Boundary Green's function, Minkowski content, and bubble measure for $\mathrm{SLE}_\kappa(\rho)$

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Background

- Schramm-Loewner evolution (SLE) is a one parameter family ($\kappa \in (0, \infty)$) of random fractal curves in plane domains, which is characterized by conformal invariance and domain Markov property (DMP).
- There are several types of SLE, among which we focus on chordal SLE and its close relative: chordal $\mathrm{SLE}_\kappa(\rho)$.
- Chordal SLE are first defined in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ using the chordal Loewner equation.

Chordal Loewner Equations

- The chordal Loewner maps g_t driven by $W \in C([0,T),\mathbb{R})$, $T \in (0,\infty]$, are the solution of

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad 0 \le t < T, \quad g_0(z) = z.$$

- For each $z \in \mathbb{C}$, the solution $t \mapsto g_t(z)$ has a maximal interval $[0, \tau_z)$. The chordal Loewner hulls driven by W are the sets $K_t = \{z \in \mathbb{H} : \tau_z \le t\}$, $0 \le t < T$.
- For each t, g_t maps $\mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$.
- If g_t^{-1} extends continuously from $\mathbb{H}$ to $\overline{\mathbb{H}}$, and $\eta(t) := g_t^{-1}(W_t)$, $0 \le t < T$, is a continuous curve, then η is called the chordal Loewner curve driven by W.

Chordal SLE

- When $W_t := \sqrt{\kappa} B_t$, where $\kappa > 0$ and B_t is a standard Brownian motion, the Loewner curve η is known to exist and is called a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞ ([Rohde-Schramm]).
- If f maps $\mathbb{H}$ conformally onto D, then $f \circ \eta$ modulo time parametrization is called a chordal SLE$_\kappa$ curve in D from $f(0)$ to $f(\infty)$ (as prime ends).
- A chordal SLE$_\kappa$ curve η in D from a to b satisfies the following DMP. If τ is a stopping time, then conditional on $\eta|_{[0,\tau]}$ and the event that η is incomplete at τ, the rest of η is a chordal SLE$_\kappa$ from $\eta(\tau)$ to b in a connected component of $D \setminus \eta[0,\tau]$.

$\text{SLE}_\kappa(\rho)$ Curves

- $\text{SLE}_\kappa(\rho)$ was introduced by Lawler, Schramm and Werner.
- To define $\text{SLE}_\kappa(\rho)$, one solves a system of SDE/ODE to find the driving process and the force point process(es).
- Let $\kappa > 0$ and $\rho \in \mathbb{R}$. Let $w \in \mathbb{R}$ and $v \in (\mathbb{R} \setminus \{w\}) \cup \{w^+, w^-\}$. Let W_t and V_t be the solution of

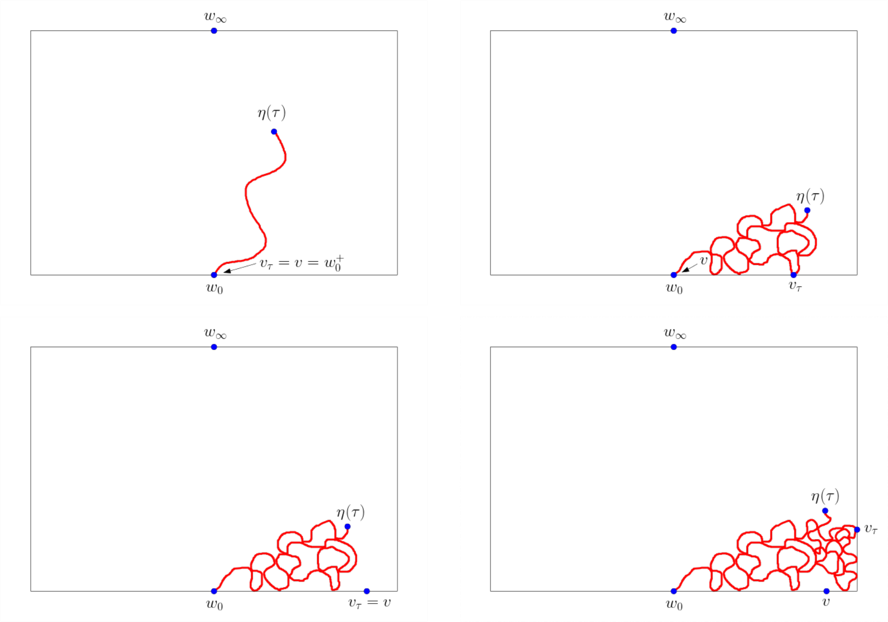
$$dW_t = \sqrt{\kappa} dB_t + \mathbb{1}_{\{W_t \neq V_t\}} \frac{\rho dt}{W_t - V_t}, \quad dV_t = \mathbb{1}_{\{W_t \neq V_t\}} \frac{2dt}{V_t - W_t}$$

