Derivative moments of CUE characteristic polynomials

Nick Simm

Joint work with Fei Wei (University of Oxford)

University of Sussex

Random matrices and related topics in Jeju

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Circular Unitary Ensemble

Let $U \in U(N)$ be chosen uniformly at random with respect to Haar measure $d\mu$ from the group of $N \times N$ unitary matrices.

This is called *Circular Unitary Ensemble* (CUE).

We define the characteristic polynomial as

$$\Lambda_N(z) = \det\Big(I - U^{\dagger} z\Big), \qquad z \in \mathbb{C}.$$

- Keating-Snaith '00: Statistical properties of $\Lambda_N(z)$ are believed to parallel analogous properties of the Riemann zeta function.
- Eigenvalues and zeros: Chance encounter between Freeman Dyson and Hugh Montgomery in 1972.

Riemann zeta function

Dirichlet series and Euler product:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \qquad \mathrm{Re}(s) > 1.$$

The ζ-function encodes the distribution of the prime numbers.

Analytic continuation to $\mathbb{C}$:

$$\zeta(s) = \left(2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\right) \zeta(1-s).$$

The *Riemann hypothesis* states that all non-trivial zeros of $\zeta(s)$ lie on the critical line $s = \frac{1}{2} + it$.

Keating-Snaith philosophy

Characteristic polynomial:

$$\Lambda_N(z) = \det\left(I - zU^\dagger\right)$$

Riemann zeta function:

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Integrals over the unitary group:

$$M_{k,N}^{\mathrm{RMT}} = \int_{U(N)} |\Lambda_N(e^{i\theta})|^{2k} d\mu(U)$$

Riemann zeta function:

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Integrals on the critical line:

$$M_{k,T}^{\mathrm{NT}} = \frac{1}{T}\int_0^T |\zeta(1/2 + it)|^{2k}\, dt$$

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Asymptotics:

$$M_{k,N}^{\mathrm{RMT}} = C_k N^{k^2}(1+o(1))$$

as $N \to \infty$.

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$$M_{k,T}^{\mathrm{NT}} = \frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\, dt$$

Conjectured ($k > 2$) asymptotics :

$$M_{k,T}^{\mathrm{NT}} = C_k'(\log T)^{k^2} a_k^{\mathrm{arith}}(1+o(1))$$

as $T \to \infty$. Proved when $k = 1, 2$.

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A conjecture of Keating and Snaith (2000) says that for all k,

$$C_k = C_k' = \prod_{n=0}^{k-1} \frac{k!}{(k+n)!} = \frac{G(1+k)^2}{G(1+2k)}.$$

Derivatives

The derivative of the CUE characteristic polynomial is motivated by several related questions in number theory:

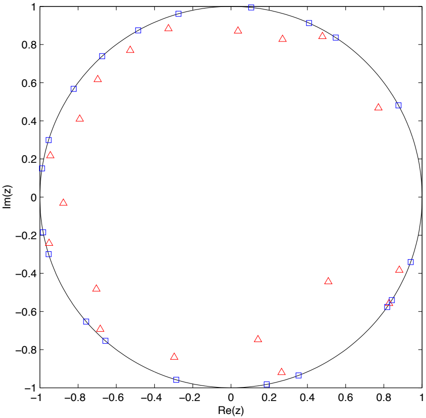
- RH is equivalent to no zeros of $\zeta'(s)$ in $0 < \sigma < \frac{1}{2}$.
- Some low order moments or joint moments of the derivative of the Riemann zeta function are known. e.g.

$$\frac{1}{T}\int_0^T \mathcal{Z}(t)\mathcal{Z}'(t)dt \sim \frac{e^2-5}{4\pi}(\log(T))^2$$

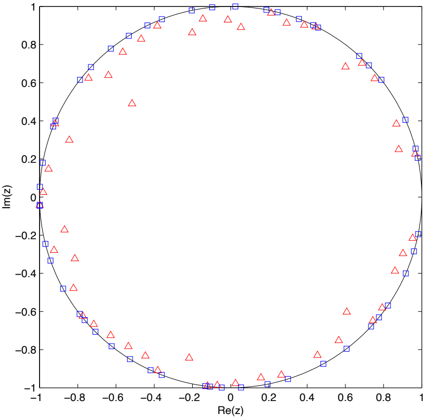
where $\mathcal{Z}(t) = e^{i\nu(t)}\zeta(1/2+it)$ is Hardy's Z-function (Conrey and Ghosh '89).

The derivative of the characteristic polynomial

Zeros are *inside* the unit circle. Non-trivial scaling of density $\rho(r)$, $r = 1 - \frac{c}{N}$, Mezzadri '02.



(a) $N = 20$



(b) $N = 50$

Derivative moments

The joint moments of the derivative are defined as

$$R_{h,k}(z) := \mathbb{E}(|\Lambda_N'(z)|^{2h}|\Lambda_N(z)|^{2k-2h}).$$

with $h \geq 0$, $k \geq h$.

Studied by several authors, mainly in the case that $|z| = 1$.

