Edge universality of sparse Erdős-Rényi directed graphs

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Undirected Erdős-Rényi graphs

$\widehat{\mathcal{G}}(N,p)$: N vertexes, every edge is connected with probability p completely independent of other edges.

Denote by $\widehat{\mathcal{A}}_{ij} = \mathbf{1}_{i\sim j}$ the adjacency matrices of the Erdős-Rényi graphs.

We consider the normalized matrix and its centering by

$$\widehat{A} := \frac{1}{\sqrt{Np(1-p)}}\widehat{\mathcal{A}}, \quad \widehat{H} := \widehat{A} - \sqrt{\frac{Np}{1-p}}\,|\mathbf{e}\rangle\langle\mathbf{e}|$$

so that $\mathrm{Var}(\widehat{A}_{ij}) = \mathbb{E}\widehat{H}_{ij}^2 = N^{-1}$.

The largest eigenvalue of $\widehat{A}$ satisfies $\widehat{\lambda}_1 \approx \sqrt{\frac{Np}{1-p}} \gg 1$. The key question is to understand the second largest eigenvalue $\widehat{\lambda}_2$.

Edge statistics

- [Erdős-Knowles-Yau-Yin, 2011], [Lee-Schnelli, 2016]
  When $p \gg N^{-2/3}$, the extreme eigenvalues of A satisfy Tracy-Widom distribution
  $$N^{2/3}\big(\widehat{\lambda}_2 - \mathbb{E}\widehat{\lambda}_2\big) \xrightarrow{d} \mathrm{TW}_1\,.$$

- [Huang-Landon-Yau, 2017], [H.-Knowles 2020]
  When $N^{-1+\varepsilon} \leqslant p \ll N^{-2/3}$, one has
  $$\frac{\widehat{\lambda}_2 - \mathbb{E}\widehat{\lambda}_2}{\sqrt{2\mathbb{E}\widehat{H}_{12}^4}} \xrightarrow{d} \mathcal{N}(0,1)\,.$$

  The sparse contribution is on the scale
  $$\sqrt{\mathbb{E}\widehat{H}_{12}^4} \asymp \frac{1}{N\sqrt{p}} \gg N^{-2/3}$$
  for $p \ll N^{-2/3}$, and it origins from
  $$\frac{1}{N}\big(\operatorname{Tr}\widehat{H}^2 - N\big) \propto \operatorname{Tr}\widehat{\mathcal{A}}^2 = \sum_{i,j}\widehat{\mathcal{A}}_{ij}\widehat{\mathcal{A}}_{ji} = \sum_{i,j}\widehat{\mathcal{A}}_{ij} = \sum_i d_i\,.$$

Edge universality

[Jaehun Lee, 2021] [Huang-Yau, 2022] For $N^{-1+\varepsilon} \leqslant p \leqslant 1/2$, we have

$$N^{2/3}\big(\widehat{\lambda}_2 - \mathbb{E}\widehat{\lambda}_2 - \mathcal{X}\big) \xrightarrow{d} \mathrm{TW}_1 \,.$$

Here

$$\mathbb{E}\widehat{\lambda}_2 = 2 + (Np)^{-1} - \frac{5}{4}(Np)^{-2} + \dots \,.$$

The first two orders of random term $\mathcal{X}$ are

$$\frac{1}{N}\Big(\sum_{ij} \widehat{H}_{ij}^2 - N\Big)$$

and

$$\frac{1}{N}\Big(\sum_{ij} \widehat{H}_{ij}^4 - \frac{1}{Np^2}\Big) + \frac{2}{N}\sum_{ijk} \widehat{H}_{ij}^2\Big(\widehat{H}_{ik}^2 - \frac{1}{N}\Big) \,.$$

The third order terms of $\mathcal{X}$ contain e.g. the fluctuation of

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$$\frac{1}{N}\Big(\sum_{ij}\widehat{H}_{ij}^2 - N\Big) \propto \sum_i d_i$$

and

$$\frac{1}{N}\Big(\sum_{ij}\widehat{H}_{ij}^4 - \frac{1}{Np^2}\Big) + \frac{2}{N}\sum_{ijk}\widehat{H}_{ij}^2\Big(\widehat{H}_{ik}^2 - \frac{1}{N}\Big) \propto \sum_i d_i + \sum_i d_i^2\,.$$

The third order terms of $\mathcal{X}$ contain e.g. the fluctuation of

$$\sum_{ijkl}\widehat{H}_{ij}\widehat{H}_{jk}\widehat{H}_{kl} \propto \sum_{jk} d_j\widehat{\mathcal{A}}_{jk}d_k\,.$$

Random regular graphs

d-regular graph: (undirected, simple) graphs on $\{1, \ldots, N\}$, each vertex has d neighbours.

