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Fermions in low dimensions and non-Hermitian random matrices

Gernot Akemann
(Bielefeld University & University of Bristol)

Random Matrices and Related Topics in Jeju
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with Sungsoo Byun, Markus Ebke and Grégory Schehr

[arXiv:2206.08815 = J Stat Phys ’23, arXiv2308.05519 = J Phys A ’23]

Outline

- **Part I: Map to Random Matrix Theory (RMT)**

  - *d*-dimensional harmonic trap: **GUE** & beyond

  - 2*d* with magnetic field / rotating trap:
    complex **Ginibre** ensemble (GinUE)

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    complex **Ginibre** ensemble (GinUE)

- **Part II: Universality and holography in 2d**

  - variance of number of Fermions $\mathcal{N}_A$ in A
    $\boxed{\mathrm{Var}\mathcal{N}_A \sim \partial A}$ for a large class of non-Hermitian RMT

  - entropy $S_q(N, A) \propto \mathrm{Var}\mathcal{N}_A$ in bulk scaling

- Summary and open questions

Fermions in a d-dimensional harmonic trap: $1d$=GUE

- N Fermions at $\vec{r}_j \in \mathbb{R}^d$, momenta $\vec{p}_j = -i\vec{\nabla}_j$, $j = 1, \ldots, N$
- **Harmonic oscillators** in d-dimensions:

  $\mathcal{H} = \sum_{j=1}^{N} \mathcal{H}_j(\vec{p}_j, \vec{r}_j)$, $\boxed{\mathcal{H}_j(\vec{p}_j, \vec{r}_j) = \frac{1}{2}(\vec{p}_j^2 + \vec{r}_j^2)}$ single particle, non-interacting

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  Hermite eigenfunctions $u_n(x) \sim H_n(x)e^{-\frac{1}{2}x^2}$, $n = 0, 1, \ldots$

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- N-particle Fermionic ground state: **Slater determinant**

$$\boxed{\Psi_0(x_1, \ldots, x_N) \sim \det \left[H_{j-1}(x_i)e^{-x_i^2/2}\right]_{i,j=1}^{N} \sim e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2} \Delta_N(\{x\})}$$

- Vandermonde determinant $\Delta_N(\{x\}) = \det \left[x_i^{j-1}\right]_{i,j=1}^{N}$

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- amplitude $|\Psi_0|^2 \sim |\Delta_N(\{x\})|^2 e^{-\sum_{i=1}^N x_i^2}$ = **GUE**
  joint eigenvalue density, determinantal point process

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$$|\Psi_0(\vec{r}_1,\ldots,\vec{r}_N)|^2 \sim \det\left[\mathcal{K}_N^{\mathrm{Fermi}}(\vec{r}_1,\vec{r}_2)\right]_{i,j=1}^{N} \text{ joint density}$$

$$R_k(\vec{r}_1,\ldots,\vec{r}_k) = \det\left[\mathcal{K}_N^{\mathrm{Fermi}}(\vec{r}_i,\vec{r}_j)\right]_{i,j=1}^{k}\ k\text{-point function}$$

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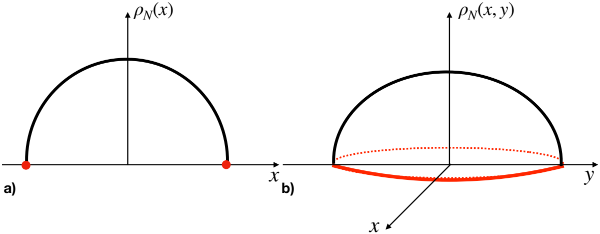
- **does not** correspond to eigenvalues of RMT for $d > 1$, but same techniques apply

Results in *d* dimensions [Dean, Le Doussal, Majumdar, Schehr '19]



- **global density**: 1*d* semi-circle (left) vs. 2*d* cup (right)

$$\boxed{\frac{1}{N^{d/2}}\mathcal{K}_N^{\mathrm{Fermi}}(\sqrt{N}\vec{r},\sqrt{N}\vec{r}) \sim \left(2-|\vec{r}|^2\right)^{d/2}} \equiv \rho(|\vec{r}|)$$

