Phase transition for the smallest eigenvalue of covariance matrices

Zhigang Bao

HKUST

Random Matrices and Related Topics

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Joint with **Jaehun Lee** and **Xiaocong Xu**

Sample covariance matrices

Sample covariance matrices

$$\mathcal{S}(Y) = YY^* = \frac{1}{N} XX^*, \qquad X = (x_{ij})_{M,N}, \qquad Y = \frac{1}{\sqrt{N}} X$$

(i) independence: x_{ij}'s are iid,

(ii) moment condition: $\mathbb{E} x_{ij} = 0$, $\mathbb{E}|x_{ij}|^2 = 1$

(iii) dimension: $M \equiv M(N)$, $c_N \equiv M/N \to c_\infty \in (0, \infty)$.

Eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_M$

Empirical spectral distribution (ESD) $\mu_Y = \frac{1}{M} \sum_{i=1}^M \delta_{\lambda_i}$

Stieltjes transform (ST) $m_Y(z) = \int (x - z)^{-1} \mu_Y(\mathrm{d}x)$, $z \in \mathbb{C}^+$.

Marchenko-Pastur Law

MP Law [Marchenko-Pastur '67, Yin '86] Assume $\mathbb{E}|x_{ij}|^2 = 1$. The ESD μ_Y is weakly approximated by

$$\mu^{\mathrm{mp}}(\mathrm{d}x) = \frac{1}{2\pi c_N x}\sqrt{(\lambda_+^{\mathrm{mp}} - x)(x - \lambda_-^{\mathrm{mp}})_+}\mathrm{d}x + (1 - c_N^{-1})_+\delta_0(x),$$

$$\lambda_\pm^{\mathrm{mp}} = (1 \pm \sqrt{c_N})^2.$$

The ST $m_Y(z), z \in \mathbb{C}^+$ is approximated by

$$m^{\mathrm{mp}}(z) = \int \frac{1}{x - z}\mu^{\mathrm{mp}}(\mathrm{d}x) = \frac{1 - c_N - z + \mathrm{i}\sqrt{(\lambda_+^{\mathrm{mp}} - z)(z - \lambda_-^{\mathrm{mp}})}}{2zc_N}.$$

Square-root in terms of ST $(c_N \to c_\infty \neq 1)$

$$\Im m^{\mathrm{mp}}(\lambda_-^{\mathrm{mp}} + E + \mathrm{i}\eta) \sim \begin{cases} \frac{\eta}{\sqrt{|E| + \eta}}, & \text{if } -\lambda_-^{\mathrm{mp}}/2 < E \leqslant 0 \\ \sqrt{|E| + \eta}, & \text{if } 0 \leqslant E \leqslant 1 \end{cases}$$

Global Law: without second moment

Let $\mathcal{S}(X) = XX^*$. Suppose that there is some $\alpha \in (0,2)$, and some slowly varying function $L(\cdot)$, s.t. $\mathbb{P}(|x_{ij}| \geqslant s) = L(s)s^{-\alpha}$. Let

$$g_\alpha(y) := \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \exp(-t^{\frac{\alpha}{2}} y) \mathrm{d}t, \qquad h_\alpha(y) = 1 - \frac{\alpha}{2} y g_\alpha(y).$$

Theorem [Belinschi-Dembo-Guionnet '09] Let μ_X^α be the ESD of $\mathcal{S}(X)/a_{N+M}^2$ with

$$a_{N+M} = \inf\left\{s : \mathbb{P}(|x_{ij}| \geqslant s) \leqslant \frac{1}{N+M}\right\} \sim L_o(N) N^{1/\alpha}.$$

In case $c_N \in (0,1]$, μ_X^α converges weakly to μ^α with with density

$$\rho_\alpha(x) = -(\pi x)^{-1} \Im(h_\alpha(Y_1(\sqrt{x}))),$$

where $(Y_1(z), Y_2(z))$ satisfies with some positive constant C_α

$$z^\alpha Y_1(z) = C_\alpha g_\alpha(Y_2(z))/(1+c_\infty), \qquad z^\alpha Y_2(z) = c_\infty C_\alpha g_\alpha(Y_1(z))/(1+c_\infty).$$

Remark The density does not vanish in neighborhood of 0.