  with initial values $W_0 = w$ and $V_0 = v$.
- The chordal Loewner curve driven by W exists ([Miller-Sheffield]), and is called an $\text{SLE}_\kappa(\rho)$ curve in $\mathbb{H}$, started from w, with the force point v, and aimed at ∞.
- The definition extends to general simply connected domains via conformal maps.

DMP for $\text{SLE}_\kappa(\rho)$

The DMP for $\text{SLE}_\kappa(\rho)$ involves the force point. Suppose η is an $\text{SLE}_\kappa(\rho)$ in D from w_0 to w_∞ with the force point v lying on the ccw (resp. cw) arc from w_0 to w_∞. Let τ be a stopping time for η. Conditionally on the part of η before τ and the event that η is incomplete at τ, the rest of η is an $\text{SLE}_\kappa(\rho)$ curve

- in the component of $D \setminus \eta[0,\tau]$ whose boundary contains w_∞
- started from $\eta(\tau)$ and aimed at w_∞,
- with the force point located at v_τ, which equals
  - v, if $\eta[0,\tau]$ does not disconnect v from w_∞;
  - the ccw(resp. cw)-most prime end visited by η by τ, if otherwise.


w_∞
$\eta(\tau)$
$v_\tau = v = w_0^+$
w_0
w_∞
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v
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w_∞
$\eta(\tau)$
w_0
$v_\tau = v$
w_∞
$\eta(\tau)$
v_τ
w_0
v

Boundary Intersection

Let η be an $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ started from w with force point $v \geq w^+$. We assume $\rho > -2$ so that η approaches ∞. There are three cases.

- (Boundary-avoiding) If $\rho \geq \frac{\kappa}{2} - 2$, then $\eta \cap (v, \infty) = \emptyset$.
- (Boundary-filling) If $\rho \leq \frac{\kappa}{2} - 4$, then $\eta \cap (v, \infty) = (v, \infty)$.
- (Boundary-touching) If $\rho \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$, then $\eta \cap (v, \infty)$ is unbounded and has no interior points.

In the third case, $\dim_{\mathcal{H}}(\eta \cap (v, \infty)) = \frac{(\rho+4)(\kappa-4-2\rho)}{2\kappa}$ by Miller and Wu. We focus on the Minkowski content of the intersection.

Minkowski Content

- Given a set $S \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $d \in (0, n)$, and $r > 0$, we define

$$\mathrm{Cont}_d(S, r) = r^{d-n}\,\mathrm{m}^n(S^r),$$

where $S^r = \{x \in \mathbb{R}^n : \mathrm{dist}(x, S) < r\}$ and m^n is the Lebesgue measure. Then $\mathrm{Cont}_d(S) := \lim_{r \to 0^+} \mathrm{Cont}_d(S, r)$ is called the d-dim Minkowski content of S, if the limit exists.

