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Orthogonal polynomials in the Ginibre random matrix model with two point insertions

Arno Kuijlaars – KU Leuven, Belgium

Random Matrices and Related Topics in Jeju,
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0 Orthogonal polynomials in the Ginibre random matrix model with two point insertions

Joint work with Mario Kieburg and Sampad Lahiry

I will also report on recent result

- **S. Berezin, A.B.J. Kuijlaars and I. Parra,
Planar Orthogonal Polynomials As Type I Multiple Orthogonal Polynomials,
SIGMA 19 (2023), 020, 18 pages.**

1 Ginibre matrix model with insertions

$$\frac{1}{Z_{n,N}} \prod_{1\leq i<j\leq n} |z_i - z_j|^2 \left(\prod_{i=1}^{n} \prod_{j=1}^{p} |z_i - a_j|^{2c_j N} \right) \prod_{i=1}^{n} e^{-N|z_i|^2}$$

- **Ginibre matrix model with p insertions (repelling charges) located at $a_1, \ldots, a_p$.**
- **Model is determinantal with correlation kernel built out of the orthogonal polynomials**

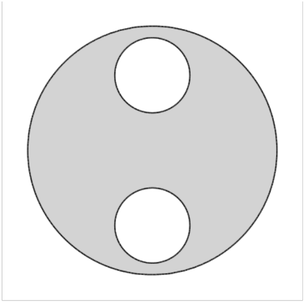
$$\int_{\mathbb{C}} P_{n,N}(z) \overline{z}^k \left(\prod_{j=1}^{p} |z - a_j|^{2c_j N} \right) e^{-N|z|^2} dA(z) = 0, \qquad k = 0, \ldots, n-1$$

where $P_{n,N}(z) = z^n + \cdots$.

1 Ginibre matrix model with insertions

$$\frac{1}{Z_{n,N}} \prod_{1\le i<j\le n} |z_i - z_j|^2 \left(\prod_{i=1}^{n} \prod_{j=1}^{p} |z_i - a_j|^{2c_j N} \right) \prod_{i=1}^{n} e^{-N|z_i|^2}$$

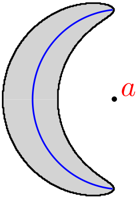
- **Limit** $n, N \to \infty$, $n/N \to t > 0$
- **Eigenvalues fill out a** $2D$ **domain, the droplet**
- **Each** a_j **has region around it that is free from eigenvalues**



Conjecture: Zeros of $P_{n,N}$ **tend to system of curves inside the convex hull of the droplet.**

1 One insertion

$$\frac{1}{Z_{n,N}} \prod_{1\le i<j\le n}^{n} |z_i - z_j|^2 \left(\prod_{i=1}^{n} |z_i - a|^{2cN} \right) \prod_{i=1}^{n} e^{-N|z_i|^2}$$



- **Zeros tend to a curve.**
  Balogh, Bertola, Lee, McLaughlin 2015
- $P_{n,N}$ **is orthogonal on a contour**

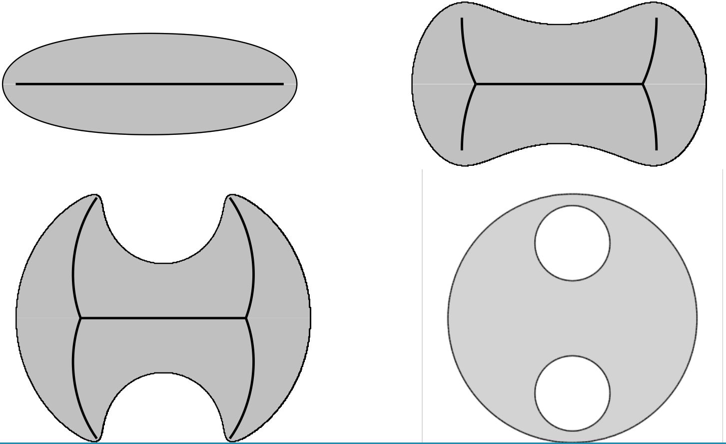
$$\frac{1}{2\pi i} \oint_\gamma P_{n,N}(z) z^k \frac{(z-a)^{Nc}}{z^{n+Nc}} e^{-aNz} dz = 0$$

for $k = 0, 1, \ldots, n-1$**.**

- $P_{n,N}$ **satisfies a Riemann-Hilbert problem of size** 2×2
- **Deift-Zhou steepest descent analysis gives strong asymptotic formulas** see also Krüger, Lee, Yang 2023

