Jack polynomials and general beta

Peter Forrester @UniMelb

Outline:

- Parameter dependent Gaussian Hermitian random matrices

$$G = |1 - e^{-2\tau}|^{1/2} H + e^{-\tau} H^{(0)}$$

- Generalised hypergeometric functions

$$\int e^{\mathrm{Tr}(\mathbf{A}\mathbf{U}^\dagger \mathbf{B}\mathbf{U})} (\mathbf{U}^\dagger d\mathbf{U}) = {}_0\mathcal{F}_0^{(2/\beta)}(\mathbf{a};\mathbf{b})$$

- Application to an RMT problem from number theory and the special value $(e^2 - 5)/(4\pi)$

(Classical) Gaussian Hermitian random matrices

Joint element PDF $\propto e^{-(\beta/2)\mathrm{Tr}\, H^2}$

Dyson index for entries
$$\begin{cases} \beta = 1, & \text{real} \\ \beta = 2, & \text{complex} \\ \beta = 4, & \text{quaternion} \end{cases}$$

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$$

Diagonalisation

$$H = U \Lambda U^\dagger$$

$$\begin{cases} \text{real orthogonal,} & \beta = 1 \\ \text{unitary,} & \beta = 2 \\ \text{unitary symplectic,} & \beta = 4 \end{cases}$$

Diagonal matrix of eigenvalues

Decomposition of measure

$$(dH) = \prod_{1 \le j < k \le N} |x_k - x_j|^\beta \, dx_1 \cdots dx_N \, (U^\dagger dU)$$

Haar measure on classical matrix groups

$\Longrightarrow$ Eigenvalue PDF

$$\propto e^{-(\beta/2)\sum_{l=1}^N x_l^2} \prod_{1 \le j < k \le N} |x_k - x_j|^\beta$$

Determinantal p.p. $\beta = 2$

Pfaffian p.p. $\beta = 1,4$

$$= e^{-\beta W(\mathbf{x})}$$

Potential energy for log-gas

Boltzmann factor for log-gas

inverse temperature

★ Parameter dependent Gaussian Hermitian random matrices

$$G = |1 - e^{-2\tau}|^{1/2} H + e^{-\tau} H^{(0)}$$

fixed

classical Gaussian

scaled eigenvalues

Joint element PDF

$$\propto e^{-(\beta/2)\mathrm{Tr}\,(G - e^{-\tau}H^{(0)})^2/(1-e^{-2\tau})} = e^{-(1/2)\mathrm{Tr}\,\tilde{\Lambda}^2} e^{-(1/2)\mathrm{Tr}\,(\tilde{\Lambda}^{(0)})^2} e^{\mathrm{Tr}\, U\tilde{\Lambda}U^\dagger\tilde{\Lambda}^{(0)}}$$

$$G = U\Lambda U^\dagger$$

Eigenvalue PDF

With $G = U\Lambda U^\dagger$

Using decomposition of measure as before

$\Longrightarrow$

Eigenvalue PDF

Stuck at this stage?

$$\propto e^{-(\beta/2)\sum_{l=1}^{N}(x_l^2+(e^{-\tau}x_l^{(0)})^2)/(1-e^{-2\tau})} \prod_{1\le j<k\le N} |x_k - x_j|^\beta \int e^{\operatorname{Tr} U\tilde{\Lambda}U^\dagger\tilde{\Lambda}^{(0)}} (U^\dagger dU)$$

HCIZ integral for the case $\beta = 2$

$$\int e^{\mathrm{Tr}(\mathbf{A}\mathbf{U}^\dagger \mathbf{B}\mathbf{U})}[\mathbf{U}^\dagger d\mathbf{U}] = \prod_{j=1}^{N} \Gamma(j) \frac{\det[e^{a_j b_k}]}{\Delta(\mathbf{a})\Delta(\mathbf{b})}$$

Vandermonde product

$\Longrightarrow$ (again) a determinantal point process

Alas, no analogous result for $\beta = 1,4$

What to do?

A generalised heat equation approach

Note that joint element PDF P_τ satisfies

$$= \begin{cases} 1, & \text{diagonal entries} \\ 1/2, & \text{off diagonal entries} \end{cases}$$

Brownian motion harmonic potential

$$\frac{\partial P_\tau}{\partial \tau} = \sum_\mu \frac{\partial}{\partial H_\mu}(H_\mu P_\tau) + \frac{1}{\beta} \sum_\mu D_\mu \frac{\partial^2 P_\tau}{\partial H_\mu^2}$$

Laplace-Beltrami operator on space of Hermitian matrices, metric

$$(ds)^2 = \mathrm{Tr}(d\mathbf{H} d\mathbf{H}) = \sum_{\mu,\nu} g_{\mu\nu} dH_\mu dH_\nu, \qquad \frac{1}{D_\mu}\delta_{\mu,\nu}$$

Fokker-Planck equation

cf. Dyson '63: Langevin eq.