Random d-regular graph: uniform probability measure on the set of d-regular graphs. Adjacency matrix $\widetilde{\mathcal{A}}_{ij} := \mathbf{1}_{i\sim j}$.

The second largest eigenvalue of $\widetilde{\mathcal{A}}$ satisfies

$$N^{2/3}\left(\frac{\widetilde{\lambda}_2 + d/N}{\sqrt{(d-1)(N-d)/N}} - 2\right) \xrightarrow{d} \mathrm{TW}_1$$

- For $N^{2/9} \ll d \ll N^{1/3}$ in [Bauerschmidt-Huang-Knowles-Yau, 2019]
- For $N^{\varepsilon} \leqslant d \ll N^{1/3}$ in [Huang-Yau, 2023]
- For $d \gg N^{2/3}$ in [H., 2022].

Erdős-Rényi digraphs

$\mathcal{G}(N,p)$: N vertexes, every directed edge (i,j) is connected with probability p completely independent of other edges.

The adjacency matrix $\mathcal{A}$ has i.i.d. entries with

$$\mathcal{A}_{ij} = \begin{cases} 1 & \text{with probability } \; p \\ 0 & \text{with probability } \; 1-p\,. \end{cases}$$

We normalize

$$A = \frac{\mathcal{A}}{\sqrt{Np(1-p)}}\,.$$

Eigenvalues of A satisfies (global) circular law:

$$\frac{1}{N}\sum_i \delta_{\lambda_i} \longrightarrow \frac{1}{\pi}\mathbf{1}_{\{z\in\mathbb{C}:|z|\leqslant 1\}}$$

as $N \to \infty$.

Local eigenvalue statistics

For dense non-Hermitian matrices, we have

- [Tao-Vu, 2012]: Under four-moment matching condition, we have universality of the local statistics in the bulk and near the edge.

Under two-moment matching

- [Bourgade-Yau-Yin, 2012]: Circular law on all mesoscopic scale.
- [Alt-Erdős-Krüger, 2016]: Spectral radius location.
- [Cipolloni-Erdős-Schröder, 2019]: Universality near the edge.
- [Cipolloni-Erdős-Xu, 2023]: Spectral radius distribution.
- [Maltsev-Osman, 2023; Dubova-Yang 2024]: Bulk universality.

There is no previous results on the local statistics of sparse non-Hermitian matrices.

Main results

For a square matrix $S \in \mathbb{C}^{N\times N}$ with eigenvalues $\lambda_1^S, ..., \lambda_N^S$, we define its k-point correlation function p_k^S through

$$\int_{\mathbb{C}^k} F(z_1, ..., z_k) p_k^S(z_1, ..., z_k) \mathrm{d}z_1 \dots \mathrm{d}z_k = \binom{N}{k}^{-1} \mathbb{E} \sum_{i_1, \dots, i_k = 1}^{N} {}^{*} F(\lambda_{i_1}^S, ..., \lambda_{i_k}^S) \, .$$

[H. 2023] (Edge Universality)
Fix $k \in \mathbb{N}_+$ and let $w_1, ..., w_k \in \mathbb{T} = \{w \in \mathbb{C}, |w| = 1\}$. For $N^{-1+\varepsilon} \leqslant p \leqslant 1/2$

$$\lim_{N\to\infty} \int_{\mathbb{C}^k} F(z_1, ..., z_k) \Big[p_k^A\Big(w_1 + \frac{z_1}{N^{1/2}}, ..., w_k + \frac{z_k}{N^{1/2}}\Big) - p_k^W\Big(w_1 + \frac{z_1}{N^{1/2}}, ..., w_k + \frac{z_k}{N^{1/2}}\Big)\Big] \mathrm{d}z_1 ... \mathrm{d}z_k = 0$$

Here $W \in \mathbb{R}^{N\times N}$ is the real Ginibre ensemble, i.e. it has i.i.d. Gaussian entries with mean 0 and variance $1/N$.

Main results (Cont)

[H. 2023] Assume $N^{-1+\varepsilon} \leqslant p \leqslant 1/2$ and let $\lambda_1, \lambda_2, ..., \lambda_N$ be the eigenvalues of A with $|\lambda_1| = \max_i |\lambda_i| \approx \sqrt{\frac{Np}{1-p}} \gg 1$.

