Large deviations of the spectrum of supercritical sparse Erdős-Rényi graphs

Fanny Augeri

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Bernoulli matrices

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where $\mu_{\mathsf{sc}} = (2/\pi)\sqrt{(1-x^2)_+}dx$.

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What is the large deviation behaviour of $\mu_{A/\sqrt{n}}$?

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- Large deviations of $\mu_{A/\sqrt{n}}$ turn out to be very difficult.

$$\text{(LDP)} \qquad \log \mathbb{P}(\mu_{A/\sqrt{n}} \simeq \nu) \overset{?}{\asymp} e^{-v_n I(\nu)}, \ \nu \neq \mu_{\mathsf{sc}},$$

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    since $n^{-2}\mathrm{tr}(A - \mathbb{E}A)^2 = 1/4$ and $\mathbb{E}A$ rank 1.
  - I cannot be universal: $I \neq I_{\mathrm{GOE}}$.

$$\delta_0 \in \mathcal{D}_I \text{ but } \delta_0 \notin \mathcal{D}_{I_{\mathsf{GOE}}}.$$

Some universality in large deviation behaviour

Let $\underline{A} = A - \mathbb{E}A$. By Cauchy interlacing inequalities

$$d_{\mathrm{KS}}(\mu_{\underline{A}/\sqrt{n}}, \mu_{A/\sqrt{n}}) \leq \frac{1}{n}\mathrm{rk}(A - \underline{A}) = \frac{1}{n}.$$

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➤ Universality is a manifestation of moderate deviation behaviour.

Sparsification as a relaxation of the problem

Take $p \ll 1$ and consider A the adjacency matrix of the Erdős-Rényi graph $G(n, p)$.

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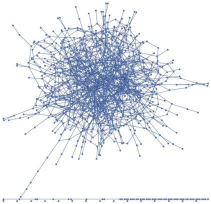
0 — $\log n$ — n

$np \lesssim \log n$	$\log n \ll np$
Inhomogeneous graph:	Homogeneous graph:
• isolated vertices • hubs • leaves	• vertices have all almost the same degrees • connected w.h.p



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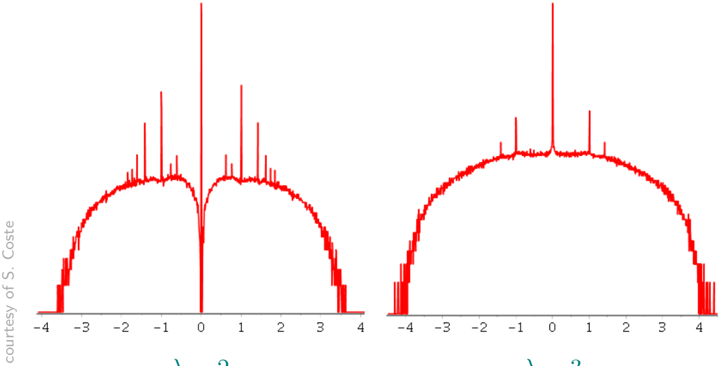

$\lambda = 2$ $\lambda = 3$

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(Bordenave-Caputo) μ_A satisfies a LDP with speed n.

➤ Obtained by contracting the LDP for $G(n, \lambda/n)$ with respect to the local weak topology.

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➤ The only possible deviations are around measures coming from Quadratic Vector Equations.

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Let $W : [0,1]^2 \to \mathbb{R}_+$ be a Borel measurable symmetric kernel s.t.

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The QVE associated with W is

$$-\frac{1}{m(z,x)} = z + \int_0^1 W(x,y)m(z,y)dy, \ z \in \mathbb{H}, x \in [0,1],$$

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(Ajanki, Erdős, Krüger): Existence and uniqueness in $\mathcal{B}^+$, the space of $m : \mathbb{H} \times [0,1] \to \mathbb{H}$ s.t. $m(z,.)$ is bounded for any $z \in \mathbb{H}$.

QVE measure of a kernel

If $m = (m(z,x))_{z\in\mathbb{H},x\in[0,1]}$ is the unique solution in $\mathcal{B}^+$ of the QVE

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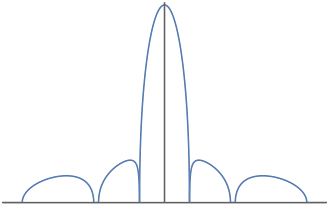

(Ajanki, Erdős, Krüger):

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$\mathrm{supp}(\upsilon_W) \subset [-2\|d_W\|_\infty^{1/2}, 2\|d_W\|_\infty^{1/2}]$.

