A random matrix model towards the quantum chaos transition conjecture

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Joint work with Bertrand Stone and Jun Yin

Random Matrices and Related Topics in Jeju

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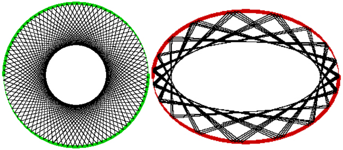
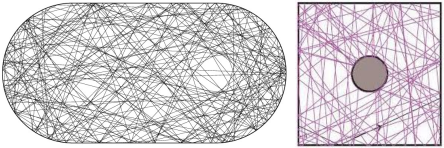
Quantum chaos

Two-dimensional classical billiard:



(a) Integrable



(b) Chaotic

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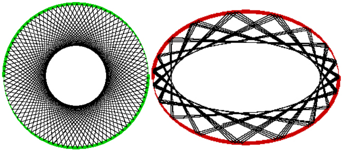

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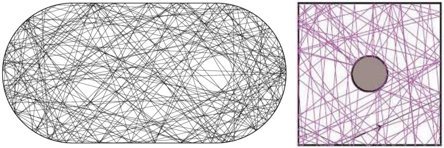


(b) Chaotic

The classical billiard is the high-energy limit of the quantum billiard:

$$-\Delta\psi = E\psi, \quad \psi|_{\partial B} = 0.$$

What is quantum chaos?

How does the quantum billiard behave if it corresponds to a chaotic (resp. integrable) classical billiard? In other words, how can we tell if a quantum system is chaotic?

Integrable v.s. Chaotic

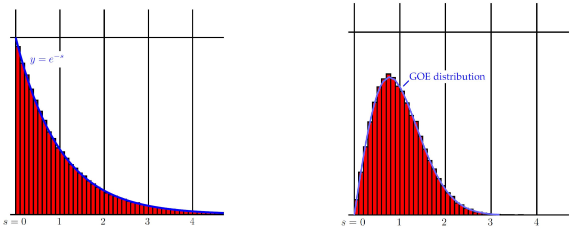


Figure: Rudnick: What is Quantum Chaos?

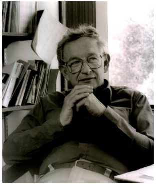
Berry-Tabor conjecture (1977)

The local energy level statistics of a generic integrable model follow a Poisson process (and the eigenvectors are localized).

Bohigas-Giannoni-Schmit conjecture (1984)

The local energy level statistics of a generic chaotic model follow the random matrix statistics (GOE/GUE) of the same symmetry (and the eigenvectors are delocalized).

QC transition of Anderson model



On $\mathbb{Z}^d$, consider the Anderson model (Anderson, 1958)

$$H = -\Delta + \lambda V$$

$-\Delta$: kinetic energy; V: i.i.d. random potential for impurities.

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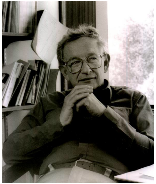
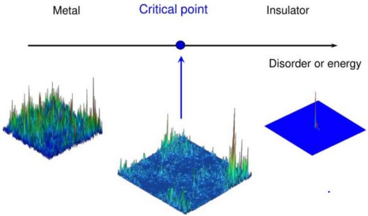
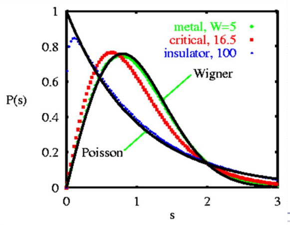


Anderson localization/delocalization conjecture

- Any $\lambda > 0$, $d = 1, 2$ or large λ, $d \geq 3$: Localization (Fröhlich-Spencer '83, Aizenman and Molchanov '93, etc.) + Poisson statistics (Minami '96).
- Small λ, $d \geq 3$: Localization near the spectrum edges, and delocalization and random matrix statistics inside the bulk. (No rigorous theory so far!)





