Sharp tail estimates for largest eigenvalues in β ensembles and some applications to last passage percolation

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Random matrices and related topics in Jeju
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A classical result

- Let S_n denote a simple random walk, then by CLT $n^{-1/2}S_n \Rightarrow N(0,1)$ and

$$\limsup \frac{S_n}{\sqrt{2n\log\log n}} = 1; \quad \liminf \frac{S_n}{\sqrt{2n\log\log n}} = -1$$

almost surely. Khinchine (1924)

- The classical proof of this relies on showing that

$$\mathbb{P}(S_n \geq t\sqrt{n}) = \exp(-\frac{t^2}{2}(1+o(1)))$$

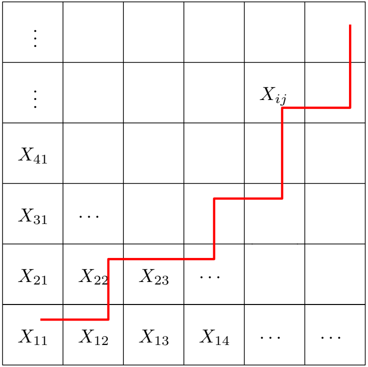
and there are $\log n$ many independent scales up to time n (i.e., the events $\{S_n \geq (1+\varepsilon)\sqrt{2n\log\log n}\}$ are approximately independent for $n = \lambda^k$.)

A similar question

- One can ask similar questions for other coupled sequence of random variables converging to a different universal law, e.g. GUE /GOE/GSE Tracy-Widom distribution.
- Consider a sequence of random variables S_n in a common probability space that converge to the Tracy-Widom distribution.
- Find $g_{\pm}(n)$ such that $\limsup S_n/g_+(n)$ and $\liminf S_n/g_-(n)$ has non-trivial constant limits.
- Inspired by a work of Paquette and Zeitounni (2017) which established a law of fractional logarithm for largest eigenvalues in GUE minor process, Ledoux asked the following question.

Exponential LPP on $\mathbb{Z}^2$

- For an up/right path γ define $T(\gamma) = \sum_{v \in \gamma} X_v$.
- Define $T_{m,n} := \max_{\gamma:(1,1)\to(m,n)} T(\gamma)$.
- Canonical exactly solvable model in the KPZ universality class.



$X_{ij} \sim$ i.i.d. $\mathrm{Exp}(1)$

Exponential LPP and Laguerre β ensemble

- For $n \in \mathbb{N}$ and $m > n-1$ and $\beta > 0$, the Laguerre β ensemble ($L\beta E_{n,m}$) is given by the joint density

$$\frac{1}{Z_{n,m,\beta}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_i \lambda^{\frac{\beta}{2}(m-n+1)-1} e^{-\beta\lambda_i/2}.$$

- The cases $\beta = 1, 2, 4$ are special.
- Johansson (1999) showed the remarkable distributional identity

$$T_{m,n} \stackrel{d}{=} \lambda_1(LUE_{n,m}),$$

  the largest eigenvalue of Laguerre Unitary ensemble ($\beta = 2$).
- For $\beta = 1$ and $\beta = 4$, similar correspondence for point-to-line exponential LPP, and half space exponential LPP (with $X_{i,j} = 0$ if $j > i$) with the largest eigenvalues of LOE and LSE respectively.
- Implicit in works of Baik-Rains (2000,2001).

Ledoux's question for exponential LPP

- It is known (Johansson (1999)) that

$$\mathcal{T}_n := \frac{T_{n,n} - 4n}{2^{4/3} n^{1/3}} \Rightarrow TW_2,$$

the GUE Tracy-Widom distribution.

- Question: Does there exist $g_+(n), g_-(n)$ and $c_1, c_2 \in (0, \infty)$ such that

$$\limsup \frac{\mathcal{T}_n}{g_+(n)} = c_1; \quad \liminf \frac{\mathcal{T}_n}{g_-(n)} = -c_2$$

almost surely?

- By analogy to classical LIL, to answer this one needs to understand (a) the tails of $\mathcal{T}_n$ and (b) the number of "independent" scales.

Tail estimates in the limit: general β

- Ramirez-Rider-Virag (2011) showed

$$\frac{\lambda_1(L\beta E_{n,n}) - 4n}{2^{4/3}n^{1/3}} \Rightarrow TW_\beta.$$

- They also established the tails of Tracy-Widom β distribution.

Theorem (Ramirez-Rider-Virag (2011), Dumaz-Virag (2013))

$$\mathbb{P}(TW_\beta \geq a) = \exp(-\frac{2\beta}{3}(1+o(1))a^{3/2}).$$

$$\mathbb{P}(TW_\beta \leq -a) = \exp(-\frac{\beta}{24}(1+o(1))a^{3}).$$

- To answer Ledoux's question, we need the pre-limiting versions of these estimates.

