \frac{1 + \log^+ \log \sup_{|u|=r_n} |\Psi_n(u)|}{r_n \sigma_n},$$

which a fortiori provides a rate for the CLT.

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- Work in the setting of zero-freeness on an infinite strip. In our applications, non-vanishing on a disk is important.
- Different flavour of quantitative Marcinkiewicz due to Golinskii and Sapogov, based on coefficients of $\log \Psi$.

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- In applications, growth of log log characteristic functions is often mild, contributing at most a log factor
- Thus, CLT with rates is reduced to a variance bounding exercise, usually we have a Gaussian limit as soon as variance grows faster than a suitable rate (e.g., log).

Application to Spin systems

Let $\Lambda \subset \mathbb{R}^d$ be a d-dimensional cube. A **spin configuration** on Λ is a map

$$\sigma_\Lambda : \Lambda \to \mathbb{R}^N$$

with

$$\sigma_x := \sigma_\Lambda(x) = (\sigma_x^1, \ldots, \sigma_x^N) \quad , \quad \forall x \in \Lambda.$$

The spins σ_x are distributed according to some compactly supported, non-negative Borel measure μ_0 on $\mathbb{R}^N$.

We consider the Hamiltonian

$$H_\Lambda(\sigma_\Lambda) := -\sum_{(x,y)\subset\Lambda} \sum_{i=1}^{N} J_{xy}^i \sigma_x^i \sigma_y^i - \sum_{x\in\Lambda} \sum_{i=1}^{N} h_x^i \sigma_x^i,$$

where $J_{xy}^i = J_{yx}^i$ are **coupling constants** and $h_x = (h_x^1, \ldots, h_x^N)$ is the **external magnetic field** at $x \in \Lambda$.

The probability measure on the space of such σ_Λ-s is given by

$$\mathbb{P}_\Lambda(\sigma_\Lambda) = \frac{1}{\mathcal{Z}_{\beta,\Lambda}(\{h_x\}_{x\in\Lambda})} \exp(-\beta H_\Lambda(\sigma_\Lambda)) \prod_{x\in\Lambda} d\mu_0(\sigma_x),$$

where $\beta > 0$ is the **inverse temperature**, and

$$\mathcal{Z}_{\beta,\Lambda}(\{h_x\}_{x\in\Lambda}) := \int_{(\mathbb{R}^N)^{\otimes|\Lambda|}} \exp(-\beta H_\Lambda(\sigma_\Lambda)) \prod_{x\in\Lambda} d\mu_0(\sigma_x)$$

is the **partition function** of the system.

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- Heisenberg model : $\mathbb{S}^2$-valued spins
- Fluctuation theory is well-studied for classical Ising model, much less understood for spin models with continuous symmetries (such as the Heisenberg model or the XY model).
- Improved rate of convergence compared to CLT for discrete statistical mechanical models by Lebowitz, Pittel, Ruelle and Speer, from $O(N^{-1/6})$ to $O(\log N \cdot N^{-1/2})$.

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The object of interest is the *magnetization* or the *total spin*, which will be denoted by S_Λ

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Under ferromagnetic conditions, S_Λ satisfies a CLT upon centering and scaling as $|\Lambda| \uparrow \infty$, with the speed of convergence at the rate $O(\log|\Lambda| \cdot |\Lambda|^{-1/2})$.

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- Rate of convergence in CLT comparable to classic $1/\sqrt{n}$ rate for i.i.d. variables, upto log factors !

A direct computation shows that

$$\mathbb{E}[e^{iu\langle S_\Lambda, e_1\rangle}] = \frac{\int_{(\mathbb{R}^N)^{\otimes|\Lambda|}} e^{iu\sum_{x\in\Lambda}\sigma_x^1} e^{-\beta H_\Lambda(\sigma_\Lambda)} \prod_{x\in\Lambda} d\mu_0(\sigma_x)}{\mathcal{Z}_{\beta,\Lambda}(\{h_x\}_{x\in\Lambda})}$$

$$= \frac{\mathcal{Z}_{\beta,\Lambda}(\{h_x + \beta^{-1} iue_1\}_{x\in\Lambda})}{\mathcal{Z}_{\beta,\Lambda}(\{h_x\}_{x\in\Lambda})},$$

where $e_1 := (1, 0, \ldots, 0) \in \mathbb{C}^N$.

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The Lee-Yang theorem is a crucial ingredient to apply our CLT version to this setting. We consider the following subset of $\mathbb{C}^N$

$$\Omega_N^+ := \{h = (h^1, \dots, h^N) \in \mathbb{C}^N : \Re h^1 > \sum_{i=2}^{N} |h^i|\}.$$

Under some conditions about the measure μ_0 and the coupling constants J_{xy}^i, we have the following

Theorem (Lee-Yang Theorem)

The partition function $\mathcal{Z}_{\beta,\Lambda}(\{h_x\}_{x\in\Lambda})$ doesn't vanish whenever $h_x \in \Omega_N^+$ for each $x \in \Lambda$.