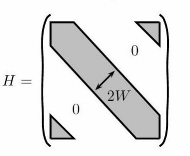
QC transition of random band matrices

Random band matrices

An $N \times N$ band matrix $H = (h_{ij})$ has centered and independent entries up to symmetry $h_{ij} = \overline{h}_{ji}$:

$$h_{ij} = 0, \quad |i-j| > W; \quad \mathbb{E}|h_{xy}|^2 = (2W+1)^{-1}, \quad |i-j| \le W,$$

where band width $W \ll N$ and the distance $|\cdot|$ is periodic.



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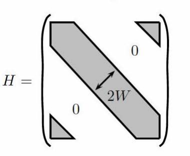


Quantum chaos transition conjecture (Casati-Molinari-Izrailev '90; Fyodorov-Mirlin '91)

Localization+Poisson if $W \ll \sqrt{L}$; delocalization+GOE/GUE if $W \gg \sqrt{L}$.

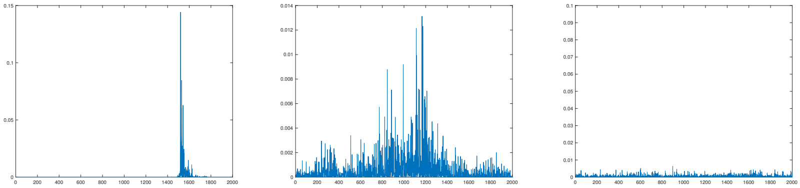


Figure: $|u_k(x)|^2$. $d = 1$, $L = 2000$, $W = L^{1/4}$, $W = L^{1/2}$ and $W = L^{3/4}$.

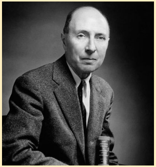
Wigner matrices

A random diagonal matrix with i.i.d. entries is the simplest (random) integrable quantum system. What is the canonical chaotic system?

Wigner matrices and GOE/GUE (Wigner 1955)

- A Wigner matrix H is an $N \times N$ Hermitian random matrix with independent entries H_{ij} of mean 0 and variance N^{-1}.
- H is called a GOE/GUE if it is real/complex Gaussian.
- The ESD of H satisfies the famous semicircle law

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}.$$



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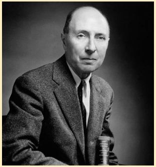
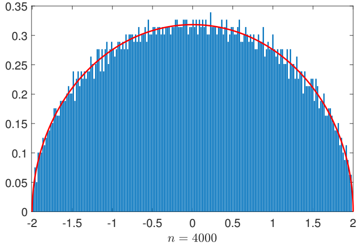


Eigenvalues around ± 2 are called edge eigenvalues, and those away from ± 2 are called bulk eigenvalues.



Quantum chaos of Wigner matrices

Wigner's universality hypothesis

The local spectral statistics of large complex quantum systems are **universal**, and given by the random matrix statistics of the same symmetry (GOE for symmetric models or GUE for Hermitian models).

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Wigner-Dyson-Gaudin-Mehta conjecture (Bourgade, Erdős, Knowles, Schlein, Tao, Vu, Yau, Yin, etc.)

Spectral statistics of Wigner ensembles do not depend on the matrix law.

- Bulk universality. Local statistics of eigenvalues $\{N(\lambda_i - E)\}$ around a bulk energy $E \in (-2, 2)$ are universal, i.e., they match those of GOE/GUE.
- QUE. For any subset I so that $|I| \gg 1$, with high probability,

$$\sum_{x \in I} \left(N |\mathbf{u}_k(x)|^2 - 1 \right) \ll |I|.$$

  In particular, the eigenvectors are completely delocalized.

Quantum unique ergodicity (Rudnick-Sarnak)

For generic strongly chaotic Hamiltonians, the high energy eigenstates become equidistributed:

$$\int f(x) |\psi_j(x)|^2 \mathrm{d}\mathcal{V}ol(x) \to \int f(x) \mathrm{d}\mathcal{V}ol(x).$$

A random matrix model

We consider the block random matrix model $H = V + \Lambda$.