Minkowski Content

Some examples:

- If B_t is an n-dim Brownian motion, $n \geq 3$, then for some constant $C_n \in (0, \infty)$, $\mathrm{Cont}_2(B[0,t]) = C_n t$ for all $t \geq 0$.
- If B_t is a 1-dim BM with the local time process L_t, then for some $C_1 \in (0, \infty)$, $\mathrm{Cont}_{1/2}([0,t] \cap B^{-1}(0)) = C_1 L_t$ for all $t \geq 0$.
- Let η be an SLE_κ curve, $\kappa \in (0,8)$, in $\mathbb{H}$ from 0 to ∞, then for any $t_2 > t_1 \geq 0$, $\mathrm{Cont}_d(\eta([t_1, t_2]))$ exists with $d = 1 + \frac{\kappa}{8}$ ([Lawler-Rezaei]). When $\kappa \in (4,8)$, for any compact real interval I, $\mathrm{Cont}_d(\eta \cap I)$ exists with $d = 2 - \frac{8}{\kappa}$ ([Lawler]).
- For $n = 2, 3$, let S be the set of cut points of an n-dim BM. Then for nice compact set $V \subset \mathbb{R}^n$, $\mathrm{Cont}_d(S \cap V)$ exists, where $d = \dim_{\mathcal{H}}(S)$ ([Holden-Lawler-Li-Sun]).

Minkowski Content Measure

Minkowski content is not a measure. In order to refer to Minkowski content as a measure, a rigorous definition is needed. For this purpose, we introduce the "Minkowski content measure".

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Definition

Let $\mathcal{S}$ denote the family of subsets of $\mathbb{R}^d$, which are intersection of a closed set and an open set. For $S \in \mathcal{S}$, a measure μ on S is called a d-dimensional Minkowski content measure (MCM) on S if for any compact set $K \subset S$,

(i) $\mu(K) < \infty$; and

(ii) if $\mu(\partial_S K) = 0$, then $\mathrm{Cont}_d(K) = \mu(K)$.

The examples in the previous slide are all MCM.

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- **Restriction.** Let $S_1 \subset S_2 \in \mathcal{S}$. If μ is the d-dim MCM on S_2, and $\mu(\partial_{S_2} S_1) = 0$, then $\mu|_{S_1}$ is the d-dim MCM on S_1.

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- **Weak limit.** If S is compact, then μ is the d-dim MCM on S iff it is the weak limit of $r^{d-n} \mathbb{1}_{S^r} \cdot \mathrm{m}^n$ as $r \to 0^+$.

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- **Weak limit.** If S is compact, then μ is the d-dim MCM on S iff it is the weak limit of $r^{d-n} \mathbb{1}_{S^r} \cdot \mathrm{m}^n$ as $r \to 0^+$.
- **Conformal covariance.** If $S \in \mathcal{S}$ has d-dim MCM μ and is contained in an open set U, and if $\phi : U \to \mathbb{R}^n$ is an injective conformal map, then $\phi_*(|\phi'|^d \cdot \mu)$ is the d-dim MCM on $\phi(S)$.

Minkowski Content Measure

Theorem

Let η be an $SLE_\kappa(\rho)$ curve in $\mathbb{H}$ started from w with force point $v \geq w^+$. Suppose ρ is in the boundary-touching region. Let $d = \dim_{\mathcal{H}}(\eta \cap (v, \infty))$. Then the d-dim MCM on $\eta \cap (v, \infty)$ exists and is atomless.

Alberts and Sheffield constructed a conformal covariant measure on the intersection of SLE_κ in $\mathbb{H}$ with $\mathbb{R}$, and conjectured that it is related to Minkowski content. This theorem confirms their conjecture.

Green's Function

- The proof follows the Green's function approach introduced by Lawler and Rezaei. Let $\alpha = 1 - d$. We need to prove the existence of the 1-point and 2-point Green's function at $x \neq y \in (v, \infty)$:

$$G(x) = \lim_{r \to 0^+} r^{-\alpha} \mathbb{P}[\mathrm{dist}(x, \eta) < r],$$
$$G(x, y) = \lim_{r,s \to 0^+} r^{-\alpha} s^{-\alpha} \mathbb{P}[\mathrm{dist}(x, \eta) < r, \mathrm{dist}(y, \eta) < s].$$

- In addition, we need some sharp estimates of the probabilities:

$$\mathbb{P}[\mathrm{dist}(x, \eta) < r] = G(x) r^{\alpha} (1 + O(r)),$$
$$\mathbb{P}[\mathrm{dist}(x, \eta) < r, \mathrm{dist}(y, \eta) < s] = G(x, y) r^{\alpha} s^{\alpha} (1 + O(r + s)).$$

Radial Bessel Processes

The work on the Green's functions relies on radial Bessel processes.