First order of extreme eigenvalues

Bai-Yin law [Yin-Bai-Krishnaiah '88, Bai-Yin '93] Assume $\mathbb{E}|x_{ij}|^4 < \infty$.

$$\lambda_1 \xrightarrow{a.s.} \lambda_+^{\mathrm{mp}}, \qquad \lambda_{M \wedge N} \xrightarrow{a.s.} \lambda_-^{\mathrm{mp}}.$$

Remark The 4-th moment condition is sufficient and necessary for the largest ev, but not necessary for the smallest ev.

Tikhomirov's law [Tikhomirov '15] Assume $\mathbb{E}|x_{ij}|^2 = 1$.

$$\lambda_{M \wedge N} \xrightarrow{a.s.} \lambda_-^{\mathrm{mp}}.$$

Second order of extreme ev: Light tail

Tracy-Widom law [Ding-Yang '18] Under the condition

$$\lim_{s\to\infty} s^4 \mathbb{P}(|x_{ij}| \geqslant s) \to 0, \qquad (*)$$

we have

$$\frac{M^{2/3}}{\sqrt{c_N}(1+\sqrt{c_N})^{4/3}}(\lambda_1 - \lambda_+^{\mathrm{mp}}) \Rightarrow \mathrm{TW}_1,$$

$$-\frac{M^{2/3}}{\sqrt{c_N}(1-\sqrt{c_N})^{4/3}}(\lambda_{M\wedge N} - \lambda_-^{\mathrm{mp}}) \Rightarrow \mathrm{TW}_1.$$

Remark 1 The condition $(*)$ is necessary for the largest ev, but not for the smallest one. The result is an extension of [Lee-Yin '14].

Remark 2 [Ding-Yang '18] derives the result under general population covariance assumption.

Remark 3 (Ref.) [Johansson '00], [Johnstone '01], [El-Karoui '07], [Pillai-Yin '14], [Bao-Pan-Zhou '15], [Lee-Schnelli '16], [Knowles-Yin '16], etc.

Second order of largest ev: Heavy tail

Fréchet law [Soshnikov '06], [Auffinger-Ben Arous-Péché '09] Suppose

$$\mathbb{P}(|x_{ij}| > s) = L(s)s^{-\alpha}, \quad 0 < \alpha < 4, \quad L(s) \text{ slowly varying}$$

and $\mathbb{E}x_{ij} = 0$ in case $2 \leqslant \alpha < 4$. Let

$$b_N = \inf\{s : \mathbb{P}(x_{ij} > s) \leqslant 1/(MN)\} \sim L_o(N)N^{2/\alpha}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}\Big(b_N^{-2}\lambda_1(\mathcal{S}(X)) \leqslant x\Big) = \exp\big(-x^{-\alpha/2}\big), \quad x > 0.$$

Remark 1 λ_1 is essentially given by the $\max_{ij}|x_{ij}|^2$.

Remark 2 In [Diaconu '23], the distribution in the case $\mathbb{P}(|x_{ij}| > s) \sim s^{-4}$ is derived for Wigner matrices.

Robustness of smallest ev: Heuristics ($c_N \leqslant 1$)

Largest eigenvalue

$$\lambda_1(\mathcal{S}(Y)) = \sup_{\mathbf{v} \in S^{M-1}} \|Y^* \mathbf{v}\|^2$$

$$\begin{bmatrix} \cdot & \cdot & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ * \\ \cdot \\ \cdot \end{bmatrix}$$

Smallest eigenvalue

$$\lambda_M(\mathcal{S}(Y)) = \inf_{\mathbf{v} \in S^{M-1}} \|Y^* \mathbf{v}\|^2$$

$$\begin{bmatrix} * & \cdot & * & \cdot & \cdot \\ \cdot & * & \cdot & \cdot & \cdot \\ \cdot & * & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ * \\ * \end{bmatrix}$$

Notations $*$: matrix entries $\gtrsim 1$, $*$: large vector entries

Heuristics:

- If $\alpha < 4$, typically there will be at least one $*$, such that σ_1 will be large

- If $\alpha > 2$, typically there will be only $o(N)$ $*$'s, such that σ_M won't be affected by these large entries too much.