- V: random diagonal block matrix consisting of D independent $N \times N$ Wigner matrices.
- Λ: interactions between neighboring subsystems, independent of V.

$$V = \begin{pmatrix} H_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & H_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & H_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & H_{D-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & H_D \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & A & 0 & \cdots & 0 & A^* \\ A^* & 0 & A & \cdots & 0 & 0 \\ 0 & A^* & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A \\ A & 0 & 0 & \cdots & A^* & 0 \end{pmatrix}.$$

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V models a union of D independent complicated systems lying on a circle. On the block level, we consider it an integrable model:

- Every eigenvector is "localized" in only one block.
- Letting $N \to \infty$ and $D \to \infty$, for any E in the bulk, the local eigenvalue statistics $\{DN(\lambda_i - E)\}$ converge to a Poisson process.

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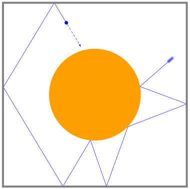
The perturbation Λ introduces interactions and brings into chaos.

Key questions

Q: Can we get a chaotic system when A increases? When the quantum chaos transition occurs?
A: It occurs at $\|\Lambda\|_{HS} = 1$.

Connection with quantum billiard

Consider the Sinai billiard on the rectangle $[0, \pi/a] \times [0, \pi/b]$.



The eigenstates of $-\Delta$ on the rectangle is

$$|n, m\rangle(x, y) = \sin(nax)\sin(mby),$$

with eigenvalue $n^2a^2 + m^2b^2$. In the basis $\{|n, m\rangle\}$, the eigenstates are localized. For high energy $E \to \infty$, the eigenvalues approximately follow a Poisson distribution if a and b are sufficiently incommensurable.

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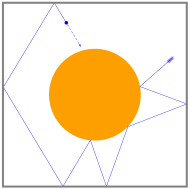

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When there is a potential $\lambda\phi$:

$$\phi_{nn'}(m, m') = \langle n, m|\phi|n', m'\rangle = \iint \phi(x, y)\sin(nax)\sin(n'ax)\sin(mby)\sin(m'by)\mathrm{d}x\mathrm{d}y.$$

Consider m and n with $n^2a^2 + m^2b^2 \sim E$. We form a block matrix where each block is labeled by a fixed momentum in the x direction.

- $\phi_{nn'}$ decreases as $|n - n'|$ increases, so we approximate it with nearest-neighbor interactions.
- ϕ_{nn} are matrices with complicated phases. We model them with random matrices.

When does quantum chaos happen as λ increases?

Main results: Integrable regime

Theorem (Stone-Y'-Yin 2023+)

Suppose D is fixed while $N \to \infty$. A is an arbitrary deterministic matrix satisfying $\|A\|_{HS} \le N^{-\varepsilon}$. Then:

- *(Localized eigenvector) Let E_a be the identity matrix restricted to the a-th block:*

$$(E_a)_{xy} = \mathbf{1}(x = y \in [aN+1, (a+1)N]).$$

The bulk eigenvectors $\mathbf{v}_k$ satisfy

$$\mathbb{P}\left(\max_{a=1}^{D} \|E_a \mathbf{v}_k\|_2^2 \ge 1 - N^{-c}\right) \ge 1 - N^{-c}.$$

- *(Perturbation of eigenvalues) The bulk eigenvalues λ_k of H are small perturbations of those of V:*

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- This theorem shows the non-chaotic behavior of H when $\|A\|_{HS}$ is small.
- If we let $N \to \infty$ followed by $D \to \infty$, the bulk eigenvalue statistics $\{DN(\lambda_i - E)\}$ converge to a Poisson process for any $E \in (-2+\epsilon, 2-\epsilon)$.