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The results are of the form

$$R_{h,k}(1) \sim N^{k^2+2h} c_{h,k}, \qquad N \to \infty,$$

where $c_{h,k}$ is a certain constant. Several works from '01 to the present day, e.g. Hughes, Conrey et al., Dehaye, Winn, Basor et al., Bailey et al., Forrester, Snaith and collaborators.

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This is not well understood when $|z| < 1$.

Main result

We consider

$$R_{h,k}(z) := \mathbb{E}(|\Lambda_N'(z)|^{2h}|\Lambda_N(z)|^{2k-2h}).$$

Theorem (S. and Wei '24)

Let $|z| < 1$, $\mathrm{Re}(h) \geq 0$ *and* $\mathrm{Re}(k-h) \geq 0$. *Then*

$$\lim_{N\to\infty} R_{h,k}(z) = \frac{e^{-k^2|z|^2}\Gamma(h+1)}{(1-|z|^2)^{k^2+2h}} \, {}_1F_1\left(h+1, 1; k^2|z|^2\right)$$

where ${}_1F_1(a, b; z)$ *is the confluent hypergeometric function*

$${}_1F_1(a, b; z) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{z^j}{j!}, \quad (a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}.$$

Results for integer moments

When $k, h \in \mathbb{N}$, the ${}_1F_1$ simplifies down to:

$$\lim_{N\to\infty} \mathcal{R}_{h,k}(z) = \frac{1}{(1-|z|^2)^{k^2+2h}} \sum_{j=0}^{h} \binom{h}{j}^2 (h-j)!\,(|z|^2k^2)^j$$
$$= \frac{1}{(1-|z|^2)^{k^2+2h}} L_h(-k^2|z|^2)$$

where $L_h(x)$ is the Laguerre polynomial of degree h.

On mesoscopic scales such that $|z| = 1 - \frac{c}{N^\alpha}$, with $0 < \alpha < 1$, we have

$$R_{h,k}(z) \sim \frac{1}{(1-|z|^2)^{k^2+2h}} L_h(-k^2).$$

Main idea

We write $\Lambda_N(z) = e^{G_N(z)}$, so that

$$|\Lambda_N'(z)|^{2h}|\Lambda_N(z)|^{2k-2h} = |G_N'(z)|^{2h}e^{k(G_N(z)+\overline{G_N(z)})}$$

where

$$G_N(z) = \log\det\left(I_N - zU^\dagger\right) = \sum_{j=1}^{\infty}\frac{\mathrm{Tr}(U^{-j})}{j}z^j.$$

Convergence to a random analytic function and log-correlated Gaussian field $G(z)$. Hughes, Keating, O'Connell '01.

Lemma

The vector $(G_N(z), G_N'(z))$ converges weakly to $(G(z), G'(z))$ where

$$G(z) = \sum_{j=1}^{\infty}\frac{z^j}{\sqrt{j}}\mathcal{N}_j, \qquad |z| < 1$$

and $\mathcal{N}_j$ are i.i.d. standard complex normal random variables.

Limiting correlation structure

Applying the Lemma, we show that as $N \to \infty$,

$$\mathbb{E}\left(|G_N'(z)|^{2h} e^{k(G_N(z)+\overline{G_N(z)})}\right) \to \mathbb{E}\left(|G'(z)|^{2h} e^{k(G(z)+\overline{G(z)})}\right),$$

where $(G(z), G'(z))$ is a bi-variate complex Gaussian.

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where $(G(z), G'(z))$ is a bi-variate complex Gaussian.

- The mean vector and relation matrix are 0.
- The covariance matrix is

$$\Gamma = \begin{pmatrix} \mathbb{E}(|G(z)|^2) & \mathbb{E}(G(z)\overline{G'(z)}) \\ \mathbb{E}(\overline{G(z)}G'(z)) & \mathbb{E}(|G'(z)|^2) \end{pmatrix}$$
$$= \begin{pmatrix} -\log\left(1-|z|^2\right) & \frac{z}{(1-|z|^2)} \\ \frac{\bar{z}}{(1-|z|^2)} & \frac{1}{(1-|z|^2)^2} \end{pmatrix}.$$

The limiting expectation is a Gaussian integral over $\mathbb{C}^2$.

Doing the Gaussian integral

The integral over G follows by completing the square. The remaining integral (over $G' = w$) is

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2w|w|^{2h}e^{-|w|^2+kzw+k\overline{zw}}$$

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$$=\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\frac{(kz)^{q_1}(k\overline{z})^{q_2}}{(q_1)!(q_2)!}\frac{1}{\pi}\int_{\mathbb{C}} d^2w\,|w|^{2h}e^{-|w|^2}(w)^{q_1}(\overline{w})^{q_2}$$

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&= \sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\frac{(kz)^{q_1}(k\overline{z})^{q_2}}{(q_1)!(q_2)!}\delta_{q_1,q_2}\Gamma\left(h+\frac{q_1+q_2}{2}+1\right)
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&= \sum_{q=0}^{\infty}\frac{\Gamma(h+q+1)}{(q!)^2}(k^2|z|^2)^q
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&= \sum_{q=0}^{\infty}\frac{\Gamma(h+q+1)}{(q!)^2}(k^2|z|^2)^q \\
&= \Gamma(p+1)\,{}_1F_1(h+1,1;k^2|z|^2).
\end{aligned}$$