Second order of smallest ev: Semi-heavy tail (I)

Assumptions

(i): x_{ij} is symmetrically distributed, with non-zero density at 0.

(ii): x_{ij} is mean 0 variance 1.

(iii): As $s \to \infty$,

$$\left| \mathbb{P}(x_{ij} > s) + \frac{\theta}{\Gamma(1-\alpha/2)} s^{-\alpha} \right| \lesssim s^{-(\alpha+\rho)}$$

for some $\alpha \in (2, 4)$, some constant $\theta > 0$ and some small $\rho > 0$.

(iv): For simplicity, we assume

$$c_N \equiv M/N \to c_\infty \in (0, 1).$$

Remark The case of $c_N = 1$ and $\alpha \in (0, 2)$ is considered in [Louvaris '22].

Second order of smallest ev: Semi-heavy tail (II)

Theorem [B.-Lee-Xu '23] Under the above assumptions, for $\alpha \in (2,4)$, there is a random variable $\mathcal{X}_\alpha$, such that the following statements hold with $\gamma_N = 1/(\sqrt{c_N}(1-\sqrt{c_N})^{4/3})$

(i): $-M^{\frac{2}{3}}\gamma_N\big(\lambda_M - \lambda_-^{\mathrm{mp}} - \mathcal{X}_\alpha\big) \Rightarrow \mathrm{TW}_1.$

(ii): $N^{\frac{\alpha}{4}}\big(\mathcal{X}_\alpha - \mathbb{E}\mathcal{X}_\alpha\big)/\sigma_\alpha \Rightarrow N(0,1), \quad \sigma_\alpha^2 = \frac{\theta c_N^{(4-\alpha)/4}(1-\sqrt{c_N})^{4(\alpha-2)}}{2}\Gamma\Big(\frac{\alpha}{2}+1\Big).$

(iii): $\mathbb{E}\mathcal{X}_\alpha = -N^{1-\frac{\alpha}{2}}\theta(1-\sqrt{c_N})^2 c_N^{(2-\alpha)/4}\Gamma\Big(\frac{\alpha}{2}+1\Big) + o(N^{1-\frac{\alpha}{2}}),$

(iv): If $\alpha = 8/3$, $-M^{\frac{2}{3}}\gamma_N\left(\lambda_M - \lambda_-^{\mathrm{mp}} - \mathbb{E}\mathcal{X}_\alpha\right) \Rightarrow \mathrm{TW}_1 + \mathcal{N}(0, \sigma^2_{\frac{8}{3},\infty}),$

with $\sigma^2_{\frac{8}{3},\infty} = \theta c_\infty^{\frac{2}{3}}(1-\sqrt{c_\infty})^{\frac{4}{3}}\Gamma\left(\frac{7}{3}\right)/3$, and $\mathrm{TW}_1 \perp\!\!\!\perp \mathcal{N}(0, \sigma_{\frac{8}{3},\infty})$.

Dynamic approach: initial condition

Gaussian divisible ensemble

$$H_t = V + \sqrt{t}\mathrm{GOE}, \qquad V \perp\!\!\!\perp \mathrm{GOE}, \qquad V \quad \text{diagonal}$$

η_***-regularity** Let $m_V(z)$ be Stieltjes transform of the spectrum of V. Let $\eta_* := N^{-\phi_*}$ with some $\phi_* \in (0, 2/3]$. We say V is η_*-regular if

1. square-root density on scale η_*

$$\Im[m_V(E + \mathrm{i}\eta)] \sim \frac{\eta}{\sqrt{|E| + \eta}}, \quad -1 \leqslant E \leqslant 0, \quad \eta_* \leqslant \eta \leqslant 10,$$
$$\Im[m_V(E + \mathrm{i}\eta)] \sim \sqrt{|E| + \eta}, \quad 0 \leqslant E \leqslant 1, \quad \eta_*^{1/2}\sqrt{|E|} + \eta_* \leqslant \eta \leqslant 10$$

2. lower bound on smallest ev $\min_i V_i \geqslant -\eta_*$

3. upper bound on largest ev We have $\|V\| \leqslant N^C$ for some $C > 0$.