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$$\upsilon_W(\tau^{2k}) = \sum_{(F,o) \in \mathcal{T}_k} t(F,W), \;\; k \in \mathbb{N}$$

where $\mathcal{T}_k$ is the set of unlabelled rooted planar trees with k edges

$$t(F,W) = \int_{[0,1]^k} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{\ell=1}^{k} dx_\ell.$$

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Take W a kernel with bounded degree function and set

$$W_{ij} = n^2 \int_{(\frac{i-1}{n},\frac{i}{n}]\times(\frac{j-1}{n},\frac{j}{n}]} W(x,y)dxdy, \quad i,j = 1,\ldots,n.$$

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➤ The optimal large deviation strategies of $\mu_{A/\sqrt{np}}$ correspond to changing $G(n,p)$ into one inhomogeous Erdős-Rényi graph $G(n,pW)$.

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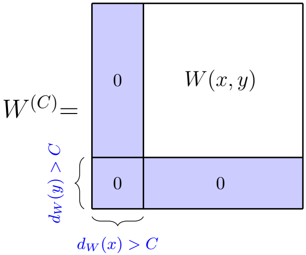
Extend the definition of the QVE measure to integrable kernels W:

$$\upsilon_W = \lim_{C \to +\infty} \upsilon_{W^{(C)}},$$

where $W^{(C)}$ is the degree truncated kernel

$$W^{(C)}(x,y) = W(x,y) \mathbb{1}_{d_W(x) \le C} \mathbb{1}_{d_W(y) \le C}, \; (x,y) \in [0,1]^2.$$



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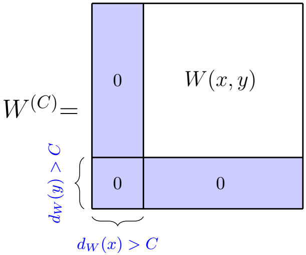

Since $d_W \in L^1$,

$$\|W^{(C)} - W\|_1 \underset{C \to +\infty}{\longrightarrow} 0.$$

If U, V are kernels with bounded degree functions

$$\mathscr{W}_2(\upsilon_U, \upsilon_V) \le \|U - V\|_1^{1/2}$$

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The rate function in the supercritical sparse case

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(A.) If $\log \ll np \ll n$, then $\mu_{A/\sqrt{np}}$ satisfies a LDP with speed $n^2 p$ and rate function I:

$$I(\nu) = \inf\left\{H(W) : \upsilon_W = \nu,\ W \in \mathcal{W}\right\},\ \nu \in \mathcal{P}(\mathbb{R}),$$

where $\mathcal{W}$ is the set of integrable kernels on $[0,1]^2$ and

$$H(W) = \int_{[0,1]^2} h(W(x,y))dxdy,\ W \in \mathcal{W},$$

with $h(u) = u\log u - u + 1$.

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- If $I(\nu) < +\infty$ then the infimum is achieved.
- Conditioned on a large deviation of its spectrum, $G(n,p)$ is expected to look like $G(n,pW)$ for some $W \in \mathcal{W}$.

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(A.) If $\log n \ll np \ll n$, $\mu_{X/\sqrt{np}}$ satisfies a LDP with speed $n^2 p$ and rate function I_L:

$$I_L(\nu) = \inf\left\{H_L(W) : \upsilon_W = \nu, W \in \mathcal{W}\right\},\ \nu \in \mathcal{P}(\mathbb{R}),$$

where $H_L(W) = \int_{[0,1]^2} h_L(W(x,y))dxdy$,

$$h_L(u) = \sup_{\theta \in \mathbb{R}}\{\theta u - L(\theta)\}, \quad L(\theta) = \mathbb{E}(e^{\theta M_{1,2}^2}) - 1,\ u, \theta \in \mathbb{R}.$$

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Thus, $(G_{ii})_{1 \le i \le n}$ satisfies a perturbed QVE:

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If the d_i's were independent, then by Bennett's inequality

$$\mathbb{P}(\Lambda_n > \delta n) \leq \exp\Big(-\delta n \log \frac{\delta n}{3\mathbb{E}(\Lambda_n)}\Big) \leq e^{-C_{\delta,\varepsilon} n^2 p\sqrt{\log(1/p)}}.$$

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- Develop a Bennett-type inequality for dependent variables.

Thank you for your attention!