Random band matrices and Anderson delocalization

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Anderson Model (In Lattice)

$$H = -\Delta + \lambda V, \qquad on \quad \mathbb{Z}^d$$

where $-\Delta$ is Laplacian, V is the random multiplication operator, and $\lambda \in \mathbb{R}$.

$$(\Delta\psi)(x) = \sum_{\|y-x\|=1} (\psi(y) - \psi(x))$$

$$(V\psi)(x) = V(x)\psi(x), \quad x \in \mathbb{Z}^d, \quad \psi \in L^2(\mathbb{Z}^d)$$

The matrix forms

$$\Delta \sim \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2 \end{bmatrix}, \quad V \sim \begin{bmatrix} V(1) & 0 & 0 & \cdots & 0 \\ 0 & V(2) & 0 & \cdots & 0 \\ 0 & 0 & V(3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V(n) \end{bmatrix}$$

Anderson Model (In Lattice)

The matrix forms

$$\Delta \sim \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2 \end{bmatrix}, \quad V \sim \begin{bmatrix} V(1) & 0 & 0 & \cdots & 0 \\ 0 & V(2) & 0 & \cdots & 0 \\ 0 & 0 & V(3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V(n) \end{bmatrix}$$

Eigenvectors

For Δ

$$\phi_k : \ \phi_k(x) = e^{ik\cdot x}, \qquad \text{extended}$$

For V

$$\phi_{x_0} : \ \phi_{x_0}(x) = \delta_{x x_0}, \qquad \text{localized}$$

How about $H = -\Delta + \lambda V$?

$$extended \iff conductor$$

$$localized \iff insulator$$

Anderson Conjecture (In Lattice)

$$H = -\Delta + \lambda V, \quad \lambda \geq 0$$

Here $V(x)$ are *i.i.d.*, $x \in \mathbb{Z}^d$. $\mathbb{E}V(x)^2 = 1$.

$d = 1, 2$

Eigenvectors are always localized.

$$\ell \ : \ \text{localization length of } \phi < \infty$$

$$\ell \sim \lambda^{-2}, \qquad d = 1, \qquad \ell \sim e^{\lambda^{-2}}, \qquad d = 2$$

For example: the function $e^{-\frac{|x|^2}{\lambda^2} + ik\cdot x}$ has localization length $\sim \lambda$.

$d \geq 3$

There is a phase transition:

$$\ell \sim +\infty \ , \quad \text{iff} \quad \lambda < \lambda_c$$

The localization length can be infinity, if randomness is small enough.

Anderson's conjecture

Eigenvalues

GOE statistics in the delocalization regime.

Poisson statistics in the localization regime

(Frohlich-Spencer, Minami, Aizenman-Malchonov, Bourgain-Kenig, Ding-Smart, ...)

Summary:

Whether the system is extended or localized depends on one parameter, i.e. λ in $H = -\Delta + \lambda \cdot V$, so as the local statistics.

So far, most of the results focused on the localization region.

Random Band Matrix

There is a similar conjecture on random band matrices.

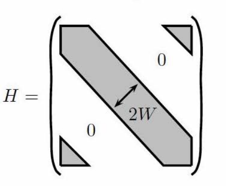
Matrix in d- dimension

The matrix (operator) H has entires H_{xy}: $x, y \in \mathbb{Z}_L^d$,
(e.g. Large box in $\mathbb{Z}^d$ with periodic boundary condition).

Random band matrices in d-dimension

$H = (H_{xy})$, with centered, independent entries up to symmetry $H = H^\dagger$, with band width $W \ll L$:

$$H_{xy} = 0, \quad |x-y| > W; \quad \mathbb{E}|H_{xy}|^2 \sim W^{-d}, \quad |x-y| \le W.$$



Band matrix conjecture

Eigenvector

$d = 1$:

In the bulk, the localization length ℓ (of the eigenvector) is $\sim W^2$.