- Let $\delta \in \mathbb{R}$. A diffusion process $\Theta_t \in [0, \pi]$ satisfying the SDE

$$d\Theta_t = dB_t + \frac{\delta - 1}{2} \cot(\Theta_t) dt$$

  is called a radial Bessel process of dim δ (RBES$^\delta$)
- The name comes from the fact that (i) it appears naturally from radial SLE$_\kappa(\rho)$ processes, and (ii) Θ behaves like a BES$^\delta$ when it is near 0 or π.
- When $\delta = n \in \mathbb{N}$, RBES^n is related to Brownian motion on sphere S^n in the same way as BES^n is related to Brownian motion on $\mathbb{R}^n$.
- The process $Z_t := \cos(\Theta_t) \in [-1, 1]$ is called a cosined radial Bessel process of dim δ (CRBES$^\delta$), which satisfies the SDE

$$dZ_t = \sqrt{1 - Z_t^2} dB_t - \frac{\delta}{2} Z_t dt.$$

Radial Bessel Processes

- A diffusion process $\Theta_t \in [0, \pi]$ satisfying the SDE

$$d\Theta_t = dB_t + \frac{\delta_+ - 1}{4} \cot(\frac{\Theta_t}{2})dt + \frac{\delta_- - 1}{4} \tan(\frac{\Theta_t}{2})dt$$

  is called a $\mathrm{RBES}^{(\delta_+, \delta_-)}$. It behaves like a Bessel process of dim δ_+ near 0 and of dim δ_- near π.
- The $\mathrm{CRBES}^{(\delta_+, \delta_-)}$ process $Z_t := \cos(\Theta_t)$ satisfies the SDE

$$dZ_t = \sqrt{1 - Z_t^2} dB_t - \frac{\delta_+}{4}(Z_t + 1)dt - \frac{\delta_-}{4}(Z_t - 1)dt.$$

- The transition density of Z is known, and is expressed as an infinite series involving Jacobi polynomials.
- When $\delta_+, \delta_- > 0$, the lifetime of Z is ∞, and has a unique stationary measure, which has a density that is proportional to $(1 - x)^{\frac{\delta_+}{2} - 1}(1 + x)^{\frac{\delta_-}{2} - 1}$.

Radial Bessel Processes and $\mathrm{SLE}_\kappa(\underline{\rho})$

Lemma

Consider an $SLE_\kappa(\rho_v, \rho_u)$ curve η in $\mathbb{H}$ started from w with force points $v < u \in [w^+, \infty)$. Define

$$Z_t = \frac{W_t + U_t - 2V_t}{U_t - W_t}, p(t) = -\frac{\kappa}{2}\log\left(\frac{U_t - V_t}{g_t'(u)(u-v)}\right), \widehat{Z}_s = Z_{p^{-1}(s)}.$$

Then $(\widehat{Z}_s)_{s\geq 0}$ is a $CRBES^{(\delta_+,\delta_-)}$ with $\delta_+ = \frac{4}{\kappa}(\rho_v + 2)$ and $\delta_- = \frac{4}{\kappa}(\kappa - 4 - \rho_v - \rho_u)$. We may recover W from $(\widehat{Z}_s)_{s\in\mathbb{R}}$ and w, v, u.

We use this lemma to study $\mathrm{SLE}_\kappa(\rho)$ with $\rho > (\frac{\kappa}{2} - 4) \vee (-2)$ by setting $\rho_v = \rho$ and $\rho_u = 0$. From $\rho > \frac{\kappa}{2} - 4$ we know that η a.s. does not hit u. The $\widehat{Z}$ a.s. hits -1 because $\delta_+ > 0$ and $\delta_- < 2$.