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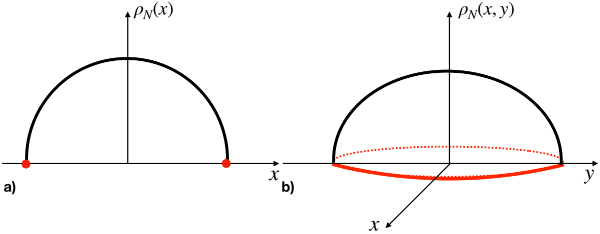

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- **local correlations: bulk kernel** at $\vec{r}_0$

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e.g. 1d: $\sim$ **sine-kernel**

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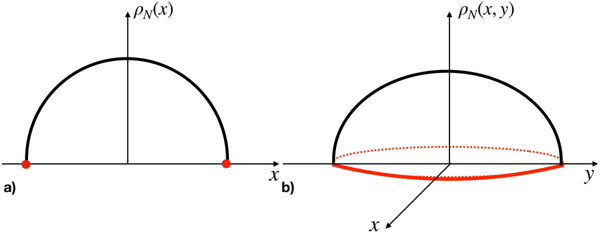

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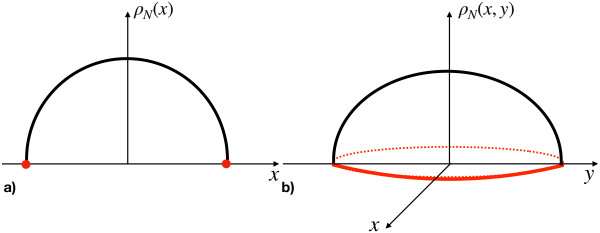

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- **rotating trap** with harmonic potential (3d confined to 2d)

$$\boxed{\mathcal{H}_i(\vec{p},\vec{r}) = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{r}^2 - \Omega\,\vec{e}_3\cdot\vec{r}\times\vec{p}}\ ,\ \vec{p}_j = -i\hbar\vec{\nabla}_j$$

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  $\neq$ Hermite in $\mathbb{R}^2$: $e^{-(x^2+y^2)/2}H_k(x)H_l(y)$

Map to complex Ginibre ensemble GinUE

- bound states: $0 \leq \Omega < \omega = 1$, no degeneracy
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- higher LL [Smith et al. '22], cf. [Haimi, Hedenmalm '13]

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  - Renyi **entanglement entropy**

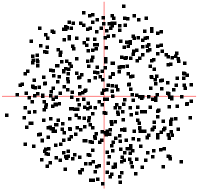
$\boxed{S_q(N, A) \equiv \frac{1}{1-q} \ln[\text{Tr}((\rho_A)^q)]}$ $q > 1 \Rightarrow$ von Neumann

$\rho_A = \text{Tr}_{A^C}[\rho]$ partial trace over complement $A^C = \mathbb{C} \setminus A$,

or express with overlap $\mathbb{A}_{kl} = \int_A d^2z \Phi_k(z)\overline{\Phi_l(z)}$

$\underline{S_q(N, A) = \frac{1}{1-q} \text{Tr}[\ln(\mathbb{A}^q + (I - \mathbb{A})^q)]}$ [Klich '06]

Results for GinUE [Lacroix-A-chez-Toine, Majumdar, Schehr '19]



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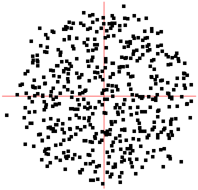

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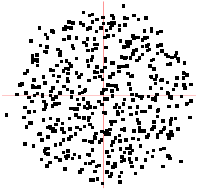

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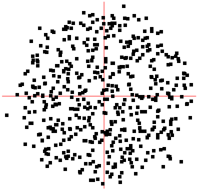

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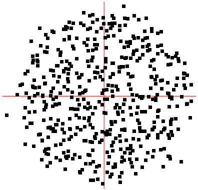

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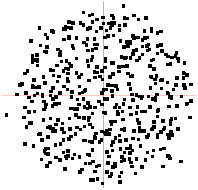

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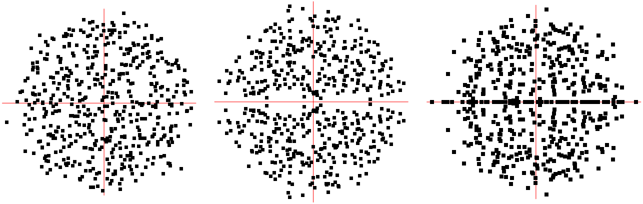
Q1 **How general** is the variance in non-Gaussian RMT – universal? (e.g. anharmonic trap?)