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- *(Bulk universality) The bulk eigenvalues statistics $\{DN(\lambda_i - E)\}$ match those of $DN \times DN$ GOE/GUE for any $E \in (-2+\epsilon, 2-\epsilon)$.*

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About the condition $\|A\| \leq N^{-\varepsilon}$:

- It simplifies the proof: the spectrum of H is a small perturbation of that of V (i.e., semicircle law).
- It is necessary: the result does not hold if A has only a few large eigenvalues.
- It is not "essential": we conjecture that the same results still hold when $\|A\| \gtrsim 1$ as long as A has "large" rank. (Currently, $\mathrm{rank}(A) \geq N^{4\varepsilon}$.)

Simulations: eigenvectors

Choose $D = 10$, $N = 200$, D independent $N \times N$ GOE, and $A = \lambda I$.

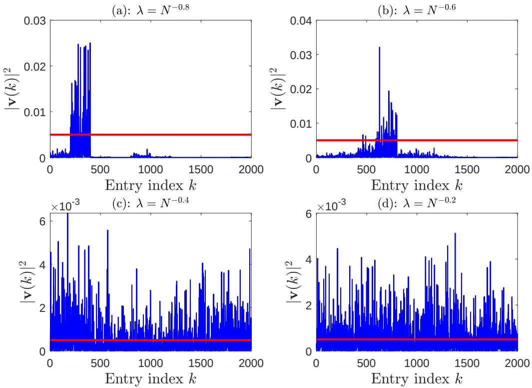

Figure: The entries $|\mathbf{v}(k)|^2$ for the $(DN/2)$-th eigenvector $\mathbf{v}$.

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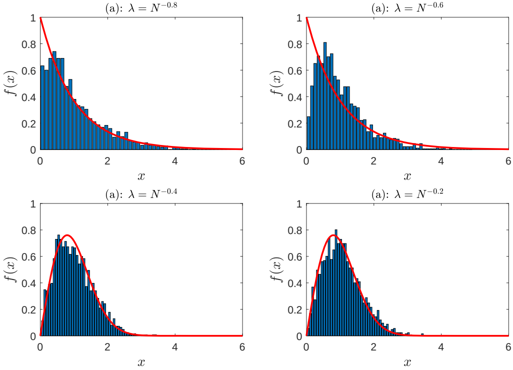


Figure: The bulk eigenvalue gap distributions for $DN\rho_{sc}(\lambda_k)(\lambda_{k+1} - \lambda_k)$; ρ_{sc} is the semicircle density.

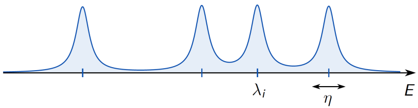
In (a) and (b), the red curves plot the PDF $f(x) = e^{-x}$. In (c) and (d), the red curves plot the Wigner surmise.

Green's function

Our proof is based on an investigation of the Green's function (or resolvent):

$$G(z) = (H - z)^{-1} = \sum_k \frac{\mathbf{v}_k \mathbf{v}_k^*}{\lambda_k - E - \mathrm{i}\eta}, \quad z = E + \mathrm{i}\eta.$$

$$\mathrm{Im}\,\langle G(z)\rangle = \frac{1}{DN} \sum_k \frac{\eta}{(\lambda_k - E)^2 + \eta^2}, \quad \langle G(z)\rangle := \frac{1}{DN} \mathrm{Tr}\, G(z).$$



$\mathrm{Im}\,\langle G(z)\rangle$ contains info of eigenvalues in a window of scale η (the spectral resolution) around E, and $\mathrm{Im}\,\langle G(z)E_a\rangle$ contains info of $\mathbf{v}_k^* E_a \mathbf{v}_k$.