2nd order perturbation theory

Main idea: change variables in the Laplace-Beltrami operator, focussing on eigenvalue dependence

Generally

$$\nabla^2 = \sum_{\mu,\nu} \frac{1}{\sqrt{\det \mathbf{g}}} \frac{\partial}{\partial H_\mu} \sqrt{\det \mathbf{g}} (\mathbf{g}^{-1})_{\mu,\nu} \frac{\partial}{\partial H_\nu}$$

generally these are the coordinates in the metric

Want to use instead coordinates in terms of eigenvalues/ eigenvectors

Eigenvalue/ eigenvector coordinates

$$(ds)^2 = \mathrm{Tr}(dGdG) \qquad \text{with} \quad G = U\Lambda U^\dagger$$

$\Longrightarrow$

Components of $(U^\dagger dU)$

$$(ds)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + \sum_{\substack{j,k=1 \\ j \neq k}}^{N} |\lambda_k - \lambda_j|^2 \sum_{s=1}^{\beta} (\delta u_{jk}^{(s)})^2$$

FP equation for log-gas

$\Longrightarrow$

$$\nabla^2 = \frac{1}{\Delta(\lambda)} \sum_{j=1}^{N} \frac{\partial}{\partial \lambda_j} \left(\Delta(\lambda) \frac{\partial}{\partial \lambda_j} \right) + O_{\mathrm{U}} \Longrightarrow \frac{\partial P_\tau}{\partial \tau} = \mathscr{L} P_\tau,$$

Vandermonde product

$$\mathscr{L} := \frac{1}{\beta} \sum_{j=1}^{N} \frac{\partial^2}{\partial \lambda_j^2} + \sum_{j=1}^{N} \frac{\partial}{\partial \lambda_j} \left(\lambda_j - \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{\lambda_j - \lambda_k} \right)$$

Involves derivatives
w.r.t. eigenvectors

Well defined for general $\beta > 0$

Green function solution

Separation of variables τ and $\mathbf{x}$

Ansatz for PDE in $\mathbf{x}$ steady state solution times a polynomial

$$G_\tau(\mathbf{x}^{(0)};\mathbf{x}) = (\sqrt{\beta/2})^{N+\beta N(N-1)/2} e^{-\beta W(\mathbf{x})} \sum_\kappa \frac{P_\kappa^{(\mathrm{H})}(\mathbf{x}^{(0)}) P_\kappa^{(\mathrm{H})}(\mathbf{x})}{\mathcal{N}_\kappa^{(\mathrm{H})}} e^{-|\kappa|\tau}.$$

Gaussian β ensemble PDF

Multivariable Hermite polynomials

Natural basis Jack polynomials

General β PDF (permits a recursive RMT construction)

partition $(\kappa_1, \ldots, \kappa_N)$

★ Jack polynomials and ${}_0\mathscr{F}_0^{(2/\beta)}(\mathbf{a};\mathbf{b})$

Polynomial eigenfunctions of Laplace-Beltrami for positive definite matrices

after change of variables to eigenvalues/ eigenvectors

$$(ds)^2 = \mathrm{Tr}(\mathbf{X}^{-1}\, d\mathbf{X}\, \mathbf{X}^{-1}\, d\mathbf{X})$$

Well defined for general $\beta > 0$

Dyson index for number field

Generating function

Consider $N = 1$ of sum over Hermite polynomials in Green function:

$$\sum_\kappa \frac{P_\kappa^{(\mathrm{H})}(\mathbf{x}^{(0)}) P_\kappa^{(\mathrm{H})}(\mathbf{x})}{\mathcal{N}_\kappa^{(\mathrm{H})}} e^{-|\kappa|\tau}.$$

The Mahler identity gives

$$\sqrt{1-z^2}\exp\left(-\frac{z^2}{1-z^2}(x^2+y^2)\right)\sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{2^n n!}z^n = \exp\left(\frac{2z}{1-z^2}(xy)\right)$$

$$\sum_{m=0}^{\infty}\frac{x^n\tilde{y}^n}{n!}t^n$$

For general N gives meaning to $\int e^{\mathrm{Tr}(\mathbf{A}\mathbf{U}^\dagger\mathbf{B}\mathbf{U})}(\mathbf{U}^\dagger d\mathbf{U}) = {}_0\mathcal{F}_0^{(2/\beta)}(\mathbf{a};\mathbf{b})$

The monomials are replaced by Jack polynomials

More on ${}_0\mathscr{F}_0^{(2/\beta)}(\mathbf{a};\mathbf{b})$

One has

$$ {}_0\mathscr{F}_0^{(2/\beta)}(\mathbf{a};c\mathbf{1}) = {}_0F_0^{(2/\beta)}(c\mathbf{a}) = \exp\Big(c\sum_{l=1}^{N} a_l\Big) $$

