Pinched-up periodic KPZ fixed point

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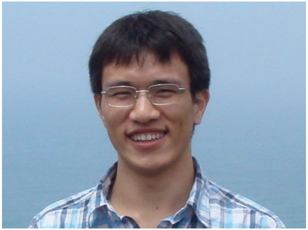


Zhipeng Liu (University of Kansas)

- Jinho Baik and Zhipeng Liu, **Pinched-up periodic KPZ fixed point**, 2024
- Zhipeng Liu and Yizao Wang, **A conditional scaling limit of the KPZ fixed point with height tending to infinity at one location**, 2022
- Jinho Baik and Zhipeng Liu, **Multi-point distribution of periodic TASEP**, JAMS, 2019

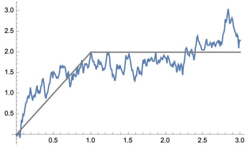
Brownian motion pinched up at $t = 1$: Conditioned that $B(1) = L$,

$$B(1+t) - B(1) \stackrel{d}{=} B^{(1)}(t) \qquad \text{for } t > 1$$

and

$$B(t) - tL \stackrel{d}{=} \mathbb{B}(t) \qquad \text{for } 0 < t < 1$$

where $\mathbb{B}(t)$ is a Brownian bridge.

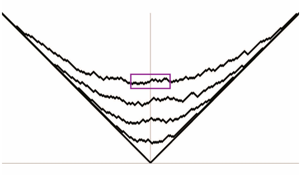


1. KPZ fixed point
2. What happens if it is pinched up at a particular (x_0, t_0)?
3. Periodic KPX fixed point
4. Pinched up case

KPZ fixed point

- 2d random field $\mathrm{H}(x,t)$, $x \in \mathbb{R}$, $t > 0$.
- It was constructed by Mateski, Quastel, Remenik 2021
- It is the conjectured universal limit of random growth models (KPZ equation, corner growth model), interacting particle systems (exclusion processes), and directed polymers,

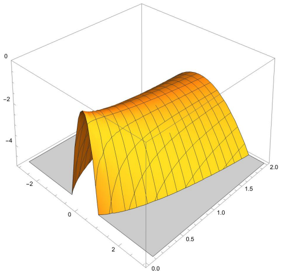
$$\frac{h(xT^{2/3}, tT) - a(x,t)T}{b(x,t)T^{1/3}} \to \mathrm{H}(x,t) \qquad \text{as } T \to \infty$$



Narrow-wedge initial condition

Assume $\mathsf{H}(0,0) = 0$ and $\mathsf{H}(x,0) = -\infty$ for $x \neq 0$.

- $\alpha \mathsf{H}(\alpha^{-2}x, \alpha^{-3}t) \overset{d}{=} \mathsf{H}(x,t)$ and $\mathsf{H}(x,t) \overset{d}{=} t^{1/3}\mathsf{H}(0,1) - \frac{x^2}{t}$
- One-point: $\mathsf{H}(x,t) \overset{d}{=} t^{1/3}\,\mathsf{TW} - \frac{x^2}{t}$
- Equal-time: $x \mapsto \mathsf{H}(x,t)$ is the Airy$_2$ process, after scaling
- Transition probabilities from $\mathsf{H}(\cdot, t_1)$ to $\mathsf{H}(\cdot, t_2)$ Mateski, Quastel, Remenik 2021
- Johansson and Rahman, and independently Liu computed the multi-point distributions $\mathbb{P}(\cap_{i=1}^{m} \mathsf{H}(x_i, t_i) \leq \beta_i)$ in 2022



$$z = \mathbb{E}[\mathsf{H}(x,t)] \approx -1.77t^{1/3} - \frac{x^2}{t}$$

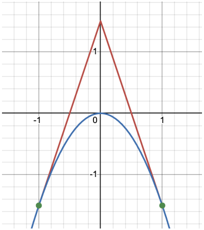
Pinch-up

What does $\mathrm{H}(x, t)$ look like if we condition that $\mathrm{H}(0, 1) = L$ and for large $L > 0$?