Dynamic approach: edge local equilibrium

Let

$$m_{\mathsf{fc},t}(z) = \frac{1}{N}\sum_i \frac{1}{V_i - z - t m_{\mathsf{fc},t}(z)}, \qquad \rho_{fc,t}(E) = \frac{1}{\pi}\lim_{\eta\downarrow 0} \Im[m_{\mathsf{fc},t}(E + \mathrm{i}\eta)],$$

and further set

$$E_-(t) := \text{left edge of } \rho_{fc,t},$$

Edge Universality [Landon-Yau '17] Suppose that $N^{-\epsilon} \geqslant t \geqslant N^{\epsilon}\sqrt{\eta_*}$. For some mild test function F, we have

$$\left|\mathbb{E}F(\gamma_-(t)N^{2/3}(\lambda_N(H_t) - E_-(t))) - \mathbb{E}F(N^{2/3}(\lambda_N(\mathsf{GOE})) + 2)\right| \leqslant N^{-c}.$$

Related Ref. [Erdős-Schlein-Yau '11], [Landon-Yau '17], [Erdős-Schnelli '17], [Landon-Sosoe-Yau '19], [Adhikari-Huang '20], [Bourgade '21], [Aggarwal-Huang '23]

Dynamic approach for covariance matrices

(Rectangular) Gaussian divisible ensemble

$$\mathcal{S}(\mathsf{D}_t) = \mathsf{D}_t\mathsf{D}_t^* = \big(\mathsf{D} + \sqrt{t}W\big)\big(\mathsf{D} + \sqrt{t}W\big)^*, \quad \{W_{ij}\} \overset{\text{iid}}{\sim} N(0, 1/N), \quad \mathsf{D} \in \mathbb{R}^{M\times N}.$$

Edge Universality (informal) [Ding-Yang '22] If the spectrum of $\mathcal{S}(\mathsf{D}) = \mathsf{D}\mathsf{D}^*$ satisfies the $\eta_*(= N^{-\phi^*})$-regularity with some $\phi_* \in (0, 2/3]$, and $N^{-\epsilon} \geqslant t \geqslant N^{\epsilon}\sqrt{\eta_*}$, the edge eigenvalue follows the Tracy-Widom law, around D-dependent edge and with D-dependent scaling factor.

Aggarwal-Loppato-Yau's decomposition

Inspired by [Aggarwal-Loppato-Yau '21] Choose $0 < \epsilon_b < (\alpha - 2)/K_1\alpha$, and $0 < \epsilon_a < \min\{\epsilon_b, 4 - \alpha\}/K_2$. Define $0 - 1$ Bernoulli ψ_{ij}'s and χ_{ij}'s

$$\mathbb{P}[\psi_{ij} = 1] = \mathbb{P}[|y_{ij}| \geq N^{-\epsilon_b}], \quad \mathbb{P}[\chi_{ij} = 1] = \frac{\mathbb{P}[|y_{ij}| \in [N^{-1/2-\epsilon_a}, N^{-\epsilon_b}]]}{\mathbb{P}[|y_{ij}| < N^{-\epsilon_b}]}$$

Let a, b, and c be random variables such that

$$\mathbb{P}[a_{ij} \in I] = \frac{\mathbb{P}[y_{ij} \in (-N^{-1/2-\epsilon_a}, N^{-1/2-\epsilon_a}) \cap I]}{\mathbb{P}[|y_{ij}| \leq N^{-1/2-\epsilon_a}]},$$

$$\mathbb{P}[b_{ij} \in I] = \frac{\mathbb{P}[y_{ij} \in \big((-N^{-\epsilon_b}, -N^{-1/2-\epsilon_a}] \cup [N^{-1/2-\epsilon_a}, N^{-\epsilon_b})\big) \cap I]}{\mathbb{P}[|y_{ij}| \in [N^{-1/2-\epsilon_a}, N^{-\epsilon_b})]},$$

$$\mathbb{P}[c_{ij} \in I] = \frac{\mathbb{P}[y_{ij} \in \big((-\infty, -N^{-\epsilon_b}) \cup (N^{-\epsilon_b}, \infty)\big) \cap I]}{\mathbb{P}[|y_{ij}| \geq N^{-\epsilon_b}]}.$$