Finite N, integer moments

This case is easier in principle. We can write for $h \in \mathbb{N}$,

$$\mathbb{E}(|\Lambda_N'(z)|^{2h}) = \partial_{z_1}\partial_{\overline{z_1}} \dots \partial_{z_h}\partial_{\overline{z_h}} \mathbb{E}\left(\prod_{j=1}^{h} \Lambda_N(z_j)\overline{\Lambda_N(z_j)}\right)\Bigg|_{z_1=z,\dots,z_h=z}$$

Then we can apply

Theorem (Akemann and Vernizzi '00)

The average of a product of k characteristic polynomials is

$$\mathbb{E}\left(\prod_{j=1}^{N}\prod_{i=1}^{h}(\lambda_j - z_i)(\overline{\lambda_j} - \overline{w_i})\right) = \frac{\det\left(\sum_{\ell=0}^{N+h-1}(z_i\overline{w_j})^{\ell}\right)_{i,j=1,\dots,h}}{\prod_{1\le i<j\le h}(z_j - z_i)\prod_{1\le i<j\le h}(\overline{w_j} - \overline{w_i})}.$$

This can be shown using the determinantal property of CUE.

Finite N results

Carrying out the derivatives and merging, we find

$$\mathbb{E}(|\Lambda_N'(z)|^{2h}) = \sum_{\vec{p},\vec{q}\in\mathcal{P}} a_{\vec{p},\vec{q}} \det\left\{(r^{p_i} f_N^{(p_i)}(r))^{(q_j)}\right\}_{i,j=1}^{h}, \qquad r = |z|^2$$

where $\mathcal{P}$ are partitions of $\frac{h^2+h}{2}$ of length h, $a_{\vec{p},\vec{q}}$ are combinatorial coefficients and

$$f_N(r) := 1 + r + r^2 + \ldots + r^{N-1} = \frac{r^N - 1}{r - 1}, \qquad r = |z|^2.$$

Works well in *meso* or *micro* regimes e.g. $|z| = 1 - \frac{c}{N}$, including $|z| = 1$.

Microscopic regime

In this regime $r = 1 - \frac{c}{N}$ and we get

$$(r^{p_i} f_N^{(p_i)}(r))^{(q_j)} \sim N^{p_i+q_j+1} \int_0^1 x^{p_i+q_j} e^{-cx} dx$$

and taking determinants

$$\det\left\{(r^{p_i} f_N^{(p_i)}(r))^{(q_j)}\right\}_{i,j=1}^{h} \sim N^{h^2+2h} \det\left\{\int_0^1 x^{p_i+q_j} e^{-cx} dx\right\}_{i,j=1}^{h}$$

which matches the order in the known case $c = 0$. There the determinant is an explicit Cauchy determinant.

For $c > 0$ and $p_i = i, q_j = j$, this is a Hankel determinant related to Painlevé V.

Summary

I discussed our results for the average

$$\mathbb{E}\left(|\Lambda_N'(z)|^{2h}|\Lambda_N(z)|^{2k-2h}\right), \quad |z| < 1.$$

To summarise:

- We approximate $\Lambda_N(z)$ by $e^{G(z)}$ where $G(z)$ is an appropriate log-correlated Gaussian field.
- Higher derivatives for $h, k \in \mathbb{N}$ can also be done using Wick's theorem for the multivariate Gaussian.

It would be interesting to obtain results uniformly in the annulus $1 - \delta < |z| \leq 1$ i.e. encompassing the different limit regimes.

Conjecture for ζ

Let $s = \sigma + it$ with $\sigma > \frac{1}{2}$ fixed and consider joint moments

$$R_{h,k}^{\mathrm{NT}}(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int_0^T |\zeta'(\sigma + it)|^{2h} |\zeta(\sigma + it)|^{2k-2h} dt.$$

From our random matrix results, it seems reasonable to expect that for $h, k \in \mathbb{N}$, we have

$$R_{h,k}^{\mathrm{NT}}(\sigma) \sim \frac{1}{(\sigma - \frac{1}{2})^{k^2+2h}} L_h(-k^2) a_{h,k}^{\mathrm{arith}}$$

as $\sigma \to \frac{1}{2}$ from above, where $L_h(x)$ is the Laguerre polynomial. $a_{h,k}^{\mathrm{arith}}$ is an arithmetic contribution the number theorists understand.

Moments and zeros

Moments

$$\frac{d}{dh}\mathbb{E}(|\Lambda_N(z)|^{2h})\Big|_{h=0} = 2\mathbb{E}(\log|\Lambda_N(z)|)$$

Jensen's formula (ρ_k zeros of $\Lambda_N'(z)$)

$$\log|\Lambda_N'(0)| = -\sum_k \log(r/|\rho_k|) + \frac{1}{2\pi}\int_0^{2\pi} \log|\Lambda_N(re^{i\theta})|d\theta$$

Therefore, in principle, moment computations may give insights into the zero distribution of $\Lambda_N'(z)$.