Q2 **Var** $\sim \partial A$ **holds** for general A, general RMT?

Complex, quaternion and real Ginibre ensemble



- GinUE (left) vs. GinSE (middle) vs. GinOE (right): uniform vs. depletion vs. accumulation on $\mathbb{R}$
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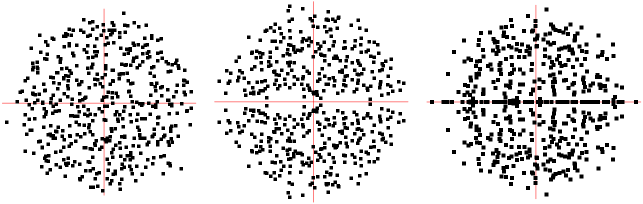

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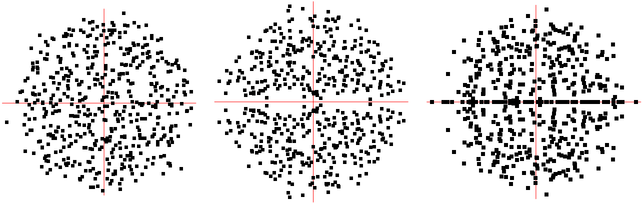

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- study further examples: **global density $\neq$ const.**
  - normal matrices $\exp[-V(|z|)]$
  - truncated $U(N+M)$ Haar distributed $\to U(N)$
  - products of m Ginibre matrices

Mean and Variance - finite N & general $V(|z|)$

- **mean** number of eigenvalues

$$\boxed{\mathbb{E}\mathcal{N}_R \equiv \int_{D_R} d^2z \; R_{1,N}(z)}$$

- **variance**

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- **example GinUE:**

$$K_N(z,u) = e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|u|^2} \sum_{j=0}^{N-1} \frac{1}{j!}(z\bar{u})^j$$ $\Rightarrow$ circular law

$\Rightarrow h_j(R)/h_j = \gamma(j+1, NR^2)/j!$

Full counting statistics - finite N & general $V(|z|)$

- higher order cumulants: centred generating function

$$\boxed{\ln\left[\mathbb{E}\exp[-\mu(\mathcal{N}_R - \mathbb{E}\mathcal{N}_R)]\right] \equiv \sum_{p=2}^{\infty} \frac{(-1)^p}{p!}\mu^p \,\mathbb{E}_c\mathcal{N}_R^p}$$

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Proposition [Lacroix-a-chez-Toine, Majumdar, Schehr '18 GinUE; A, Byun, Ebke, Schehr '23 GinSE]

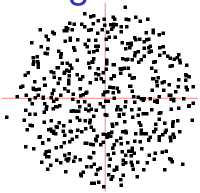
For $V(|z|)$ admissable the cumulants read:

$$\mathbb{E}_c\mathcal{N}_R^p = (-1)^{p+1}\sum_{j=0}^{N-1} \mathrm{Li}_{1-p}\left(1 - h_j/h_j(R)\right)$$

for GinUE and index $j \to 2j+1$ for GinSE class,

where $\mathrm{Li}_q(x) = \sum_{k=1}^{\infty} k^{-q}x^k$ is the polylogarithm.