Conjectured by Casati-Molinari-Izrailev '90; Feingold-Leitner-Wilkinson '91 and Fyodorov-Mirlin, 1991

$d \geq 3$:

A phase transition occurs.

In the bulk, Localization, $W \leq W_c$; Delocalization, $W \geq W_c$ (for any L).
The localization length $\ell = \infty$ if $W \geq W_c$.

Eigenvalue

Localization + Poisson *v.s.* Delocalization + GUE

It is the **same** conjecture as the one for Anderson's model.

Main results

$d = 1$, conjecture: $\ell \sim W^2$ in bulk

$\ell \leq W^{4+\epsilon}$: Cipolloni, Peled, Schenker, Shapiro (2022) Chen and Smart (2022)

$\ell \geq W^{4/3-\epsilon}$: Bourgade, Yau, Yang and Y. (2018)

Edge $L \sim W^{6/5}$: Sodin (2008) on the extreme eigenvalue distribution.

Supersymmetry $\ell \geq W^{2-\epsilon}$: Gaussian entries and $\mathbb{E}|H_{xy}|^2 = (W^2\Delta - 1)^{-1}_{xy}$.
Shcherbina and Shcherbina 2017, 2019, Disertori, Lohmann, Sodin 2018

$W \gg L^{3/4} \implies$ GUE/GOE: Bourgade, Yau, Yang and Y. (2018)

$W \ll L^{1/7} \implies$ Poisson: Hislop, Krishna (2021)

Delocalization (QUE)

Theorem (Yang, Yau, Y' (2021) Xu, Yang, Yau, Y' (2022))

For any $d \geq 7$ and fixed large $n > 0$, $\ell \geq W^n$ for large enough W.

More precisely if $L \leq W^n$, then

All bulk eigenvectors are delocalized.

Furthermore, they are delocalized in the sense of QUE:

$$|u_k(\cdot)|^2 * \chi \to 1/L^d$$

here $\chi(x) = (2L')^{-d}\mathbf{1}(|x| \leq L')$, $L' = L^{\frac{5}{d-2}-\epsilon}$. (local average).

The previous best result is $\ell \geq W^{d/2}$ (Yang, Y. 2017)

Xu, Yang, Yau, Y. (2022)

Suppose $W \geq L^a$, $a > 0$, $d \geq 150/a$, GUE

Quantum diffusion

Resolvent

$$G = (H - z)^{-1}, \quad z \in \mathbb{C}$$

It studies the spectrum of H at $E := Re\ z$ in the scale of $\eta := Im\ z$.

Theorem (Quantum diffusion of the resolvents)

$$|G_{xy}|^2 \quad \sim \quad \frac{1}{W^2|x-y|^{d-2}}$$

It is called diffusion since:

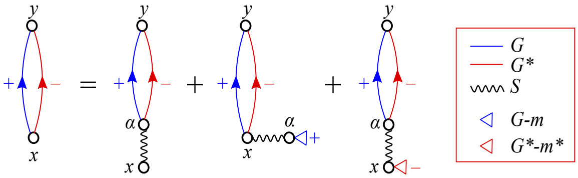
$$\sum_x \mathbb{E}\,|G_{0x}|^2\, e^{\mathrm{i}p\cdot x} \sim \frac{1}{\eta + W^2\left(p\cdot \mathcal{D}_{eff}\cdot p\right)},$$

It was predicted in physics literature (see Spencer's review book)

About the proof.

* The whole proof has 200+ pages. It relies a series of very complicated graphical expansions. The number of the graphs required for proving $\ell \gg W^n$ is $O(n^{O(n)})$.

About expansion, here is one example of basic local expansion is:



$$m : m + 1/m + z = 0, \quad S_{xy} = \mathbb{E}|H_{xy}|^2$$

About the proof.