Key ideas of proof

Recall that

$$\Psi_{\langle S_\Lambda, e_1\rangle}(u) = \frac{\mathcal{Z}_{\beta,\Lambda}(\{h_x + \beta^{-1} i u e_1\}_{x\in\Lambda})}{\mathcal{Z}_{\beta,\Lambda}(\{h_x\}_{x\in\Lambda})}.$$

Since $\Omega_N^+ = \left\{\Re h^1 > \sum_{i=2}^N |h^i|\right\}$ is an open set, if $|u|$ small enough (let's say, $|u| \le r_\Lambda$) then

$$h_x + \beta^{-1} i u e_1 \in \Omega_N^+ \quad , \quad \forall x \in \Lambda.$$

This will imply $\Psi_{\langle S_\Lambda, e_1\rangle}(u)$ does not vanish on $\overline{\mathbb{D}(0, r_\Lambda)}$. Under some assumptions about the external magnetic field h, we can simplify the situation to the case of the same radius r for all S_Λ.

Hence the non-vanishing condition for the characteristic function is satisfied.

Key ideas of proof

Thus, to obtain a CLT, we only have to verify

$$\lim_{|\Lambda|\uparrow\infty} \frac{\log^+ \log |\mathbb{E}[e^{r_\Lambda |\langle S_\Lambda, e_1\rangle|}]|}{r_\Lambda \sqrt{Var(\langle S_\Lambda, e_1\rangle)}} = 0.$$

Since μ_0 is compactly supported, there is a positive number M (depending only on μ_0) such that

$$|S_\Lambda| = |\sum_{x\in\Lambda} \sigma_x^1| \leq M \cdot |\Lambda|$$

This implies

$$\log^+ \log |\mathbb{E}[e^{r_\Lambda |S_\Lambda|}]| \lesssim \log |\Lambda|.$$

Key ideas of proof

To control the variance of $\langle S_\Lambda, e_1 \rangle$, we define

$$\Lambda_O := \{(x_1, \dots, x_d) \in \Lambda : x_1 + \dots + x_d \text{ is odd}\}$$

and

$$\Lambda_E := \{(x_1, \dots, x_d) \in \Lambda : x_1 + \dots + x_d \text{ is even}\}.$$

For a vertex $x \in \Lambda$, we denote by ∂x the set of all the vertices that are connected to x. Then if $x \in \Lambda_E$, $\partial x \subset \Lambda_O$, and vice versa. Thus

$$\mathbb{P}_\Lambda(\sigma_x, x \in \Lambda_E | \sigma_y, y \in \Lambda_O) \propto \exp\Big(\beta\Big(\sum_{x\in\Lambda_E}\sum_{y\in\partial x}\sum_{i=1}^{N} J^i_{xy}\sigma^i_x\sigma^i_y + \sum_{x\in\Lambda_E}\sum_{i=1}^{N} h^i_x\sigma^i_x\Big)\Big)\prod_{x\in\Lambda_E} d\mu_0(\sigma_x)$$

$$= \prod_{x\in\Lambda_E}\exp\Big(\beta\Big(\sum_{y\in\partial x}\sum_{i=1}^{N} J^i_{xy}\sigma^i_x\sigma^i_y + \sum_{i=1}^{N} h^i_x\sigma^i_x\Big)\Big) d\mu_0(\sigma_x).$$

In other words, given the spins on Λ_O, the spins on Λ_E are independent.

Key ideas of proof

By the law of total variance, one has

$$\begin{aligned} Var[\langle S_\Lambda, e_1\rangle] &\geq \mathbb{E}[Var[\langle S_\Lambda, e_1\rangle | \sigma_y, y \in \Lambda_O]] \\ &= \mathbb{E}\Big[Var\Big[\sum_{x\in\Lambda} \sigma_x^1 | \sigma_y, y \in \Lambda_O\Big]\Big] \\ &= \mathbb{E}\Big[Var\Big[\sum_{x\in\Lambda_E} \sigma_x^1 | \sigma_y, y \in \Lambda_O\Big]\Big] \\ &= \sum_{x\in\Lambda_E} \mathbb{E}[Var[\sigma_x^1 | \sigma_y, y \in \Lambda_O]] \geq \frac{c}{2}|\Lambda|. \end{aligned}$$

Lemma

If μ_0 is compactly supported, the coupling constants J^i_{xy} and the external magnetic field h^i_x are uniformly bounded (as the system size $|\Lambda|$ grows), then there exists a constant $c > 0$ (not depending on Λ) such that

$$Var[\sigma_x^1 | \sigma_y, y \in \partial x] \geq c \quad a.s. \quad \forall x \in \Lambda.$$

Key ideas of proof

Putting all ingredients together

$$\frac{\log^+ \log |\mathbb{E}[e^{r_\Lambda |S_\Lambda|}]|}{r_\Lambda \sqrt{Var(S_\Lambda)}} \lesssim \frac{\log |\Lambda|}{|\Lambda|^{1/2}},$$

as desired.