The large-N Limit: Global density and scaling limits



eigenvalues of GinUE

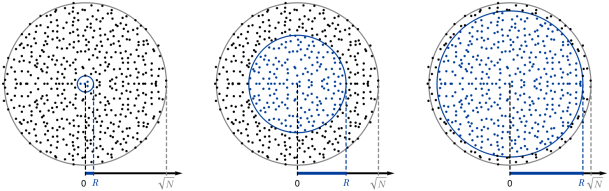
- rescaled weight $w(z) = e^{-NV(|z|)}$, $V(|z|)$ admissible:

  global density $\boxed{\rho(z) \equiv \lim_{N\to\infty} R_{1,N}(z) = \frac{2}{\beta}\Delta_z V(|z|) \cdot \chi_D}$

  Frostman's equilibrium measure on droplet D,
  for Ginibre $V(|z|) = |z|^2$ circular law on disc $D_{R=1}$

- **scaling regimes** for variance of particles in D_R, radius R
  - origin limit: $R = t/N^{\delta}$ non-universal
  - bulk: $0 < R < 1$ fixed
  - edge: $R = 1 - \frac{S}{c\sqrt{N}} \to 1$, S fixed

Mean and Variance - 3 scaling regimes



- **origin** (left) vs. **bulk** (middle) vs. **edge** (right) scaling

The large-N Limit: Universal Bulk and Edge

Theorem [LMS '19 / A, Byun, Ebke '22]

For GinUE / GinSE class ($\beta = 2 \ / \ 4$) with V admissible:

(i) **(Bulk)** For $R \in (0,1)$ fixed, we have

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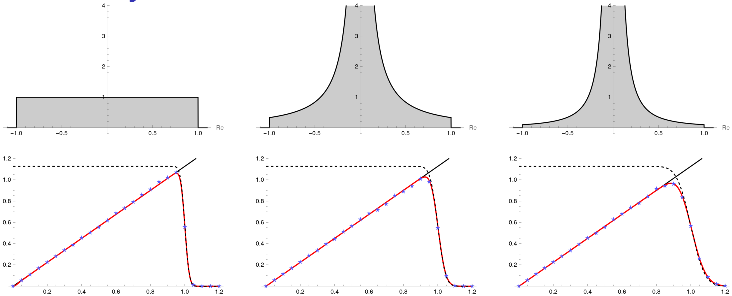
- expect same for GinOE: origin at large R, numerics

Comparison GinOE/UE/SE: limiting mean at origin

	GinOE	GinUE	GinSE
$\mathbb{E}\mathcal{N}_R$	$R^2 + \frac{1}{2} - \frac{1}{2}e^{2R^2}\text{erfc}(\sqrt{2}R)$	R^2	$R^2 - \frac{1}{4} + \frac{1}{4}e^{-4R^2}$
$R \to 0$	$\sqrt{\frac{2}{\pi}}R + O(R^3)$	R^2	$2R^4 + O(R^6)$
$R \to \infty$	$R^2 + \frac{1}{2} + O(\frac{1}{R})$	R^2	$R^2 - \frac{1}{4} + O(e^{-4R^2})$

Universality: Product of m Random Matrices



- **top**: **global density** for product of $m = 1, 3, 10$ GinUE

  $\boxed{\rho(z) = \frac{1}{m}|z|^{\frac{2}{m}-2}}$ divergence at origin

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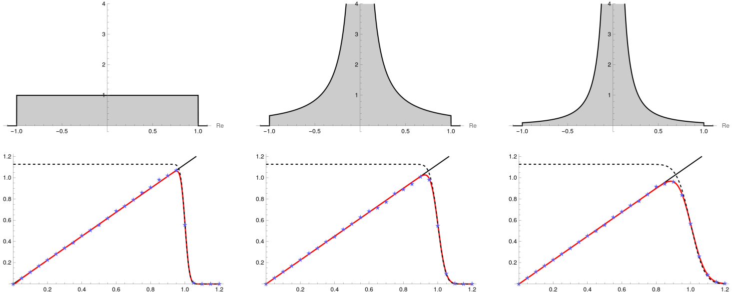