One-point Green's Function

- Applying the lemma to $\rho_v = \rho$ and $\rho_u^* = \kappa - 8 - 2\rho$, we get $\widehat{Z}^*$ with dim (δ_+, δ_-^*), where $\delta_-^* = \frac{4}{\kappa}(\rho + 4) = 4 - \delta_- > 2$.
- Note that δ_-^* is the dual dimension of δ_-. The $\widehat{Z}^*$ could be understood as the $\widehat{Z}$ conditioned to not hit -1, and the $\mathrm{SLE}_\kappa(\rho, \kappa - 8 - 2\rho)$ curve could be understood as the $\mathrm{SLE}_\kappa(\rho)$ curve conditioned to pass through u.
- The above facts could be used to derive the 1-pt Green's function for the $\mathrm{SLE}_\kappa(\rho)$. In fact, the Green's function is closely related to the RN process between $\mathrm{SLE}_\kappa(\rho, \kappa - 8 - 2\rho)$ and $\mathrm{SLE}_\kappa(\rho)$.

Two-point Green's Function

For the 2-point Green's function, we use following philosophy introduced by Lawler and Werness. Assume that $u_2 > u_1 > v$.

- In order for η to reach two discs $D_1 = D(u_1, r_1)$ and $D_2 = D(u_2, r_2)$, it is very likely that η visits D_1 before D_2.
- The probability that η reaches D_1 is approximated by 1-pt Green's function.
- Then we could use DMP of $\mathrm{SLE}_\kappa(\rho)$, Koebe's distortion theorem and 1-pt Green's function to estimate the conditional probability that the rest of η reaches D_2.
- We need to show that, with a high probability, η is not close to u_2 when it reaches D_1. This requires the most technical work.

Decomposition Formula

- Let $\mu^{\mathbb{H}}_{(w,v)\to\infty}$ denote the law of the above $\mathrm{SLE}_\kappa(\rho)$ curve.
- For $u \in I = (v, \infty)$, let $\mu^{\mathbb{H}}_{(w,v)\to u\to\infty}$ denote the law of a random curve with two arms: the first arm from w to u is an $\mathrm{SLE}_\kappa(\rho, \kappa - 8 - 2\rho)$ curve with force points (v, u), and the second arm conditioned on the first is an $\mathrm{SLE}_\kappa(\rho)$ curve from u to ∞.
- We understand $\mu^{\mathbb{H}}_{(w,v)\to u\to\infty}$ as the $\mathrm{SLE}_\kappa(\rho)$ conditioned to hit u.
- Let $\mathcal{M}_{\eta\cap I}$ denote the d-dim MCM on $\eta \cap I$.
- Let $G(w, v; u)$ be the 1-pt Green's function for $\mu^{\mathbb{H}}_{(w,v)\to\infty}$ at u.
- Then we have the following decomposition formula:

$$\mu^{\mathbb{H}}_{(w,v)\to\infty}(d\eta)\overrightarrow{\otimes}\mathcal{M}_{\eta\cap I}(du) = \mu^{\mathbb{H}}_{(w,v)\to u\to\infty}(d\eta)\overleftarrow{\otimes}G(w, v; u)\cdot \mathrm{m}(du).$$

SLE Loop Measures

We now discuss $\mathrm{SLE}_\kappa(\rho)$ bubble measures, which resemble SLE loop measures.

- SLE loop measures are **single**-loop measures, which may be rooted or unrooted.
- Werner constructed (2008) unrooted $\mathrm{SLE}_{8/3}$ loop measures in $\widehat{\mathbb{C}}$.
- Sheffield and Werner constructed (2012) rooted SLE_κ bubbles for $\kappa \in (\frac{8}{3}, 4]$ in the exploration of CLE_κ (**multi**-loop measures).
- Miller and Sheffield constructed (2012) space-filling SLE_κ for $\kappa > 4$ from ∞ to ∞.
- Benoist and Dubédat constructed (2016) unrooted SLE_2 loops.

