THE UNIVERSITY OF MELBOURNE

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The exciting world of singular values and eigenvalues

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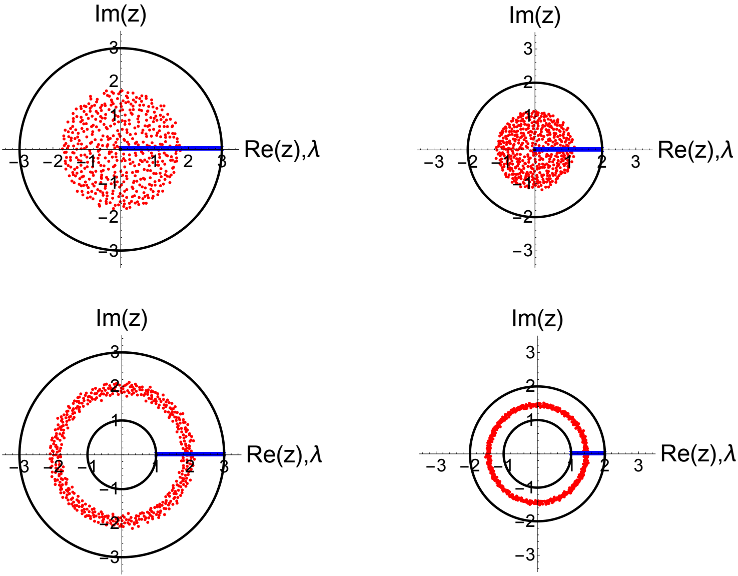
Random Matrices and Related Topics,
Jeju, 6th of May 2024

Outline

- SV vs. EV
- Cross Correlations
- Polynomial Ensembles
- Outlook

SV vs. EV



Consider a complex matrix $X = \mathbb{C}^{n\times n}$

Computes eigenvalues $z_j \in \mathbb{C}$ and its singular values $\lambda_j \geq 0$.

Relations for Fixed Matrices

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n; \qquad |z_1| \geq |z_2| \geq \ldots \geq |z_n|$$

1. **One exact equality:**

$$|\det X| = \prod_{j=1}^{n} \lambda_j = \prod_{j=1}^{n} |z_j|.$$

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2. **Schur's Inequality:**

$$\operatorname{tr} XX^{\dagger} = \sum_{j=1}^{n} \lambda_j^2 \geq \sum_{j=1}^{n} |z_j|^2.$$

(1909)

Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen.

Von

I. Schur in Berlin.

Ist eine algebraische Gleichung in der Form

$$\begin{vmatrix} a_{11}-x, & a_{12} & , \cdots, & a_{1n} \\ a_{21} & , a_{22}-x, & \cdots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & , a_{n2} & , \cdots, & a_{nn}-x \end{vmatrix} = 0$$

gegeben und sind $\omega_1, \omega_2, \cdots, \omega_n$ ihre Wurzeln, so läßt sich, wie Herr A. Hirsch*) gezeigt hat, in sehr einfacher Weise eine obere Grenze für die absoluten Beträge der ω_ν angeben; bedeutet nämlich a die größte unter den n^2 Zahlen $|a_{\varkappa\lambda}|$, so ist

$$|\omega_\nu| \leqq na.$$

In der vorliegenden Arbeit möchte ich auf einige andere, wesentlich schärfere Ungleichungen aufmerksam machen, die für die absoluten Beträge der ω_ν bestehen. Die einfachste und zugleich wichtigste unter ihnen ist die Ungleichung

$$\sum_{\nu=1}^{n} |\omega_\nu|^2 \leqq \sum_{\varkappa,\lambda} |a_{\varkappa\lambda}|^2.$$

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3. **Weyl's Inequalities:**

$$\sum_{j=1}^{k} \phi(\lambda_j) \geq \sum_{j=1}^{k} \phi(|z_j|)$$

for all $k = 1, \ldots, n$.

(1949)

INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION

BY HERMANN WEYL

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \qquad \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n.$$

I shall prove the following

THEOREM. *Let $\varphi(\lambda)$ be an increasing function of the positive argumen λ, $\varphi(\lambda) \geq \varphi(\lambda')$ for $\lambda \geq \lambda' > 0$, such that $\varphi(e^{\xi})$ is a convex function of ξ anc $\varphi(0) = \lim_{\lambda \to 0} \varphi(\lambda) = 0$. Then the eigenvalues λ_i and κ_i in descending orde satisfy the inequalities*

$$\varphi(\lambda_1) + \ldots + \varphi(\lambda_m) \leq \varphi(\kappa_1) + \ldots + \varphi(\kappa_m) \qquad (m = 1, 2, \ldots, n), \quad (1)$$

in particular

$$\lambda_1^s + \ldots + \lambda_m^s \leq \kappa_1^s + \ldots + \kappa_m^s \qquad (m = 1, 2, \ldots, n) \qquad (2)$$

for any real exponent $s > 0$.