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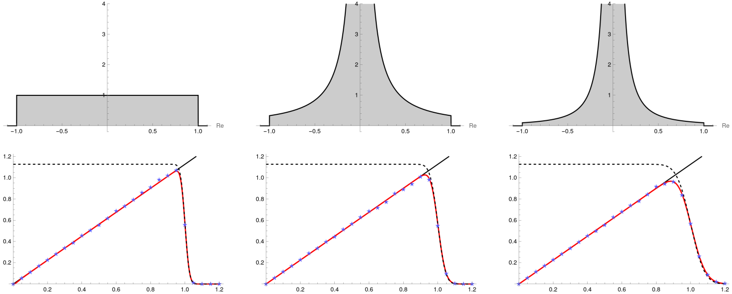

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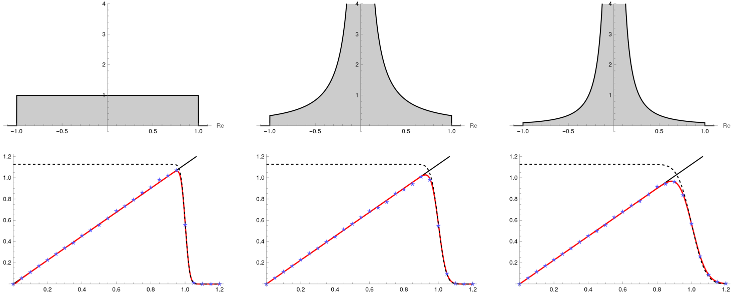

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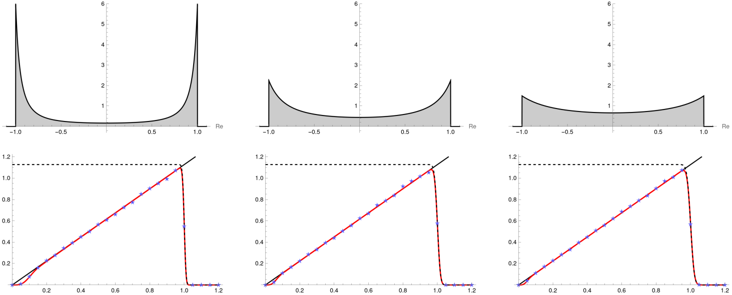
- similar plots for quaternionic matrices

Universality: Truncated Unitary Symplectic Matrices



- **top**: **global density** for truncated unitary symplectic matrices $USp(2N(1+c)+1) \to USp(2N)$, $c = 0.2, 0.8, 2.0$

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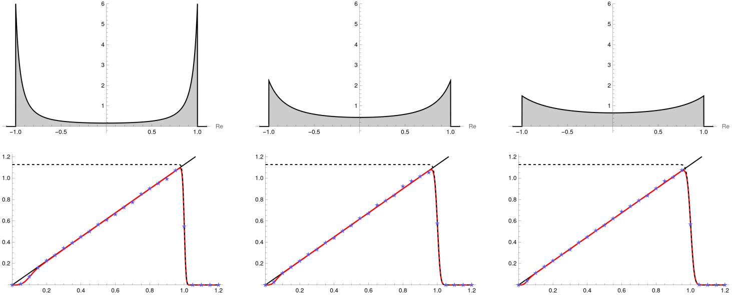

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- **bottom**: lim Var$\mathcal{N}_R$ in **bulk** (line) & **edge** regime (dotted) vs. finite $N = 250$ (red) and 10 k samples (blue stars)
- **origin** behaviour $\sim T^4$ is visible, $R = T/N^{1/2}$
- similar plots for unitary matrices

Summary & Open Questions

- N **non-interacting Fermions** in harmonic trap
  in $1d$ / in $2d$ in rotating trap $\Leftrightarrow$ with $\vec{B}$
  mapped to Gaussian RMT with $\mathbb{R}$ / $\mathbb{C}$ **eigenvalues**
- **universal asymptotic** $\forall$ cumulants in GinUE class & GinSE class for V admissible **in bulk and edge limit**
- bulk relation **entropy - variance** beyond Gaussian RMT
- real matrices GinOE: mean and variance at origin $\surd$
  $\to$ bulk (and edge) conjectured to be universal
- **variance** $\sim \partial A$ for more **general** A in elliptic GinUE
  $\longrightarrow$ talk by Leslie Molag