* Sum zero property

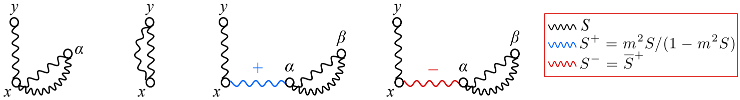
The graphs don't cancel each other in the expanding.

$$\sum_{\mu} \Gamma^{(\mu)} \neq 0$$

The main questions/difficulties of the expansion:
What are the sizes of these graphs? Will the sizes of the new graphs always be less or equal to the initial graphs?

Fortunately, we found that some subgraphs satisfy the sum zero property.

$$\sum_{x} \sum_{\mu} \Gamma^{(\mu)}_{xy} \approx 0$$



$$(\mathcal{E}_4)_{xy} = (|m|^4 + 2\Re(m^4)) S_{xy} \sum_{\alpha} S^2_{x\alpha} + 2\Re(m^2) S^2_{xy} + m^4 S_{xy} \sum_{\alpha,\beta} S^{+}_{x\alpha} S^2_{\alpha\beta} + m^4 S_{xy} \sum_{\alpha,\beta} S^{-}_{x\alpha} S^2_{\alpha\beta}$$

$$\sum_{x} (\mathcal{E}_4)_{xy} = \Re\left(|m|^4 + 2m^2/(1-m^2)\right) \sum_{x} S^2_{xy} = O(\eta W^{-d})$$

About the proof.

* **Sum zero property** is the key property of our proof. With this property, the leading terms of many estimates would disappear.

$$\sum_x \Theta_{0x} \sum_\mu \Gamma_{xy}^{(\mu)} = \left(\Theta_{0y} + \nabla\Theta_{0y} \cdot (x-y) + \cdots\right) \sum_x \sum_\mu \Gamma_{xy}^{(\mu)}$$
$$= \Theta_{0y} \sum_x \sum_\mu \Gamma_{xy}^{(\mu)} + \nabla\Theta_{0y} \cdot (x-y) \sum_x \sum_\mu \Gamma_{xy}^{(\mu)} + \cdots$$

Usually the 2nd term would also disappear due to the symmetry.

We used a ”renormalization” method in our proof to obtain this property.

Basic idea

1. Sum zero property relies on the structure of graphs, not parameters.
2. For different Ls and Ws, we use the same expansion (graphs).

About the proof.

*__Doubly connected property__.

To make the expansion effective, we need another important property, i.e, so called doubly connected property. Roughly speaking in graph Γ, for any two vertices x and y, there exist two edge-paths connecting x and y, which do not have common edges. It will imply a key estimate:

$$\Gamma_{xy} \ll |x-y|^{-d}$$

Together with sum zero property, one can obtain the correct sizes of the graphs.

$$\sum_x \Theta_{0x} \sum_\mu \Gamma^{(\mu)}_{xy}$$
$$= \Theta_{0y} \sum_x \sum_\mu \Gamma^{(\mu)}_{xy} + \nabla\Theta_{0y} \cdot (x-y) \sum_\mu \Gamma^{(\mu)}_{xy} + \cdots$$

Note: we proved doubly connected property directly, but we can not prove the sum zero property directly.

Band Anderson model

Can we apply this method on Anderson model?

$$H = \lambda \cdot V + \Delta$$

Note that V is the special case of band matrix i.e., $W = 1$.

$$H_{Anderson} = \lambda \cdot H_{diagonal\ matrix} + \Delta$$

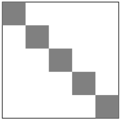
Or something like Anderson model?

$$H = \lambda \cdot H_{band\ matrix} + \Delta$$

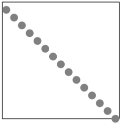
But it does not have the spirit of Anderson conjecture.

Block Anderson model

Let $V^{(B)}$ be a (random) block random matrix, with block size W.



$V^{(B)}$ for Block Anderson



V for Anderson

$$H = -\Delta + \lambda \cdot V^{(B)}, \qquad H = -\Delta + \lambda \cdot V$$

* $V^{(B)} = V$ when $W = 1$
* The eigenvectors of $V^{(B)}$ are localized (in the block).
*

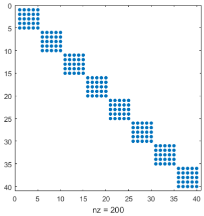
Δ v.s. $V^{(B)}$

delocalization v.s. localization.