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for all $k = 1, \ldots, n$.

4. **Horn's Theorem:** (1)+ (3) for $\phi(x) = x$ $\Rightarrow \exists X \in \mathbb{C}^{n \times n}$ with EV z_j and SV λ_j.

(1954)

ON THE EIGENVALUES OF A MATRIX WITH PRESCRIBED SINGULAR VALUES[1]

ALFRED HORN

The singular values of a matrix A (with complex elements) are the non-negative square roots of the eigenvalues of A^*A, where A^* is the conjugate transpose of A. Let us call a pair of n-tuples $(\lambda_1, \cdots, \lambda_n)$, $(\alpha_1, \cdots, \alpha_n)$ *allowable* if there exists an nth order matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ and singular values $\alpha_1, \cdots, \alpha_n$. (Weyl [1] has proved that if $(\lambda_1, \cdots, \lambda_n)$, $(\alpha_1, \cdots, \alpha_n)$ is allowable and if

(1) $\qquad \alpha_1 \geqq \cdots \geqq \alpha_n \geqq 0 \quad$ and $\quad |\lambda_1| \geqq \cdots \geqq |\lambda_n|,$

then

(2) $\qquad |\lambda_1| \cdot \cdots \cdot |\lambda_k| \leqq \alpha_1 \cdot \cdots \cdot \alpha_k \qquad$ for $1 \leqq k \leqq n$,

(3) $\qquad |\lambda_1| \cdot \cdots \cdot |\lambda_n| = \alpha_1 \cdot \cdots \cdot \alpha_n.$

THEOREM 1. *If the pair* $(\lambda_1, \cdots, \lambda_n)$, $(\alpha_1, \cdots, \alpha_n)$ *satisfies* (1), (2), (3) *and if* $\alpha_n > 0$ *and* $\lambda_i > 0$ *for* $1 \leqq i \leqq n$, *then this pair is allowable.*

Relations for Random Matrices

1. No exact relation for arbitrary ensembles on $\mathbb{C}^{n\times n}$ apart from what we know from the previous slide.

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2. Add bi-unitary invariance, meaning the probability measure on $\mathbb{C}^{n\times n}$ satisfies

$$d\mu(UXV^{-1}) = d\mu(X) \qquad \text{for all } U, V \in \mathrm{U}(n).$$

$\Rightarrow$ The complex spectrum is isotropic $\Rightarrow$ focus on the distribution of the squared eigenradii $r = |z|^2$.

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$\Rightarrow$ The complex spectrum is isotropic $\Rightarrow$ focus on the distribution of the squared eigenradii $r = |z|^2$.

3. For $n \to \infty$, Haagerup-Larsson theorem (2000):

$$S_{\mathrm{SV}}\left(\int_0^r d\mu_{\mathrm{EV}}(r') - 1\right) = \frac{1}{r},$$

where S is the S-transform of free probability theory.

Bijection between SV and EV Distributions

Theorem: (MK, Kösters '16)

1. Random Matrix $X \in \mathbb{C}^{n\times n}$ is bi-unitarily invariant.
2. The jpdfs of the EV z_j and squared SV a_j are $f_{\rm EV}$ and $f_{\rm SV}$, respectively.
3. Then, they are bijectively related via the integral

$$f_{\rm EV}(z) \propto |\Delta_n(z)|^2 \int_{\times_{j=1}^n (i\mathbb{R}+j)} d[s] \int_{\mathbb{R}_+^n} \frac{d[a]}{\det a} f_{\rm SV}(a)$$
$$\mathrm{Perm}[|z_b|^{-2s_c}]_{b,c=1,\ldots,n} \frac{\det[a_b^{s_c}]_{b,c=1,\ldots,n}}{\Delta_n(s)\Delta_n(a)}$$

with the Vandermonde determinant Δ_n

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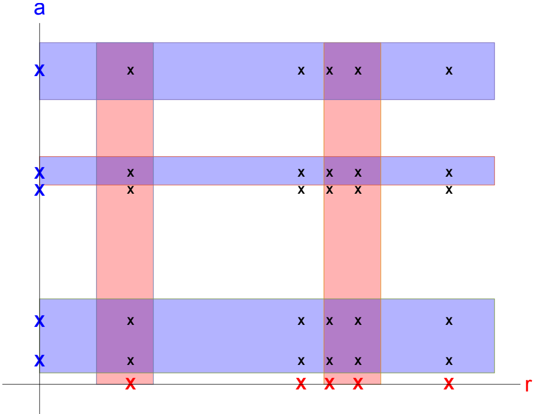
- The s integral is actually regularised so that the integral is absolutely convergent.
- There is an inverse of this integral transformation.