Local law

In RMT literature, the following local law has been proved for $G(z)$:

Theorem (Ajanki-Erdős-Krüger 2019; Erdős-Krüger-Schröder 2019; He-Knowles-Rosenthal 2018)

For all $E \in (-2+\varepsilon, 2-\varepsilon)$, $\eta \gg N^{-1+\varepsilon}$ and any deterministic unit vectors $\mathbf{u}, \mathbf{v}$, with high probability,

$$\left|\mathbf{u}^* \left[G(z) - M(z)\right] \mathbf{v}\right| \le N^c (N\eta)^{-1/2}$$

for any small constant $c > 0$.

Define $m(z)$ as the unique solution to the equation

$$m(z) = \frac{1}{DN} \operatorname{Tr} \frac{1}{\Lambda - z - m(z)}, \quad \text{with} \quad \operatorname{Im} m(z) > 0.$$

$m(z)$ is actually the Stieltjes transform of the free convolution of the empirical measure of Λ and the semicircle law.

Then, we define $M(z) = \left[\Lambda - z - m(z)\right]^{-1}$. It is a deterministic matrix with $\|M(z)\| \sim 1$.

Integrable regime: eigenvalue

In the integrable regime, the estimate on eigenvalues follows from a simple perturbation approach. Define the interpolating matrices

$$H(\theta) = V + \theta\Lambda, \quad \theta \in [0,1], \quad \text{with} \quad H(0) = V, \quad H(1) = H.$$

Then, by the perturbation theory of eigenvalues,

$$\lambda_k(1) - \lambda_k(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\lambda_k(\theta)\mathrm{d}\theta = \int_0^1 \mathbf{v}_k(\theta)^*\Lambda\mathbf{v}_k(\theta)\mathrm{d}\theta,$$

from which we derive that

$$\mathbb{E}\,|\lambda_k(H) - \lambda_k(V)| \le \int_0^1 \mathbb{E}\left|\mathbf{v}_k(\theta)^*\Lambda\mathbf{v}_k(\theta)\right|\mathrm{d}\theta \le \int_0^1 \left(\mathbb{E}\left|\mathbf{v}_k(\theta)^*\Lambda\mathbf{v}_k(\theta)\right|^2\right)^{1/2}\mathrm{d}\theta.$$

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With the spectral decomposition of $\operatorname{Im} G(\theta)$,

$$\operatorname{Tr}\left[(\operatorname{Im} G)\Lambda(\operatorname{Im} G)\Lambda\right] = \sum_{k,l} \frac{\eta^2\left|\mathbf{v}_k^*\Lambda\mathbf{v}_l\right|^2}{|z-\lambda_k|^2|z-\lambda_l|^2}, \quad z = E + \mathrm{i}\eta,$$

if we let $E = \lambda_k$ and $\eta = N^{-1+c}$, then

$$\mathbb{E}|\mathbf{v}_k^*\Lambda\mathbf{v}_k|^2 \le \eta^2\mathbb{E}\operatorname{Tr}\left[(\operatorname{Im} G)\Lambda(\operatorname{Im} G)\Lambda\right].$$

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$$\operatorname{Tr}\left[(\operatorname{Im} G)\Lambda(\operatorname{Im} G)\Lambda\right] = \sum_{k,l} \frac{\eta^2\left|\mathbf{v}_k^*\Lambda\mathbf{v}_l\right|^2}{|z-\lambda_k|^2|z-\lambda_l|^2}, \quad z = E + \mathrm{i}\eta,$$

if we let $E = \lambda_k$ and $\eta = N^{-1+c}$, then

$$\mathbb{E}|\mathbf{v}_k^*\Lambda\mathbf{v}_k|^2 \le \eta^2\mathbb{E}\operatorname{Tr}\left[(\operatorname{Im} G)\Lambda(\operatorname{Im} G)\Lambda\right].$$

Need to show $\mathbb{E}\operatorname{Tr}\left[(\operatorname{Im} G)\Lambda(\operatorname{Im} G)\Lambda\right] \le N^c\|A\|_{HS}^2 \le N^{-2\varepsilon+c}$ ← a crucial technical part!