Suppose that the above r.v.s are all independent,

$$Y = \mathsf{A} \text{ (small) } + \mathsf{B} \text{ (medium) } + \mathsf{C} \text{ (large) } =: \mathsf{A} + \mathsf{D},$$

$$\mathsf{A}_{ij} = (1 - \psi_{ij})(1 - \chi_{ij})a_{ij}, \quad \mathsf{B}_{ij} = (1 - \psi_{ij})\chi_{ij}b_{ij}, \quad \mathsf{C}_{ij} = \psi_{ij}c_{ij}$$

Gaussian divisible model

Original model $Y = \mathsf{A} + \mathsf{D}$

Gaussian divisible model $Y_t = \sqrt{t}\mathsf{W} + \mathsf{D}$

$$t = N\mathbb{E}|\mathsf{A}_{ij}|^2 \sim N^{-2\epsilon_a}, \qquad \mathsf{W}_{ij} \sim N(0, N^{-1}), \qquad \mathsf{W} \perp\!\!\!\perp Y.$$

Proof route

—Step 1: prove the results for $\mathcal{S}(Y_t) = Y_t Y_t^*$ via a dynamic approach.

—Step 2: prove the results for $\mathcal{S}(Y) = YY^*$ via a Green function comparison.

Gaussian divisible model: minor argument

Deviate sets Let $\Psi = (\psi_{ij})$ and $[k] = \{1, \cdots, k\}$. We define the index sets

$$\mathcal{D}_r \equiv \mathcal{D}_r(\Psi) := \Big\{ i \in [M] : \sum_{j=1}^{N} \psi_{ij} \geqslant 1 \Big\}, \quad \mathcal{D}_c \equiv \mathcal{D}_c(\Psi) := \Big\{ j \in [N] : \sum_{i=1}^{M} \psi_{ij} \geqslant 1 \Big\}$$

Minors For any $M \times N$ matrix H

—Row minor $H^{(\mathcal{D}_r)}$: $\mathcal{D}_r$-rows removed

—Column minor $H^{[\mathcal{D}_c]}$: $\mathcal{D}_c$-columns removed

Cauchy interlacing Let $\mathcal{S}(H) = HH^*$

$$\lambda_M(\mathcal{S}(\mathsf{D}^{[\mathcal{D}_c]})) \leqslant \lambda_M(\mathcal{S}(\mathsf{D})) \leqslant \lambda_{M-|\mathcal{D}_r|}(\mathcal{S}(\mathsf{D}^{(\mathcal{D}_r)}))$$

implies

$$\lambda_M(\mathcal{S}(\mathsf{B}^{[\mathcal{D}_c]})) \leqslant \lambda_M(\mathcal{S}(\mathsf{D})) \leqslant \lambda_{M-|\mathcal{D}_r|}(\mathcal{S}(\mathsf{B}^{(\mathcal{D}_r)}))$$

Gaussian divisible model: mesoscopic regularity

Matrix with bounded support B, $\mathsf{B}^{[\mathcal{D}_c]}$, $\mathsf{B}^{(\mathcal{D}_r)}$, i.e., $|\mathsf{B}_{ij}| \leqslant N^{-\epsilon_b}$.