Cross Correlations

Def.: For any measurable test function ϕ depending on $\mathbb{R}_+^{2k}$ the (j,k)-point correlation measure is the marginal measure

$$\mathbb{E}\left[\frac{(n-j)!(n-k)!}{(n!)^2}\sum_{\substack{1\le l_1,\dots,l_j,p_1,\dots,p_k\le n\\ l_\alpha\neq l_\beta,\ p_\alpha\neq p_\beta \text{ when } \alpha\neq\beta}}\phi(r_{l_1}(X),\dots,r_{l_j}(X);a_{p_1}(X),\dots,a_{p_k}(X))\right]$$

$$=\int_{\mathbb{R}_+^{j+k}} d\mu_{j,k}(r_1,\dots,r_j;a_1,\dots,a_k)\phi(r_1,\dots,r_j;a_1,\dots,a_k)$$



For $n = 2$

Proposition: (Allard, MK '24)

1. The $(2, 2)$-point correlation function is

$$d\mu_{2,2}(r_1, r_2; a_1, a_2) = \Theta(a_{\max} - r_{\max})\,\Theta(r_{\min} - a_{\min}) \times \frac{f_{\mathrm{SV}}(a_1, a_2)}{2|a_1 - a_2|}(r_1 + r_2)\,\delta(r_1 r_2 - a_1 a_2) dr_1 dr_2 da_1 da_2.$$

• Even when we start with a density the (n, n)-point correlation function will not have a density!

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2. The $(1, 2)$-point correlation function is

$$f_{1,2}(r; a_1, a_2) = \Theta(a_{\max} - r)\,\Theta(r - a_{\min}) \frac{f_{\mathrm{SV}}(a_1, a_2)}{2|a_1 - a_2|}\left(1 + \frac{a_1 a_2}{r^2}\right).$$

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3. The $(2,1)$-point correlation function is
$$f_{2,1}(r_1, r_2; a) = \left[\Theta(a - r_{\max}) + \Theta(r_{\min} - a)\right]\frac{r_1 + r_2}{2|a^2 - r_1 r_2|} \times f_{\mathrm{SV}}\left(a, \frac{r_1 r_2}{a}\right).$$

General $n > 2$ and general Ensembles

Theorem: (Allard, MK '24)

1. The $(1,1)$-point correlation function is

$$f_{1,1}(r;a_1) \propto \sum_{j=1}^{n} \int_{j+i\mathbb{R}} \frac{ds}{2\pi i} \int_{\mathbb{R}_+^{n-1}} \prod_{b=2}^{n} da_b \; r^{j-1-s} f_{\mathrm{SV}}(a) \frac{\det \begin{bmatrix} a_b^{s-1} \\ a_b^{\tau_c(j)-1} \end{bmatrix}}{\Delta_n(s,\tau(j))\Delta_n(a)},$$

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We need more integrable structures!

Polynomial Ensembles

Definition: (Kuijlaars, Stivigny '14)

A bi-unitarily invariant random matrix is called a polynomial ensemble if the jpdf of the squared singular values has the form

$$f_{\mathrm{SV}}(a) = \frac{\Delta_n(a)\det\left[w_{k-1}(a_j)\right]_{j,k=1}^{n}}{n!\ \det\left[\mathcal{M}w_{k-1}(j)\right]_{j,k=1}^{n}} \geq 0 \quad \text{for all } a \in \mathbb{R}_+^n,$$

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Remark: polynomial ensembles exhibit a determinantal point process

$$R_k(a_1,\ldots,a_k) = \det[K(a_b,a_c)]_{a,b=1,\ldots,k}, \qquad \rho_{\rm SV}(a) = \frac{1}{n}K(a,a)$$

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The kernel is unique if it is a polynomial in the second entry!

General $n > 2$ and Polynomial Ensembles

Theorem: (Allard, MK '24)

1. The level density of the complex eigenvalues is

$$\rho_{\mathrm{EV}}(r) = \frac{1}{n}\int_0^1 dt \int_0^\infty \frac{dv}{v} h\left(\frac{r}{v}, t\right) K\left(v, \frac{rt}{t-1}\right)$$

   with

$$h(x,t) = \Theta(1-x)\frac{n}{x}\left(\frac{1}{x}-1\right)^{n-2}\left[\frac{n}{x}-1-(n+1)t\left(\frac{1}{x}-1\right)\right](t-1)^{n-1}.$$

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2. The $(1,1)$-point correlation function is

$$f_{1,1}(r;a) = \rho_{\mathrm{SV}}(a)\rho_{\mathrm{EV}}(r) + \mathrm{cov}(r;a)$$

   with covariance density

$$\begin{aligned}\mathrm{cov}(r;a) =& \frac{1}{n^2}\int_0^1 dt \frac{1}{a} h\left(\frac{r}{a}, t\right) K\left(a, \frac{rt}{t-1}\right)\\ & - \frac{1}{n^2}\int_0^\infty \frac{dv}{v}\int_0^1 dt\, h\left(\frac{r}{v}, t\right) K\left(a, \frac{rt}{t-1}\right) K(v,a)\end{aligned}$$