1. (Spectral radius) We have

$$\max_{2 \leqslant i \leqslant N} |\lambda_i| = 1 + O(N^{-1/2+\varepsilon}) \tag{1}$$

   with very high probability.

2. (Delocalization) Suppose $\mathbf{u} \in \mathbb{C}^N$ satisfies $A\mathbf{u} = \lambda \mathbf{u}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 2$. Then

$$\|\mathbf{u}\|_\infty = O(N^{-1/2+\varepsilon}\|\mathbf{u}\|)$$

   with very high probability.

Observation. No phase transition! The edge statistics of non-Hermitian random matrices are more robust than those of Hermitian random matrices.

Simple explanation: the sparse contribution is on the scale

$$\sqrt{\mathbb{E}\widehat{H}_{12}^4} \asymp \sqrt{\mathbb{E}(A_{12} - \mathbb{E}A_{12})^4} \asymp \frac{1}{N\sqrt{p}}$$

for both cases, while Tracy-Widom is on scale $N^{-2/3}$, and the extreme eigenvalues of A fluctuates on scale $N^{-1/2}$.

Detailed explanation: proof outline.

Proof outline

Girko's Hermitization

$$\frac{1}{N}\sum_i f(\sqrt{N}(\lambda_i - w_*)) = \frac{\mathrm{i}}{4\pi N}\int_{\mathbb{C}}\int_0^\infty \nabla_w^2 f(\sqrt{N}(w-w_*))\,\mathrm{Tr}(H_w - \mathrm{i}\eta)^{-1}\mathrm{d}\eta\,\mathrm{d}^2 w$$

with $|w_*| = 1$, and

$$H_w := \begin{pmatrix} 0 & A - w \\ A^* - \overline{w} & 0 \end{pmatrix} \in \mathbb{C}^{2N\times 2N}\,.$$

The key is to study $g(w, \mathrm{i}\eta) := \frac{1}{2N}\,\mathrm{Tr}(H_w - \mathrm{i}\eta)^{-1}$. It is close to a deterministic $m \equiv m(w, \mathrm{i}\eta)$, which solves

$$P(m) := m^3 + 2\mathrm{i}\eta m^2 + (1 - \eta^2 - |w|^2)m + \mathrm{i}\eta = 0\,.$$

Estimate g through

$$g - m = O\left(\frac{P(g)}{P'(m)}\right),$$

and near the edge it is greatly affected by the cusp singularity

$$P'(m) = 3m^2 + 4\mathrm{i}\eta m + (1 - \eta^2 - |w|^2) \asymp |1 - |w|| + \eta^{2/3}.$$

It origins from the smallness of m

$$m = \mathrm{i}\,\mathrm{Im}\, m = O\big(|1 - |w||^{1/2} + \eta^{1/3}\big).$$

This requires very precise estimates of $P(g)$.

Observation: cusp singularity is an advantage on estimating higher order terms in the sparse regime. For instance

$$\mathbb{E}P(g) = O\Big(\frac{\mathbb{E}g^3}{Np} + \frac{\mathbb{E}g^5}{N^2p^2} + \dots\Big)$$

Due to the smallness of m

$$\frac{\mathbb{E}g^3}{Np} \approx \frac{m^3}{Np} = O\bigg(\frac{|1-|w||^{3/2}+\eta}{Np}\bigg) = O\bigg(P'(m)\frac{|1-|w||^{1/2}+\eta^{1/3}}{Np}\bigg).$$

Sparsity does not affect the estimate!

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Sparsity does not affect the estimate!

In the Hermitian case, $\widehat{g} = N^{-1}\operatorname{Tr}(\widehat{A}-z)^{-1}$ is close to the solution of $1 + z\widehat{m} + \widehat{m}^2 = 0$, while

$$\mathbb{E}(1 + z\widehat{g} + \widehat{g}^2) = O\Big(\frac{\mathbb{E}\widehat{g}^4}{Np} + \frac{\mathbb{E}\widehat{g}^6}{N^2p^2} + \dots\Big)$$

and

$$\frac{\mathbb{E}\widehat{g}^4}{Np} \approx \frac{\widehat{m}^4}{Np} = O\Big(\frac{1}{Np}\Big), \quad \text{as } |\widehat{m}| \approx 1!$$

We can prove that

$$|g - m| \lesssim \frac{1}{N\eta} + \frac{\eta^{1/3}}{Np} \qquad (*)$$

with very high probability. For $\eta \leqslant N^{-3/4}$, we get $|g - m| \lesssim \frac{1}{N\eta}$.