Tail estimates at the matrix level: general β

- Estimates without the correct constants in front of the exponent were known.

For $\beta \geq 1$ and $1 \ll a \ll n^{2/3}$

$$\exp(-c'a^{3/2}) \leq \mathbb{P}(\lambda_1(L\beta E_{n,n}) \geq 4n + an^{1/3}) \leq \exp(-ca^{3/2}).$$

$$\exp(-c'a^{3}) \leq \mathbb{P}(\lambda_1(L\beta E_{n,n}) \leq 4n - an^{1/3}) \leq \exp(-ca^{3}).$$

Ledoux-Rider (2010)

B.-Ganguly-Hegde-Krishnapur (2021)

- Similar results are known for $G\beta E_n$ and $L\beta E_{n,m}$ for m/n bounded.

Tail estimates for the matrix level: $\beta = 1, 2$

- For the special case $\beta = 2$ (which is determinantal) upper tail is well understood by analysing a Fredholm determinant formula cf. Borodin-Ferrari-Sasamoto (2008).
- Right tail for GOE_n, GUE_n has also been dealt with in Paquette-Zeitouni (2017), Erdős-Xu (2023).
- Lower tail is in general more difficult.
- Riemann-Hilbert analysis has been used in a few related problems (Geometric and Poissonian LPP): Lowe-Merkl (2001), Lowe-Merkl-Rolles (2002), Baik-Deift-McLaughlin-Miller-Zhou (2001).

Back to the LIL problem

- Recall
$$\mathcal{T}_n := \frac{T_{n,n} - 4n}{2^{4/3} n^{1/3}} \Rightarrow TW_2.$$
- Question: Does there exist $g_+(n), g_-(n)$ and $c_1, c_2 \in (0, \infty)$ such that
$$\limsup \frac{\mathcal{T}_n}{g_+(n)} = c_1; \quad \liminf \frac{\mathcal{T}_n}{g_-(n)} = -c_2$$
almost surely?
- Using the non-sharp moderate deviation results one can show
$$g_+(n) := (\log\log n)^{2/3}, g_-(n) := (\log\log n)^{1/3}$$

Ledoux (2018)

B.-Ganguly-Hegde-Krishnapur (2021)

- Constants were obtained for KPZ equation and KPZ fixed point: Das-Ghosal (2023), Das-Ghosal-Lin (2022).

Our results: moderate deviations

Theorem (Baslingker, B., Bhattacharjee, Krishnapur (2024+))

For $\beta \geq 1$, $1 \ll a \ll n^{\delta}$ and n large

$$\mathbb{P}\left(\frac{\lambda_1(L\beta E_{n,n}) - 4n}{2^{4/3}n^{1/3}} \geq a\right) = \exp(-\frac{2\beta}{3}(1+o(1))a^{3/2}).$$

$$\mathbb{P}\left(\frac{\lambda_1(L\beta E_{n,n}) - 4n}{2^{4/3}n^{1/3}} \leq -a\right) = \exp(-\frac{\beta}{24}(1+o(1))a^{3}).$$

- Similar result holds for $\lambda_1(L\beta E_{m,n})$ for $\frac{m}{n}$ bounded.
- Similar result holds for $G\beta E_n$.
- Proofs use the tridiagonal matrix model for β ensembles: Dumitriu-Edelman (2002), ideas from Ramirez-Rider-Virag (2011).

Our results: LIL for exponential LPP

Theorem (Baslingker, B., Bhattacharjee, Krishnapur (2024+))

Almost surely,

$$\limsup \frac{\mathcal{T}_n}{(\log\log n)^{2/3}} = (\frac{3}{4})^{2/3}; \quad \liminf \frac{\mathcal{T}_n}{(\log\log n)^{1/3}} = -(12)^{1/3}.$$

- Similar results for point-to-line, and half space (point-to-point) LPP.
- Using the sharp tail estimates, proof of the limsup result is almost same as the proof of classical LIL.
- The lower tail is substantially more complicated: old and new results about understanding the geometry of geodesics in LPP.

Other applications of sharp deviation estimates

- Exact rate for decay of correlations in the Airy_1 process: B.-Busani-Ferrari (2022). This only required the upper tail upper bound in $\beta = 1$ and $\beta = 2$ cases which were available before.
- Limit theorems for maxima and minima of Airy_1 and Airy_2 processes, extending a result of Pu (2023) for maxima of Airy_1 process: B.-Bhattacharjee (2024+).
- Endpoint distribution in point-to-line geodesic (or midpoint distribution in point-to-point geodesic) in exponential LPP: Agarwal-B. (2024+). Stronger results in directed landscape by Das-Dauvergne-Virag (2024+).

Thank You

Questions?