Covariance Density

1. For two observables ψ_{EV} of one squared eigenradius and ϕ_{SV} of one squared singular value, it is

$$\mathrm{Cov}(\psi_{\mathrm{EV}}, \phi_{\mathrm{SV}}) = \int_{\mathbb{R}_+^2} dr\, da\, \mathrm{cov}(r; a)\psi_{\mathrm{EV}}(r)\phi_{\mathrm{SV}}(a).$$

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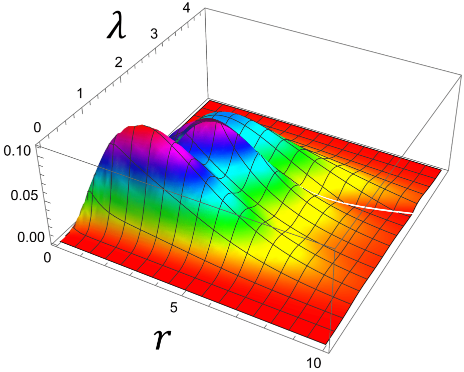
$$\mathrm{Cov}(\psi_{\rm EV},\phi_{\rm SV})=\int_{\mathbb{R}_+^2} dr\,da\,\mathrm{cov}(r;a)\psi_{\rm EV}(r)\phi_{\rm SV}(a).$$

2. Its interpretation is for an $A\subset\mathbb{R}_+^2$:
   - $\int_A dr\,da\,\mathrm{cov}(r;a)\gg 0$: it is likely to find an EV when one found a SV in A and vice versa.
   - $\int_A dr\,da\,\mathrm{cov}(r;a)\ll 0$: it is unlikely to find an EV when one found a SV in A and vice versa.
   - $\int_A dr\,da\,\mathrm{cov}(r;a)\approx 0$: the occurrence of EV and SV in A is almost uncorrelated.

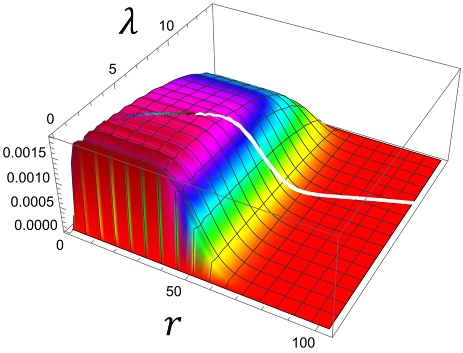
Example:

Induced Ginibre/Wishart-Laguerre Ensemble

(1,1)-point correlation function $2\lambda f_{1,1}(r,\lambda^2)$



$n = 3$

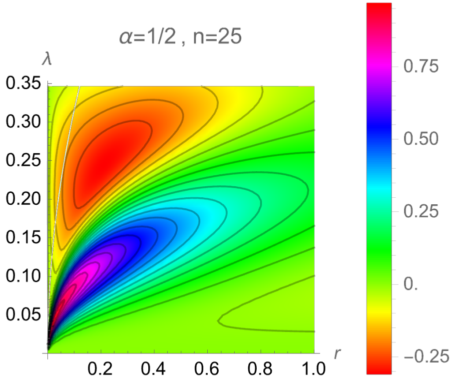


$n = 50$

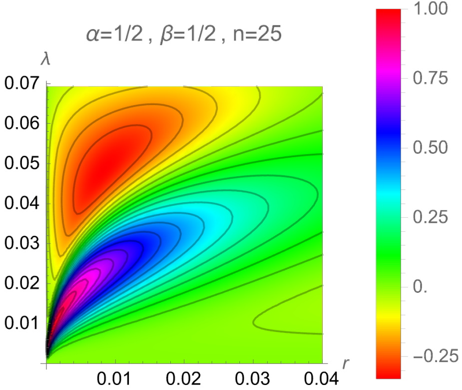
Outline

1. Non-trivial hard statistics (currently writing up)

covariance density $2\lambda\mathrm{cov}(r, \lambda^2)$



Laguerre ensemble



Jacobi ensemble

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2. Computation of the finite *n*-correction of the Haagerup-Larsson Theorem and other Free Probability results

3. Computation of mixed correlation functions for $j, k > 1$

4. Computation of the correlation between the largest (smallest) eigenradius and singular value.

5. etc.

Many Thanks
for your attention!

MK, Holger Kösters:
Exact Relation between Singular Value and Eigenvalue Statistics
Random Matrices: Theory Appl. **05**, 1650015 (2016)
arXiv:1601.02586

Matthias Allard, MK:
Correlation functions between singular values and eigenvalues
arXiv:2403.19157