SLE Loop Measures

- Several types of SLE_κ loop measures for $\kappa \in (0, 8)$ were systematically constructed (Z. 2021), which include
  - Rooted SLE_κ loops in $\widehat{\mathbb{C}}$.
  - Unrooted SLE_κ loops in $\widehat{\mathbb{C}}$.
  - Boundary-avoiding unrooted SLE_κ loops in arbitrary domains.
  - Rooted SLE_κ bubbles in simply connected domains.
  - Unrooted SLE_κ bubbles for $\kappa \in (4, 8)$.
- For $\kappa \in (\frac{8}{3}, 4]$, the unrooted SLE_κ loop measure in $\widehat{\mathbb{C}}$ is some constant times the intensity measure of whole-plane CLE_κ ([Kemppainen-Werner, 15], [Ang-Sun, 22]).

Domain Markov Property

An SLE_κ loop η in $\widehat{\mathbb{C}}$ rooted at w satisfy DMP as follows. Let τ be a positive stopping time. Then conditionally on the part of η before τ, and the event that η is incomplete at τ, the rest of η is an SLE_κ

- in the connected component of $D \setminus \eta[0,\tau]$ whose boundary contains both $\eta(\tau)$ and w
- from (the prime end) $\eta(\tau)$ to (the prime end) w.

The rooted SLE_κ loop measures are infinite measures, and we have to use some symbols to make the above statement rigorous.

Domain Markov Property

- Let $\mu^D_{a\to b}$ denote the law of a chordal SLE$_\kappa$ in D from a to b.
- Let $\Sigma = \bigcup_{\widehat{T}\in(0,\infty]} C([0,\widehat{T}),\widehat{\mathbb{C}})$.
- For any stopping time τ, let $\Sigma_\tau = \{f \in \Sigma : \tau(f) < \widehat{T}_f\}$.
- The SLE$_\kappa$ loop measure in $\widehat{\mathbb{C}}$ rooted at w is a sigma-finite measure μ_w on Σ, which satisfies that, for any stopping time $\tau > 0$,

$$\mathbb{1}_{\Sigma_\tau}\mu_w = \mathbb{1}_{\Sigma_\tau}\mu_w(d\eta) \oplus_\tau \mu^{\widehat{\mathbb{C}}(\eta[0,\tau];w)}_{\eta(\tau)\to w},$$

where $\widehat{\mathbb{C}}(\eta[0,\tau];w)$ is the unique connected component of $\widehat{\mathbb{C}} \setminus \eta[0,\tau]$ whose boundary contains w, and $\eta(\tau)$ and w are identified with two prime ends of $\widehat{\mathbb{C}} \setminus \eta[0,\tau]$.

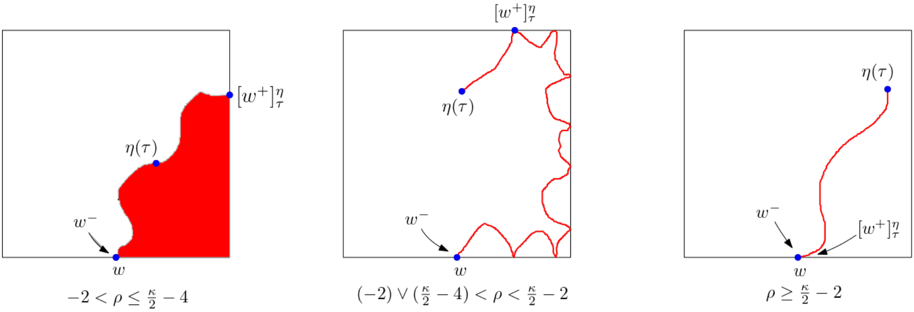
DMP for $\mathrm{SLE}_\kappa(\rho)$ Bubbles

$\mathrm{SLE}_\kappa(\rho)$ bubbles are closely related to SLE_κ bubbles. There are ccw and cw bubbles. A ccw $\mathrm{SLE}_\kappa(\rho)$ bubble η in D rooted at $w \in \partial D$ satisfies the following DMP. Let τ be a positive stopping time. Then conditional on the part of η before τ and the event that η is incomplete at τ, the rest of η is an $\mathrm{SLE}_\kappa(\rho)$ curve

- in the connected component of $D \setminus \eta[0,\tau]$ which shares with D a prime end arc lying immediately to the left of w
- from (the prime end) $\eta(\tau)$ to (the prime end) w^-
- with the force point $[w^+]^\eta_\tau$ being the ccw-most prime end visited by η by time τ.