<u>For $t = 1$:</u>

Heuristics from the Airy process (e.g., [Ganguly and Hedge 2022]) indicates that

$$\mathbb{E}[\mathrm{H}(u\sqrt{L}, 1)] \approx \begin{cases} (1 - 2|u|)L & |u| \leq 1 \\ -u^2 L & |u| > 1 \end{cases}$$



KPZ fixed point conditional to be large at one point

For $t > 1$:

$\mathsf{H}(x,1) - \mathsf{H}(0,1)$ looks like the narrow-wedge initial condition. From the Markovian property, we expect that conditional on $\mathsf{H}(0,1) = L \to \infty$,

$$\mathsf{H}(x, 1+t) - \mathsf{H}(0,1) \approx \mathsf{H}^{(1)}(x,t) \qquad \text{for } t > 0$$

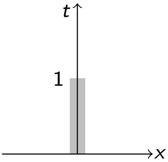
This was proved for one-point distributions in 2023 by Ron Nissim and Ray Zhang.

For $0 < t < 1$:

Liu and Wang showed in 2022 that, conditional on $\mathsf{H}(0,1) = L \to \infty$,

$$\frac{\mathsf{H}(\frac{x}{\sqrt{2}L^{1/4}}, t) - tL}{\sqrt{2}L^{1/4}} \to (\mathbb{B}_1(t) - x) \wedge (\mathbb{B}_2(t) + x)$$

for $(x,t) \in \mathbb{R} \times (0,1)$ in the finite-dimensional distributions sense where $\mathbb{B}_1, \mathbb{B}_2$ are independent Brownian bridges.

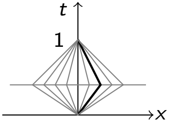


Heuristics on the scaling

Property of KPZ fixed point

$$\mathrm{H}(0,1) \overset{d}{=} \max_{x\in\mathbb{R}}\{\mathrm{H}(x,t)+\mathrm{H}^{(1)}(x,1-t)\}$$

Geodesic $x = x_{\max}(t)$



Consider the probability

$$P(x,y) := \mathbb{P}(x_{\max}=x,\ \mathrm{H}(x_{\max},t)-tL=y|\mathrm{H}(0,1)=L)$$

When $L>0$ is large, heuristically,

$$P(x,y) \approx \frac{\mathbb{P}(\mathrm{H}(x,t)=tL+y,\ \mathrm{H}^{(1)}(x,1-t)=(1-t)L-y)}{\mathbb{P}(\mathrm{H}(0,1)=L)}$$

$$\approx \frac{\mathbb{P}\left(\mathrm{TW}=(tL+y+\frac{x^2}{t})t^{1/3}\right)\mathbb{P}\left(\mathrm{TW}=((1-t)L-y+\frac{x^2}{1-t})(1-t)^{1/3}\right)}{\mathbb{P}(\mathrm{TW}=L)}$$

Right tail of the TW distribution

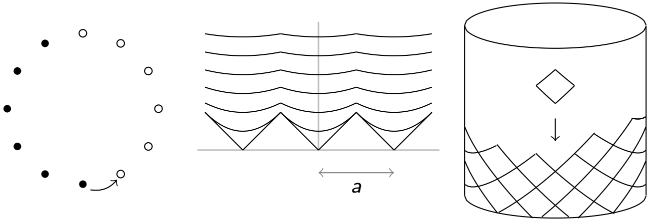
$$\mathbb{P}(\mathrm{TW}=y)\mathrm{d}y \approx \frac{e^{-\frac{4}{3}y^{3/2}}}{8\pi y}\mathrm{d}y \qquad y\to+\infty$$

After Taylor expansion,

$$\log P(x,y) \approx -\frac{1}{t(1-t)}\left[2x^2L^{1/2}+\frac{y^2}{2L^{1/2}}\right]$$

This is $O(1)$ when $x=O(L^{-1/4})$ and $y=O(L^{1/4})$.