Integrable regime: eigenvectors

For simplicity, we suppose $D = 2$. Then,

$$H \begin{pmatrix} \mathbf{u}_k \\ \mathbf{w}_k \end{pmatrix} = \begin{pmatrix} H_1 & A \\ A^* & H_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_k \\ \mathbf{w}_k \end{pmatrix} = \lambda_k \begin{pmatrix} \mathbf{u}_k \\ \mathbf{w}_k \end{pmatrix} \Rightarrow \mathbf{w}_k = -G_2(\lambda_k) A^* \mathbf{u}_k, \quad \mathbf{u}_k = -G_1(\lambda_k) A \mathbf{w}_k$$

where $G_1(z) := (H_1 - z)^{-1}$, $G_2(z) := (H_2 - z)^{-1}$.

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We know $\mathrm{dist}(\lambda_k, \mathrm{spec}(H_1)) \le N^{-1-c}$ or $\mathrm{dist}(\lambda_k, \mathrm{spec}(H_2)) \le N^{-1-c}$ with probability $1 - o(1)$. But, by bulk universality of Wigner matrices (Bourgade-Erdős-Yau-Yin 2016),

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We claim that if $\mathrm{dist}(\lambda_k, \mathrm{spec}(H_1)) \ge N^{-1-c/2}$, then $\|\mathbf{u}_k\| = \|G_1(\lambda_k) A \mathbf{w}_k\|$ is small.

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With the spectral decompositions of $\mathrm{Im}\, G$ and $\mathrm{Im}\, G_1$, if we let $z = \lambda_k + \mathrm{i}\eta$ with $\eta = N^{-1+c}$, then

$$\begin{aligned} \|G_1(\lambda_k) A \mathbf{w}_k\|^2 &\le \left(\frac{\eta}{\mathrm{dist}(\lambda_k, \mathrm{spec}(H_1))} \right)^2 \mathbb{E}\mathrm{Tr} \left[\begin{pmatrix} 0 & 0 \\ 0 & A^* \mathrm{Im}\, G_1(z) A \end{pmatrix} \mathrm{Im}\, G(z) \right] \\ &\le N^{3c} \mathbb{E}\mathrm{Tr} \left[(\mathrm{Im}\, G_{\theta=0}(z)) \Lambda (\mathrm{Im}\, G(z)) \Lambda \right] \end{aligned}$$

Need to show $\mathbb{E}\mathrm{Tr} \left[(\mathrm{Im}\, G_{\theta=0}(z)) \Lambda (\mathrm{Im}\, G(z)) \Lambda \right] \le N^c \|A\|_{HS}^2$ $\leftarrow$ a similar bound as before!

Chaotic regime: QUE

By the spectral decompositions of $\operatorname{Im} G$, we have

$$\operatorname{Tr}\left[\operatorname{Im} G(z)(E_a - D^{-1}I)\operatorname{Im} G(z)(E_a - D^{-1}I)\right] = \eta^2 \sum_{i,j} \frac{|\mathbf{v}_i^*(E_a - D^{-1})\mathbf{v}_j|^2}{|\lambda_i - z|^2 |\lambda_j - z|^2}.$$

Choosing $z = \lambda_k + \mathrm{i}\eta$ with $\eta = N^{-1+c}$, we get

$$\mathbb{E}|\mathbf{v}_k^*(E_a - D^{-1})\mathbf{v}_k|^2 \le \eta^2 \mathbb{E}\operatorname{Tr}\left[\operatorname{Im} G(z)(E_a - D^{-1}I)\operatorname{Im} G(z)(E_a - D^{-1}I)\right].$$