In Hermitization

$$\frac{1}{N} \sum_i f(\sqrt{N}(\lambda_i - w_*)) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \int_0^\infty \nabla_w^2 f(\sqrt{N}(w - w_*))\, g(w, \mathrm{i}\eta)\, \mathrm{d}\eta\, \mathrm{d}^2 w\,,$$

we still need to treat

$$\frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \int_{N^{-3/4}}^\infty \nabla_w^2 f(\sqrt{N}(w - w_*))\, g(w, \mathrm{i}\eta) \mathrm{d}\eta\, \mathrm{d}^2 w\,,$$

while (*) is unimprovable.

Idea: Integration by-parts for the shift.

$$\int_{\mathbb{C}} \int_{N^{-3/4}}^{\infty} \nabla_w^2 f(\sqrt{N}(w-w_*))\ g(w,\mathrm{i}\eta)\,\mathrm{d}\eta\mathrm{d}^2 w$$

$$= -4 \int_{\mathbb{C}} \int_{N^{-3/4}}^{\infty} \partial_{\bar{w}} f(\sqrt{N}(w-w_*))\ \partial_w g(w,\mathrm{i}\eta)\,\mathrm{d}\eta\mathrm{d}^2 w$$

$$= -\frac{2\mathrm{i}}{N} \int_{\mathbb{C}} \int_{N^{-3/4}}^{\infty} \partial_{\bar{w}} f(\sqrt{N}(w-w_*)) \sum_{i=1}^{N} (G^2)_{i+N,i}\,\mathrm{d}\eta\mathrm{d}^2 w \qquad G := (H_w - \mathrm{i}\eta)^{-1}$$

$$= \frac{2\mathrm{i}}{N} \int_{\mathbb{C}} \partial_{\bar{w}} f_{w_*} \sum_{i=1}^{N} G_{i+N,i}(\mathrm{i}N^{-3/4})\,\mathrm{d}^2 w\,.$$

Idea: Integration by-parts for the shift.

$$\int_{\mathbb{C}}\int_{N^{-3/4}}^{\infty} \nabla_w^2 f(\sqrt{N}(w-w_*))\; g(w,\mathrm{i}\eta)\,\mathrm{d}\eta\,\mathrm{d}^2 w$$
$$= -4\int_{\mathbb{C}}\int_{N^{-3/4}}^{\infty} \partial_{\bar w} f(\sqrt{N}(w-w_*))\; \partial_w g(w,\mathrm{i}\eta)\,\mathrm{d}\eta\,\mathrm{d}^2 w$$
$$= -\frac{2\mathrm{i}}{N}\int_{\mathbb{C}}\int_{N^{-3/4}}^{\infty} \partial_{\bar w} f(\sqrt{N}(w-w_*))\sum_{i=1}^{N}(G^2)_{i+N,i}\,\mathrm{d}\eta\,\mathrm{d}^2 w \qquad G := (H_w-\mathrm{i}\eta)^{-1}$$
$$= \frac{2\mathrm{i}}{N}\int_{\mathbb{C}} \partial_{\bar w} f_{w_*}\sum_{i=1}^{N} G_{i+N,i}(\mathrm{i}N^{-3/4})\,\mathrm{d}^2 w\,.$$

From $\nabla_w^2 f(\sqrt{N}(w-w_*))g(w,\mathrm{i}\eta)$ to $\partial_{\bar w} f(\sqrt{N}(w-w_*))\frac{1}{N}\sum_{i=1}^N (G^2)_{i+N,i}$, the (naive) scale change is

$$\frac{1}{\sqrt{N}}\times\frac{1}{\eta} = N^{1/4} \quad \text{for } \eta = N^{-3/4}.$$

However, we can show that

$$\frac{1}{N}\sum_i G_{i+N,i} + \frac{1+m^2}{w} \lesssim \frac{1}{N^2\eta^2} \; \left(\text{instead of } g-m \lesssim \frac{1}{N\eta}\right).$$

Hence the actual scale change is

$$\frac{1}{\sqrt{N}}\times\frac{1}{\eta}\times\frac{1}{N\eta} = 1 \quad \text{for } \eta = N^{-3/4}.$$

Thank you!