1 Two insertions at $\pm ia$

$$\frac{1}{Z_{n,N}} \prod_{1\le i<j\le n} |z_i - z_j|^2 \left(\prod_{i=1}^{n} |z_i^2 + a^2|^{2cN} \right) \prod_{i=1}^{n} e^{-N|z_i|^2}$$



2 Potential theory

$$I(\mu) = \iint \log \frac{1}{|z-w|} d\mu(z) d\mu(w)$$
$$U^{\mu}(z) = \int \log \frac{1}{|z-w|} d\mu(w)$$

Droplet Ω_t in Ginibre model with insertions is characterized by equilibrium problem in potential theory

- **Minimize**

$$I(\nu) + \frac{1}{t} \int \left(|z|^2 - \sum_{j=1}^{p} c_j \log |z - a_j| \right) d\nu(z)$$

among probability measures on $\mathbb{C}$.

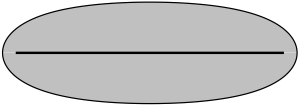
- **Minimizer is** $\boxed{\chi_{\Omega_t} \frac{dA}{\pi t}}$

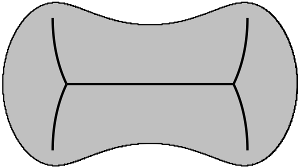
2 Motherbody

- **Motherbody (or potential theoretic skeleton) is probability measure μ_t on curve Δ_t satisfying**

$$U^{\mu_t}(z) = U^{\chi_{\Omega_t} \frac{dA}{\pi t}}(z), \qquad z \in \mathbb{C} \setminus \Omega_t.$$

with inequality $\geq$ in Ω_t.





2 Theorem on motherbody

In the model with two insertions with $\boxed{a^2 > 2c}$ **and t small.**

Theorem

The motherbody exists and is supported on an interval $[-x_1, x_1]$. It is characterized by a vector equilibrium problem
Minimize

$$I(\mu_1)+I(\mu_2)-I(\mu_1,\mu_2)-\frac{c}{t}\int \log(x^2+a^2)d\mu_1(x)+\frac{t+2c}{t}\int \log|x|d\mu_2(x)$$

among pairs (μ_1, μ_2) of measures with

- $\mathrm{supp}(\mu_1) \subset [-x_1, x_1]$ ***and*** $\int d\mu_1 = 1$***,*** μ_1 ***is motherbody***
- $\mathrm{supp}(\mu_2) \subset \mathbb{R}$ ***and*** $\int d\mu_2 = \frac{t+c}{t}$***,***
- $\boxed{d\mu_2(x) \leq \frac{a}{\pi t}dx}$ ***(upper constraint)***

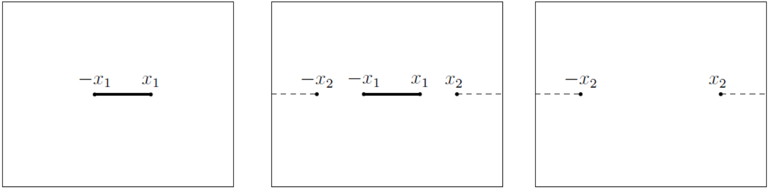
2 Consequence 1: Riemann surface

For minimizer (μ_1, μ_2) we have

$$\mathrm{supp}(\mu_1) = \Delta_1 = [-x_1, x_1]$$
$$\mathrm{supp}(\frac{a}{\pi t}dx - \mu_2) = \Delta_2 = (-\infty, -x_2] \cup [x_2, \infty)$$

From Δ_1 and Δ_2 we build a three sheeted Riemann surface with sheets

$$\mathcal{R}_1 = \mathbb{C} \setminus \Delta_1, \quad \mathcal{R}_2 = \mathbb{C} \setminus (\Delta_1 \cup \Delta_2), \quad \mathcal{R}_3 = \mathbb{C} \setminus \Delta_2,$$



2 Consequence 2: Schwarz function

$$
\begin{aligned}
S_1(z) &= t\int \frac{d\mu_1(s)}{z-s} + \frac{2cz}{z^2+a^2} \\
S_2(z) &= -t\int \frac{d\mu_1(s)}{z-s} + t\int \frac{d\mu_2(s)}{z-s} \pm ia, \qquad \text{for } \pm \operatorname{Im} z > 0, \\
S_3(z) &= -t\int \frac{d\mu_2(s)}{z-s} + \frac{t+2c}{z} \mp ia,
\end{aligned}
$$

defines a meromorphic function on the Riemann surface.