Using $\operatorname{Im} G = \frac{G(z) - G(\bar{z})}{2\mathrm{i}}$ and $I = \sum_b E_b$, the RHS can be written as a linear combination of

$$\mathcal{L}_{aa}(z_1, z_2) - \frac{2}{D}\sum_{b=1}^{D} \mathcal{L}_{ab}(z_1, z_2) + \frac{1}{D^2}\sum_{b,b'=1}^{D} \mathcal{L}_{bb'}(z_1, z_2),$$

where $z_1, z_2 \in \{z, \bar{z}\}$ and $\mathcal{L}_{ab}(z_1, z_2) = \mathbb{E}\operatorname{Tr}\left[G(z_1)E_a G(z_2)E_b\right]$.

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Theorem (Multi-resolvent local law, Stone-Y'-Yin 2023+)

There exists a constant c_A depending on ε such that with high probability,

$$\max_{a,b,a',b'=1}^{D} |\mathcal{L}_{ab} - \mathcal{L}_{a'b'}| \le N^{-c_A}\eta^{-2}.$$

In other words, the matrix $\mathcal{L}$ is flat. This is the most technical part of the paper.

Chaotic regime: QUE $\Rightarrow$ bulk universality

Need to show the invariance of the joint moments of $\operatorname{Im}\langle G(E+\frac{x_i}{N}+\mathrm{i}\eta)\rangle$, $i=1,\ldots,k$, for some $\eta \ll N^{-1}$.

The three-step strategy (Bourgade, Erdős, Schlein, Yau, Yin, etc. 2009-2015)

1. A local law on the Green's function down to $\eta \geq \eta_* = N^{-1+c}$.
2. Short-time relaxation of Dyson Brownian Motion (DBM):

$$\mathrm{d}H_t = -\frac{1}{2}H_t\,\mathrm{d}t + \frac{1}{\sqrt{N}}\mathrm{d}B_t, \quad B_t := \text{symmetric independent BM}$$

$$H_t \overset{\mathrm{d}}{=} e^{-t/2}H + \sqrt{1-e^{-t}}\,\mathrm{GUE}$$

$$\text{Dyson Brownian Motion: } \mathrm{d}\lambda_k = \frac{1}{\sqrt{N}}\mathrm{d}B_k + \frac{1}{N}\sum_{l\neq k}\frac{1}{\lambda_k-\lambda_l}\mathrm{d}t - \lambda_k\,\mathrm{d}t$$

Theorem (Optimal relaxation of DBM to equilibrium, Landon-Yau 2017; Landon-Sosoe-Yau 2019)

The local spectral statistics of H_t in the bulk match those of GOE/GUE if $t \gg \eta_*$.

3. Green's function comparison: the law of $\operatorname{Im}\langle G_t\rangle$ is unchanged from $t=0$ to $t=\eta_*$:

$$\mathbb{E}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\langle G_t\rangle \leq N^{-c}\eta_*^{-1}.$$

This part uses Itô's formula and depends on the QUE of eigenvectors!

Some remarks

1. We only assume some high moment conditions for the distribution of the random entries. For the investigation of quantum chaos, it is desirable to extend Wigner matrices to "quasi-random matrices" (seems to be challenging!).
2. Our proof uses the block translation invariance of Λ. It can extend to non-nearest-neighbor interactions as long as the translation symmetry is maintained.
3. For the translation symmetry, we only need the interactions in different block to have (almost) the same distribution. Specifically, A can be replaced by i.i.d. random matrices. When the off-diagonal blocks are lower triangular random matrices, we obtain the random band matrices.
4. We have considered the small D case. But, it is important to extend the theory to large D cases. In particular, the most relevant case is when $D \sim N$ (corresponding to $n^2a^2 + m^2b^2 \sim E$), which also corresponds to critical random band matrices with $W \sim N^{1/2}$.
5. Similar to the Anderson model, one can further consider the higher dimensional cases, i.e., the subsystems lie on $\mathbb{Z}^d$ lattices instead of a cycle.

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Thank you!