Block Anderson Model

Let $V^{(B)}$ be random block matrices, the blocks are i.i.d. Winger matrices (whose spectrum has order 1)

$$H := -\lambda \cdot \Delta + \cdot V^{(B)}$$



Conjecture: There is a phase transition when $d \geq 3$
The phase transition occurs at

$$\lambda_c := W^{-\frac{d-1}{2}}, \quad d \geq 3$$

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Block Anderson Model

Yang and Y. (2024)

Delocalization of block Anderson model.

For $d \geq 7$ and $\lambda \gg W^{-\frac{d-2}{4}}$, $\ell \gg W^m$ for any m.

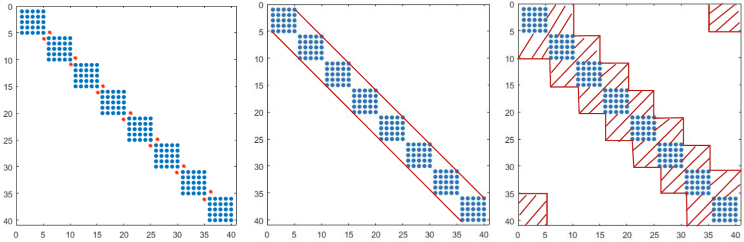
Localization of block Anderson model. (Simiar results: Peled - Schenker - Shamis - Sodin, 2019)

For any $d \geq 1$, if $\lambda \ll \lambda_c \sim W^{-\frac{d-1}{2}}$, $\ell = W$.

Note: A very small connection can change the phase.

Similar models

We have similar results for



(1) block Anderson model

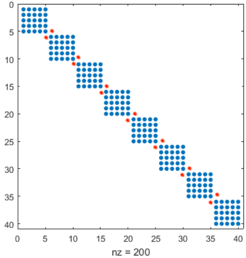
$$H := -\lambda \cdot \Delta + \cdot V^{(B)}$$

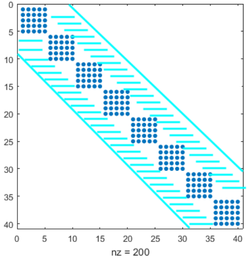
(2) Anderson orbital model

$$H := -\lambda \cdot \Delta \otimes I_W + \cdot V^{(B)}$$

(3) Wegner orbital model.

Review Band matrix





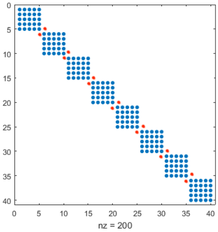
The non-diagonal-block entries (light blue part) of the band matrix can be much smaller, and the matrix is still delocalized. e.g.

$$\mathbb{E}H_{xy}^2 = W^{-d} \longrightarrow \mathbb{E}H_{xy}^2 = W^{-2d+\epsilon}$$

Universal Model:

Block band matrix + Perturbation

Block Anderson Conjecture



* The condition

$$\lambda_c := W^{-\frac{d-1}{2}}, \quad d \geq 3$$

comes from the universal model, i.e., block band matrix + perturbation, in which the condition is $\|Pertubation\|_{HS} \sim 1$ (in one block). This condition is also the condition for $\ell \gg W$, i.e., sub-system starts to mix with neighbors.

* There was no result on the delocalization of this model.

Block Anderson Conjecture

* We also used the sum zero property and doubly connected property of these models. The sum zero property seems very basic, but we still can not see it directly.

* Furthermore, we start to use the more general sum zero property:

$$\sum_{x_1}\sum_{\mu}\Gamma^{(\mu)}_{x_1x_2x_3x_4}\approx 0$$

It is similar to the spontaneous renormalization of interacting vertex in Feymann diagram.

Thank you!