Theorem [Ding-Yang '18, Hwang-Lee-Schnelli '19] For covariance matrices with bounded support, for any $z \in \{E + \mathrm{i}\eta : C^{-1}\lambda_-^{\mathsf{mp}} \leqslant E \leqslant \lambda_+^{\mathsf{mp}} + 1, 0 < \eta < 3\}$,

$$|m_{\mathsf{B}}(z) - m_{\mathsf{mp}}^{(t)}(z)| \prec N^{-\epsilon_b} + (N\eta)^{-1},$$

where m_{B} is the Stieltjes transform of the spectrum of $\mathcal{S}(\mathsf{B})$ and $m_{\mathsf{mp}}^{(t)}(z) = (1-t)^{-1} m_{\mathsf{mp}}(z/(1-t))$. Further,

$$|\lambda_M(\mathcal{S}(\mathsf{B})) - (1-t)\lambda_-^{\mathsf{mp}}| \prec N^{-2\epsilon_b} + N^{-2/3}$$

Similar results hold for $\mathcal{S}(\mathsf{B}^{[\mathcal{D}_c]})$ and $\mathcal{S}(\mathsf{B}^{(\mathcal{D}_r)})$.

Consequence Together with interlacing, with high probability, we have

left edge regularity of $\mathcal{S}(\mathsf{D})$ on scale $\eta_* = N^{-\epsilon_b} \ll t^2$

Gaussian divisible model: Tracy-Widom part

Recall $Y_t = \sqrt{t}W + \mathsf{D}$. Let μ_{D} and m_{D} be the ESD and ST of $\mathcal{S}(D)$.

Rectangular free convolution Let

$$m_t(z) := (1 + c_N m_t(z)) m_{\mathsf{D}}(\zeta_t(z)),$$
$$\zeta_t(z) := (1 + c_N t m_t(z))^2 z - t(1 - c_N)(1 + c_N t m_t(z)),$$
$$\rho_t(E) := \frac{1}{\pi} \lim_{\eta \downarrow 0} \Im m_t(E + \mathrm{i}\eta).$$

First conditioning on D and applying the DBM mechanism driven by $\sqrt{t}W$,

$$\lim_{N\to\infty} \mathbb{E} f\Big[\gamma_N M^{2/3}\big(\lambda_M(\mathcal{S}(Y_t)) - \lambda_{-,t}\big)\Big] = \lim_{N\to\infty} \mathbb{E}\Big[f\big(M^{2/3}(\mu_M^{\mathsf{GOE}} + 2)\big)\Big],$$

where

$$\gamma_N = 1/(\sqrt{c_N}(1 - \sqrt{c_N})^{4/3}) + o_{\mathbb{P}}(1)$$

and

$$\lambda_{-,t} = \text{left edge of } \rho_t$$

Characterizing $\lambda_{-,t}$

Set

$$\Phi_t(\zeta) := (1 - c_N t m_{\mathsf{D}}(\zeta))^2 \zeta + (1 - c_N) t (1 - c_N t m_{\mathsf{D}}(\zeta)).$$

Let $\zeta_{-,t}$ be the smallest positive preimage of local extremas of $\Phi_t(x)$ on $\mathbb{R} \backslash \{0\}$.

$$\lambda_{-,t} = \Phi_t(\zeta_{-,t}) \Longleftrightarrow \zeta_{-,t} = \zeta_t(\lambda_{-,t}).$$

Estimate of $\zeta_{-,t}$

$$\zeta_{-,t} = (1-t)\lambda_-^{\mathsf{mp}} - \sqrt{c_N} t^2 + \text{smaller randomness}$$

Expansion of $\lambda_{-,t}$

$$\lambda_{-,t} = \lambda_{\mathsf{shift}} + \lambda_{\mathsf{fluct}} + o_{\mathbb{P}}(N^{-\frac{\alpha}{4}})$$

with

$$\lambda_{\mathsf{shift}} = (1 - c_N t \mathbb{E} m_{\mathsf{D}}(\zeta_e))^2 \zeta_e + (1 - c_N) t (1 - c_N t \mathbb{E} m_{\mathsf{D}}(\zeta_e))$$
$$\lambda_{\mathsf{fluct}} = 2 c_N \lambda_-^{\mathsf{mp}} t \big(m_{\mathsf{D}}(\zeta_e) - \mathbb{E} m_{\mathsf{D}}(\zeta_e) \big), \qquad \zeta_e := \mathbb{E} \zeta_{-,t}$$