Generally

$$ {}_pF_q^{(2/\beta)}\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix}\,\middle|\,\mathbf{a}\right) $$

is a multivariable Jack polynomial based generalisation of ${}_pF_q\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix}\,\middle|\,a\right)$

This occurs in β ensemble averages (requires more theory)

Dual Cauchy product

e.g. $$ \left\langle \prod_{l=1}^{N}\prod_{k=1}^{m}(x_l - t_k)\right\rangle_{\mathrm{L}\beta\mathrm{E}_N} = {}_1F_1^{(\beta/2)}\left(\begin{matrix}-N\\ a+2m\beta\end{matrix}\,\middle|\,t_1,\dots,t_m\right) $$

★ Application to an RMT problem from number theory

Joint moment $-\frac{1}{2} < s < h + \frac{1}{2}$

real part of characteristic polynomial

circular β ensemble PDF

$$\mathscr{M}_{N,\beta}(s,h) = \frac{1}{C_{N,\beta}} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \left| V_U(\theta)|_{\theta=0} \right|^{2(s-h)} \left| \frac{d}{d\theta} V_U(\theta) \Big|_{\theta=0} \right|^{2h} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^{\beta}$$

Moment of a singular statistic

$$= \frac{2^{-2h}}{C_{N,\beta}} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{l=1}^{N} |1 - e^{i\theta_l}|^{2s} \left| \sum_{j=1}^{N} \frac{\cot \theta_j}{2} \right|^{2h} |\Delta(\mathbf{e}^{\mathbf{i}\theta})|^{\beta}.$$

The special value $(e^2 - 5)/(4\pi)$

In the case $s = 1, h = \frac{1}{2}, \beta = 2$ a result from number theory implied

Conrey and Gosh

$$\lim_{N\to\infty} \frac{1}{N^2} \mathscr{M}_{N,\beta}(s, h) = (e^2 - 5)/(4\pi)$$

RMT proofs Winn '12

Assiotis et al'21

Expression as a Cauchy ensemble average

Moment of trace statistic

$$\mathcal{M}_{N,\beta}(s,h) \propto \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \prod_{j=1}^{N} \frac{1}{(1+x_j^2)^{\alpha_1}} |x_1 + \cdots + x_N|^{2h} \left| \Delta_N(\mathbf{x}) \right|^{\beta} d\mathbf{x}$$

$\beta(N-1)/2 + 1 + s$

Note: Trace statistic for Gaussian and Lagerre β ensemble is easy

- Introduce characteristic function $\quad |x_1 + \cdots + x_N|^{2h} \mapsto e^{it(x_1 + \cdots + x_N)}$
- Establish a structural property function for $\quad s \in \mathbf{Z}_{\geq 0}$

Characteristic function $\quad = e^{-N|t|} Q_{sN}(|t|)$

polynomial degree sN

- The coefficients of the polynomial are shifted Cauchy moments

  Compute using a relation to Jacobi moments (PJF and AAR)

$$Q_{sN}(t) = {}_1F_1^{(2/\beta)}\left(\begin{matrix} -s \\ -2s \end{matrix} \middle| (2t)^N \right)$$

N variables set equal

$$\frac{1}{(1+x^2)^\alpha} \mapsto \frac{1}{(1+ix)^\alpha (1-ix)^\gamma}$$

- Duality formula

$$Q_{sN}(t) = {}_1F_1^{(\beta/2)}\left(\begin{matrix} -N \\ 2s/\beta \end{matrix} \middle| (-4t/\beta)^s \right)$$

s variables set equal

- Scaled limit

  Characteristic function trace statistic $\Big|_{t \mapsto t/N}$

$$\to e^{-|t|} \; {}_0F_1^{(\beta/2)}\left(\begin{matrix} - \\ 4s/\beta \end{matrix} \middle| (4|t|/\beta)^s \right)$$

classical for $s = 1$

- $\implies$

$$\lim_{N\to\infty} \frac{1}{N^{(2/\beta)+1}} \mathscr{F}_{N,\beta}(s,h)\Big|_{s=1,h=1/2} = \frac{\Gamma(2/\beta)}{\Gamma(4/\beta)} \left(\frac{1}{\pi} - \frac{1}{2\pi} \frac{1}{1+\beta/4}\, {}_2F_2(1,1;3,4/\beta+2;4/\beta) \right)$$

$$= \begin{cases} (3e^4 - 103)/(768\pi), & \beta = 1, \\ (e^2 - 5)/(4\pi), & \beta = 2, \\ (e - \sqrt{2})/\sqrt{\pi}, & \beta = 4. \end{cases}$$

References

B. Winn, CMP '12

T. Assiotis et al, PMP '21

PJF, PMP '22