Theorem

S_1 ***is the Schwarz function for the droplet, i.e.,***

$$S_1(z) = \overline{z} \qquad \textit{for } z \in \partial\Omega_t.$$

3 Riemann Hilbert problem

Recall planar orthogonality

$$\int_{\mathbb{C}} P_{n,N}(z)\overline{z}^k \left(\prod_{j=1}^{p} |z-a_j|^{2c_j N}\right) e^{-N|z|^2} dA(z) = 0, \quad k = 0, \ldots, n-1.$$

- **Planar orthogonal polynomials are multiple orthogonal of type II on a contour around the origin** **Lee, Yang, 2019**

There are p weights $w_1, \ldots, w_p$ (varying with n and N) such that

$$\frac{1}{2\pi i} \oint_{\gamma} P_{n,N}(z) z^k w_j(z) dz = 0, \qquad j = 1, \ldots, p, k = 0, \ldots, n_j - 1$$

with $\lfloor \frac{n}{p} \rfloor \leq n_j \leq \lceil \frac{n}{p} \rceil$, $\sum_j n_j = n$.

- **It comes with Riemann Hilbert problem of size $p+1 \times p+1$.**

Van Assche, Geronimo, K, 2001

3 Alternative

- **In case $c_j N$ is an integer for every j, then $P_{n,N}$ is also multiple orthogonal of type I** **Berezin, Kuijlaars, Parra 2023**

There are auxiliary polynomials $Q_1, \ldots, Q_p$, $\deg Q_j \leq c_j N - 1$ such that

$$\frac{1}{2\pi i} \oint_\gamma \left(P_{n,N}(z) \frac{\prod_{j=1}^p (z-a_j)^{c_j N}}{z^{n+|c|N}} + \sum_{j=1}^p Q_j(z) \frac{e^{N\overline{a_j}z}}{z^{n+|c|N}} \right) z^k dz = 0,$$

for $k = 0, \ldots, n + |c|N - 1$ with $|c| = \sum_{j=1}^p c_j$.

- **It comes with alternative type I Riemann Hilbert problem of size $p+2 \times p+2$, that (in the present setting) can be reduced to size $p+1 \times p+1$.**

4 Asymptotic analysis

Ginibre model with two insertions

- Type II weights are expressed as Bessel functions of second kine in case of model with two insertions
- Type I weights are simpler, and we are able to apply the Deift/Zhou steepest descent analysis to the type I RH problem.
- Major role is played by the g-functions

$$g_j(z) = \int \log(z-s)d\mu_j(s), \quad j = 1, 2,$$

that are associated with the minimizers (μ_1, μ_2) of the vector equilibrium problem.

4 Strong asymptotics of polynomials

Ginibre model with two insertions, $a^2 > 2c > 0$ and small t.

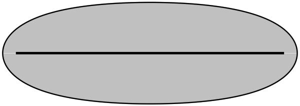
Theorem

Suppose $n, N \to \infty$ with $n - tN$ bounded and cN integer. Then

$$P_{n,N}(z) = M_{11}(z)e^{ng_1(z)}\left(1 + \mathcal{O}(n^{-1})\right) \qquad \textit{as } n \to \infty$$

uniformly for z in compact subsets of $\mathbb{C} \setminus [-x_1, x_1]$.
Zeros of $P_{n,N}$ tend to the interval $[-x_1, x_1]$ with μ_1 as limiting distribution.

Agrees with known asymptotics in exterior region see e.g. Hedenmalm, Wennman, 2021



Also asymptotic formulas on $(-x_1+\varepsilon, x_1-\varepsilon)$ and near $\pm x_1$ in terms of Airy functions.

4 Overview of RH analysis

Type I RH problem for analytic $Y : \mathbb{C} \setminus \gamma \to \mathbb{C}^{3\times 3}$

- $Y_+ = Y_- J_Y$ **on** γ **with**

$$J_Y(z) = \begin{pmatrix} 1 & 0 & \frac{(z^2+a^2)^{cN} e^{-iaNz}}{z^{n+2cN}} \\ 0 & 1 & \frac{e^{-2iaNz}}{z^{n+2cN}} \\ 0 & 0 & 1 \end{pmatrix}$$

- **Asymptotic condition** $\left(I_3 + \mathcal{O}(z^{-1})\right) \operatorname{diag}\left(z^n, z^{cN}, z^{-n-cN}\right)$ **as** $z \to \infty$
- **Then** $P_{n,N}(z) = Y_{1,1}(z)$

Sequence of transformations $\boxed{Y \mapsto X \mapsto T \mapsto S \mapsto R}$
for OP: Deift, Kriecherbauer, McLaughlin, Venakides, Zhou, 1999

4 Transformation $Y \mapsto X$ (preliminary step)

$Y \mapsto X$ takes different form in four components of $\mathbb{C} \setminus (\gamma \cup \mathbb{R})$.