Periodic KPZ fixed point



The spatial correlations of KPZ universality models scale as $T^{2/3}$. Thus, random growth processes on a ring are not affected by the ring size a if $a \gg T^{2/3}$. The interesting limit arises when

$$T, a \to \infty \text{ such that } p := \frac{2a}{T^{2/3}} \text{ is fixed}$$

In this "relaxation time scale", the multi-point distributions were obtained for

- continuous-time TASEP [Baik and Liu 2019]
- discrete-time TASEP [Liao 2022]
- PushASEP [Li and Saenz 2023]
- PNG and Brownian directed last passage percolation [Tripathi 2024]

p-periodic KPZ fixed point

- Periodicity:
$$\mathcal{H}_p(\gamma + p, \tau) = \mathcal{H}_p(\gamma, \tau)$$
We assume the periodic narrow wedge initial condition.
- It is expected to interpolate the KPZ fixed point and the Brownian motion:
$$\mathcal{H}_p(2\gamma, \tau) \to \mathrm{H}(\gamma, \tau) \qquad \text{as } p \to \infty \text{ (long period)}$$
and
$$\frac{\sqrt{2}p^{1/4}}{\pi^{1/4}} \left(\mathcal{H}_p(\gamma, \tau) + p^{-1}\tau\right) \to \mathrm{B}(\tau) \qquad \text{as } p \to 0 \text{ (short period)}$$
- The one-point distribution case when $\gamma = 0$ was proved by Baik, Liu, and Silva 2022

Results for pinched-up periodic KPZ fixed point

Theorem [Baik and Liu]: Let

$$\widetilde{\mathcal{H}}_p(x,t) := \frac{\mathcal{H}_p(\frac{\sqrt{2}x}{\ell^{1/4}}, t) - t\ell}{\sqrt{2}\ell^{1/4}} \quad \text{conditional on } \mathcal{H}_p(0,1) = \ell$$

The following results hold for $(x,t) \in \mathbb{R} \times (0,1)$ in the sense of convergence of finite-dimensional distributions as $\ell \to \infty$:

- ("long" period)

$$\widetilde{\mathcal{H}}_p(x,t) \to (\mathbb{B}_1(t) - x) \wedge (\mathbb{B}_2(t) + x) \qquad \text{if } \ell^{-1/4} \ll p \text{ and } \log p \ll \ell^{3/2}$$

- (critically shrinking period)

$$\widetilde{\mathcal{H}}_p(x,t) \to \mathsf{M}_{\mathsf{r}}(x,t) \qquad \text{if } p = \mathsf{r}\sqrt{2}\ell^{-1/4} \text{ for } \mathsf{r} > 0.$$

- (short period)

$$\widetilde{\mathcal{H}}_p(x,t) \to \frac{1}{\sqrt{2}}\mathbb{B}(t) \qquad \text{if } \ell^{-1}\log\ell \ll p \ll \ell^{-1/4}$$

Note: The long period result is expected to hold without assuming $\log p \ll \ell^{3/2}$.

Random field $M_r(x, t)$ for the critically shrinking case

For every $r > 0$, let

$$w_r(x, y) = \max\left\{x + r\left[\frac{y-x}{2r}\right], y - r\left[\frac{y-x}{2r}\right] - r\right\}$$

Let $\mathbb{B}_1^{(r)}(t)$ be a Brownian bridge on the circle $S^1(r) = \mathbb{R}/\sqrt{2}r\mathbb{Z}$. Let $\mathbb{B}_2(t)$ be an independent standard Brownian bridge.
Define

$$M_r(x, t) = w_r\left(\frac{\mathbb{B}_1^{(r)}(t) + \mathbb{B}_2(t)}{\sqrt{2}} - x, \frac{-\mathbb{B}_1^{(r)}(t) + \mathbb{B}_2(t)}{\sqrt{2}} + x\right)$$

for $(x, t) \in \mathbb{R} \times (0, 1)$, where $\mathbb{B}_1^{(r)}(t)$ is a Brownian bridge on the circle $S^1(r) = \mathbb{R}/\sqrt{2}r\mathbb{Z}$ and $\mathbb{B}_2(t)$ is an independent standard Brownian bridge.