Gaussian divisible model: Gaussian part

Claim $m_{\mathsf{D}}(\mathbb{E}\zeta_{-,t})$ is a linear eigenvalue statistics of $\mathcal{S}(\mathsf{D})$ on mesoscopic scale

$$|\lambda_M(\mathcal{S}(\mathsf{D})) - (1-t)\lambda_-^{\mathsf{mp}}| \prec N^{-2\epsilon_b} \ll (1-t)\lambda_-^{\mathsf{mp}} - \mathbb{E}\zeta_{-,t} \sim t^2.$$

Strategy for CLT (Martingale CLT) Similar to CLT of linear eigenvalue statistics for semi-heavy tailed on global scale [Benaych Georges-Maltsev '16], [Malysheva '23]

Mesoscopic CLT We have

$$N^{\frac{\alpha}{4}}\lambda_{\mathsf{fluct}}/\sigma_\alpha \Rightarrow N(0,1), \quad \sigma_\alpha^2 = \frac{\theta c_N^{(4-\alpha)/4}(1-\sqrt{c_N})^4(\alpha-2)}{2}\Gamma\Big(\frac{\alpha}{2}+1\Big)$$

and

$$\lambda_{\mathsf{shift}} = \lambda_-^{\mathsf{mp}} - N^{1-\frac{\alpha}{2}}\theta(1-\sqrt{c_N})^2 c_N^{(2-\alpha)/4}\Gamma\Big(\frac{\alpha}{2}+1\Big) + o(N^{1-\frac{\alpha}{2}})$$

Green function comparison

Task Compare

$$\mathbb{P}\Big(M^{2/3}(\lambda_M(\mathcal{S}(Y)) - \lambda_{-,t}) \leqslant s\Big) \quad \text{and} \quad \mathbb{P}\Big(M^{2/3}(\lambda_M(\mathcal{S}(Y_t)) - \lambda_{-,t}) \leqslant s\Big)$$

Green function comparison [Erdős-Yau-Yin '12] It suffices to compare

$$\mathbb{E}F\Big(N\int_{E_1}^{E_2} \mathrm{d}E\, \Im m_{Y_t}(\lambda_{-,t} + E + \mathrm{i}\eta)\Big) \quad \text{and} \quad \mathbb{E}F\Big(N\int_{E_1}^{E_2} \mathrm{d}E\, \Im m_{Y}(\lambda_{-,t} + E + \mathrm{i}\eta)\Big)$$

for $E_1 < E_2$, $|E_1|, |E_2| \leqslant N^{-\frac{2}{3}+\epsilon}$ and $\eta = N^{-\frac{2}{3}-\epsilon}$, where

$$m_{Y_*}(z) = M^{-1}\mathrm{Tr}G_{Y_*}(z) = M^{-1}\mathrm{Tr}(\mathcal{S}(Y_*) - z)^{-1}, \qquad * = t \text{ or null}$$

Let

$$Y^\gamma = \gamma \mathsf{A} + (1-\gamma^2)^{1/2} t^{1/2} W + \mathsf{D},$$
$$m^\gamma(z) = M^{-1}\mathrm{Tr}G^\gamma(z) = M^{-1}\mathrm{Tr}(\mathcal{S}(Y^\gamma) - z)^{-1}$$

Green function comparison

It suffices to bound

$$\mathrm{d}\mathbb{E}F\left(N\int_{E_1}^{E_2}\Im m^{\gamma}(z_{-,t})\mathrm{d}E\right)/\mathrm{d}\gamma, \qquad z_{-,t}:=\lambda_{-,t}+E+\mathrm{i}\eta$$

Typical form of terms

$$\sum_{ij}\mathbb{E}\Big[\big(\mathsf{A}_{ij}-\frac{\gamma t^{1/2}}{(1-\gamma^2)^{1/2}}W_{ij}\big)f_{ij}(Y^{\gamma})\Big]$$

where $f_{ij}(Y^{\gamma})$ is a function of Y^{γ} and G^{γ} entries.

Rough idea Split off A_{ij} and W_{ij} from $f_{ij}(Y^{\gamma})$, resulting in an expansion, and use moment matching to gain cancellation.

Major difficulty A_{ij} is dependent of D_{ij}; boundedness of G^{γ}_{ab}; random $\lambda_{-,t}$, etc.

THANK YOU!