Application to α-determinantal processes

For a matrix $A \in \mathbb{C}^{n\times n}$, define

$$\det_\alpha[A] := \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} A_{i\sigma(i)},$$

where $\nu(\sigma)$ is the number of cycles in $\sigma \in S_n$.

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Let μ be a non-negative Borel measure on $\mathbb{R}^d$ and let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a symmetric kernel, the α-**determinantal point process** on $\mathbb{R}^d$ with kernel K and background measure μ is a random locally finite point set on $\mathbb{R}^d$ such that :

For any finite subset $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$, the probability (with respect to $\mu^{\otimes n}$) of having points at these locations is given by the n-point correlation function

$$\rho_n(x_1, \dots, x_n) = \det_\alpha[(K(x_i, x_j))_{1\le i,j\le n}].$$

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• $\alpha = -1$ is determinantal point process • $\alpha = 0$ is Poisson point process • $\alpha = +1$ is permanental point process.

Application to α-determinantal processes

Let X be a α-determinantal point process on $\mathbb{R}^d$. We will be interested in the distribution of its linear statistics. For a test function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support, the linear statistic $\Lambda(\varphi)$ is given by

$$\Lambda(\varphi) := \sum_{x \in X} \varphi(x).$$

We will investigate the family of random variables given by the linear statistics

$$\{\Lambda(\varphi_L)\}_{L>0} \quad , \qquad \text{where } \varphi_L(\cdot) := \varphi(\cdot/L).$$

Application to α-determinantal processes

Under mild conditions about the background measure μ and the kernel K, we have the following CLT for linear statistics:

Theorem (Dinh, G., Tran, Tran)

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a bounded compactly supported test function with $\|\varphi\|_2 > 0$. Then $\Lambda(\varphi_L)$ satisfies a CLT upon centering and scaling as $L \uparrow \infty$ with the speed of convergence at the rate $O(\log L \cdot L^{-\eta})$ for some positive number η depending only on the model.

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- Covers a broad class of kernels, in particular kernels with slow spatial decay (such as the Bessel kernel) in general dimensions
- Obtain explicit rates of convergence in a wide spectrum of models.

Application to α-DPPs : particular cases

Suppose $d > 2$ and

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \|x - y\|^3 |K(x, y)|^2 dy < \infty \tag{1}$$

and

$$\inf_{x \in \mathbb{R}^d} \inf_{u \in \mathbb{S}^{d-1}} \int_{\mathbb{R}^d} |\langle y - x, u \rangle|^2 |K(x, y)|^2 dy > 0, \tag{2}$$

and $\varphi \in C_c^2$.Then

Result

[Under above conditions] $\Lambda(\varphi_L)$ satisfies a CLT upon centering and scaling as $L \uparrow \infty$ with the speed of convergence at the rate $O(\log L \cdot L^{-\eta})$, where $\eta = \frac{1}{2}(d - 2)$.

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Result

[Bessel process] For the Bessel kernel point process in $d > 1$, we have $\Lambda(\varphi_L)$ satisfies a CLT upon centering and scaling as $L \uparrow \infty$ with the speed of convergence at the rate $O(\log L \cdot L^{-\eta})$, where $\eta = d - 1$.

Key ingredients

Theorem (Shirai, Takahashi)

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a bounded, compactly supported test function. Then for every $u \in \mathbb{D}(0, r)$ (r depends only on $\|\varphi\|_\infty$), we have

$$\mathbb{E}[\exp(-u\Lambda(\varphi))] = Det[I + \alpha M_{\varphi,u} K]^{-1/\alpha}$$

where Det is the Fredholm determinant, $M_{\varphi,u}$ is the multiplication operator by the function $1 - \exp(-u\varphi(x))$ and I is the identity operator on $L^2(\mu)$. Further, the Laplace transform is holomorphic in u on $\mathbb{D}(0, r)$.

To obtain a CLT, r_L can be taken to be $O(\|\varphi_L\|_\infty^{-1}) = O(1)$, and we have to verify

$$\lim_{L\to\infty} \frac{\log^+ \log |\mathbb{E}[e^{r_L|\Lambda(\varphi_L)|}]|}{r_L \sigma_L} = 0.$$

We show that

$$\log^+ \log |\mathbb{E}[e^{r_L|\Lambda(\varphi_L)|}]| \lesssim \log L \quad \text{and} \quad \sigma_L \gtrsim L^\eta$$

for some $\eta > 0$ depending only on the model, as desired.

THANK YOU !!