Note

$$\lim_{r\to\infty}\left\lceil\frac{\alpha}{2r}\right\rceil=\begin{cases}-1 & \alpha<0\\ 0 & \alpha\geq 0\end{cases}\quad\text{and}\quad\lim_{r\to 0} r\left\lceil\frac{\alpha}{2r}\right\rceil=\frac{\alpha}{2}$$

Lemma

$w_r(x,y)=\max\left\{x+r\left\lceil\frac{y-x}{2r}\right\rceil, y-r\left\lceil\frac{y-x}{2r}\right\rceil-r\right\}$ *satisfies*

$$\lim_{r\to\infty} w_r(x,y)=x\wedge y \quad \textit{and} \quad \lim_{r\to 0} w_r(x,y)=\frac{x+y}{2}.$$

Thus, $M_r(x,t)=w_r\big(\frac{\mathbb{B}_1^{(r)}(t)+\mathbb{B}_2(t)}{\sqrt{2}}-x, \frac{-\mathbb{B}_1^{(r)}(t)+\mathbb{B}_2(t)}{\sqrt{2}}+x\big)$ satisfies:

- As $r\to\infty$,

$$M_r(x,t)\to\min\left\{\frac{\mathbb{B}_1^{(\infty)}(t)+\mathbb{B}_2(t)}{\sqrt{2}}-x, \frac{-\mathbb{B}_1^{(\infty)}(t)+\mathbb{B}_2(t)}{\sqrt{2}}+x\right\}$$
$$\stackrel{d}{=}(\mathbb{B}_1^{(1)}(t)-x)\wedge(\mathbb{B}_2^{(2)}(t)+x)$$

- As $r\to 0$, $M_r(x,t)\to\frac{1}{\sqrt{2}}\mathbb{B}_2(t)$

The proof relies on the exact formula of the multi-point distributions of Baik and Liu 2019,

$$\mathbb{P}\left(\cap_{i=1}^{m-1}\{\mathcal{H}_p(\gamma_i,\tau_i)\geq\beta_i\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\leq\beta_m\}\right)$$
$$=\frac{(-1)^{m-1}}{(2\pi\mathrm{i})^m}\oint\cdots\oint C(z)\det(1-\mathrm{K}_z)\prod_{i=1}^{m}\frac{\mathrm{d}z_i}{z_i}$$

where the contours circles satisfying $0<|z_1|<\cdots<|z_m|<1$ and

$$\det(1-\mathrm{K}_z)=1+\sum_{n_1,\cdots,n_m=0}^{\infty}A(n_1,\cdots,n_m)$$

To compute the conditional distribution, we evaluate the limit of

$$\frac{\frac{\partial}{\partial\beta_m}\mathbb{P}\left(\cap_{i=1}^{m-1}\{\mathcal{H}_p(\gamma_i,\tau_i)\geq\beta_i\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\leq\beta_m\}\right)}{\frac{\partial}{\partial\beta_m}\mathbb{P}\left(\mathcal{H}_p(\gamma_m,\tau_m)\leq\beta_m\right)}$$

where $\gamma_m=0,\tau_m=1,\beta_m=L$. The main contribution comes from

$$A(1,\cdots,1)$$

In the end, the limit for the "long" period case involves the integral

$$\frac{1}{(2\pi \mathrm{i})^m}\int\cdots\int\prod_{i=2}^{m}\frac{1}{\xi_i-\xi_{i-1}}\prod_{i=1}^{m}e^{\frac{(a_i-a_{i-1})}{2}\xi_i^2-(b_i-b_{i-1})\xi_i}\mathrm{d}\xi_i$$

which is equal to

$$\mathbb{P}\left(\mathrm{B}(a_1)\geq b_1,\cdots,\mathrm{B}(a_{m-1})\geq b_{m-1},\mathrm{B}(a_m)=b_m\right),$$

The critically shrinking period case $p=\mathrm{r}\sqrt{2}\ell^{-1/4}$ involves

$$\sum_{k\in\mathbb{Z}^m}\frac{\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{\mathrm{B}_1(a_i)-b_i\geq \mathrm{r}k_i\}\cap\{\mathrm{B}_1(a_m)-b_m=\mathrm{r}k_m\}\right)}{w_1^{k_1}\cdots w_m^{k_m-k_{m-1}}}$$

for complex numbers $w_1,\cdots,w_m$. An integral over $w_1,\cdots,w_m$ involving this multi-sum above has a probability interpretation.

Thank you for attention