- **$X_+ = X_- J_X$ on $\gamma \cup \mathbb{R}$ with**

$$J_X = \begin{pmatrix} 1 & 0 & e^{n(V_1 - V_2)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \gamma,$$

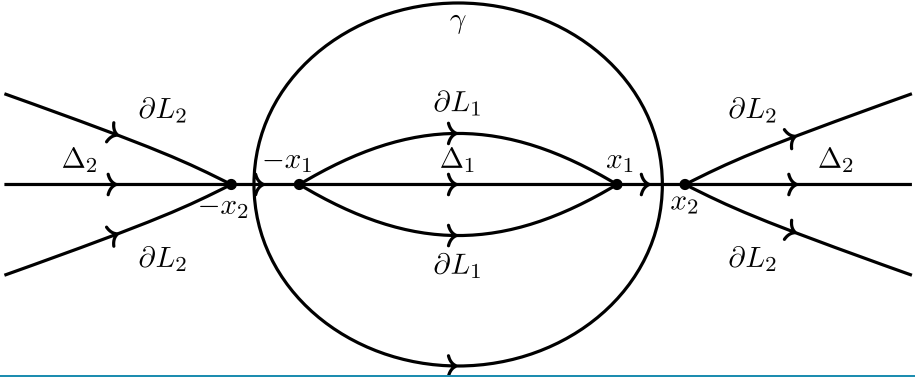
$$J_X = \begin{pmatrix} 1 & e^{nV_1} & 0 \\ 0 & 1 & 0 \\ 0 & -e^{nV_2} & 1 \end{pmatrix} \quad \text{on part of } \mathbb{R} \text{ inside } \gamma$$

$$J_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{nV_2} & 1 \end{pmatrix} \quad \text{on part of } \mathbb{R} \text{ outside } \gamma$$

with $V_1(z) = \frac{c}{t} \log(z^2 + a^2)$ and $V_2(z) = \frac{t+2c}{t} \log z$.

4 Transformations $X \mapsto T \mapsto S$

- $X \mapsto T$ uses the g-functions. It normalizes the RH problem (almost) at infinity.
- $T \mapsto S$ is opening of lenses L_j around Δ_j for $j = 1, 2$
- S has jumps on following contour:



4 Parametrices and transformation $S \mapsto R$

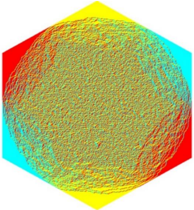
- **Construction of global parametrix**
- **Construction of Airy parametrices around $\pm x_1$ and $\pm x_2$**
- $S \mapsto R$ **leads to** R **with all jump matrices** $I_3 +$ **small as** $n \to \infty$.

Conclusion:

$$R(z) = I_3 + \mathcal{O}\left(\frac{1}{(1+|z|)n}\right) \text{ as } n \to \infty$$

uniformly for $z \in \mathbb{C}$.

Thank you for your attention !

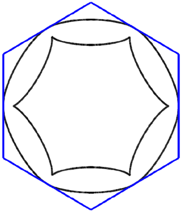


Joint PhD position KU Leuven – UCLouvain

Project title:

Correlations and Gap Probabilities for Random Tilings

Promotors: Tom Claeys (UCLouvain) and Arno Kuijlaars (KU Leuven)



When?	Fall 2024 – Fall 2028.
Where?	In Belgium, 2 years at UCLouvain, 2 years at KU Leuven.
Apply before?	End of June 2024.
How to apply?	Applicants should their application file to tom.claeys@uclouvain.be and arno.kuijlaars@kuleuven.be in the form of a single pdf-file, containing: - a detailed curriculum vitae; - a motivation letter and description of research interests; - the name and e-mail address of at least one academic referee.

More information? Contact tom.claeys@uclouvain.be or arno.kuijlaars@kuleuven.be, or see

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