The above statement could be easily made rigorous using symbols.

DMP for $\mathrm{SLE}_\kappa(\rho)$ Bubbles



The above three figures respectively represent the boundary-filling, boundary-touching and boundary-avoiding cases.

Construction of the $\mathrm{SLE}_\kappa(\rho)$ Bubbles

- It suffices to construct ccw $\mathrm{SLE}_\kappa(\rho)$ bubble in $\mathbb{H}$ rooted at 0, which could be intuitively understood as an $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ from 0 to 0^- with the force point 0^+.
- Before the construction, we make the following observation. Let $x < 0 < 0^+ \le v$, and η is an $\mathrm{SLE}_\kappa(\rho)$ curve in $\mathbb{H}$ started from 0, aimed at x, and with the force point v.
- Let σ be the hitting time at $\{\infty\} \cup (-\infty, x]$. Then by SLE coordinate changes ([Schramm-Wilson]), the part of η before σ is a two-force-point $\mathrm{SLE}_\kappa(\rho, \kappa - 6 - \rho)$ curve in $\mathbb{H}$ started from 0, with the force points (v, x), and stopped when it swallows x.

Construction of $\mathrm{SLE}_\kappa(\rho)$ Bubbles

- We first consider the easy case: $\rho \in (-2, \frac{\kappa}{2} - 4]$. Then $\kappa - 6 - \rho \geq \frac{\kappa}{2} - 2 > -2$. Thus, we may define an $\mathrm{SLE}_\kappa(\rho, \kappa - 6 - \rho)$ curve γ in $\mathbb{H}$ started from 0 with the force points $(0^+, 0^-)$.
- The γ ends at ∞ and does not intersect $\mathbb{R}_-$ as $\kappa - 6 - \rho \geq \frac{\kappa}{2} - 2$.
- To complete the construction of the bubble, we simply continue it with an $\mathrm{SLE}_\kappa(\rho)$ curve in the remaining domain from ∞ to 0 with the force point being ∞^+.

Radial Bessel Processes and $\mathrm{SLE}_\kappa(\underline{\rho})$

For the case $\rho > (\frac{\kappa}{2} - 4) \vee (-2)$, we will use the following lemmas

Lemma

Let η be an $SLE_\kappa(\rho_+, \rho_-)$ curve started from 0 with force points $v^+ \in [0^+, \infty)$ and $v^- \in (-\infty, 0)$. Define

$$Z_t = \frac{2W_t - V_t^+ - V_t^-}{V_t^+ - V_t^-}, \quad p(t) = \frac{\kappa}{2} \log\left(\frac{V_t^+ - V_t^-}{v^+ - v^-}\right), \quad \widehat{Z}_s = Z_{p^{-1}(s)}.$$

Then $(\widehat{Z}_s)_{s \geq 0}$ is a $CRBES^{(\delta_+, \delta_-)}$, where $\delta_\pm = \frac{4}{\kappa}(\rho_\pm + 2)$. We may recover W using $\widehat{Z}$ and w, v^+, v^-.

This lemma will be applied to the case $\rho_+ = \rho$ and $\rho_- = \kappa - 6 - \rho$. In that case, we have $\delta_+ > 0$ and $\delta_- < 2$.

Radial Bessel Processes and $\mathrm{SLE}_\kappa(\underline{\rho})$

Another lemma we need is the following.

Theorem

Let η be an $SLE_\kappa(\rho_+, \rho_-^*)$ curve η in $\mathbb{H}$ started from 0 with force points $0^+, 0^-$, where $\rho_+, \rho_-^* > -2$. Define

$$Z_t = \frac{2W_t - V_t^+ - V_t^-}{V_t^+ - V_t^-}, \quad p(t) = \frac{\kappa}{2}\log(V_t^+ - V_t^-), \quad \widehat{Z}_s = Z_{p^{-1}(s)}.$$

Then $(\widehat{Z}_s)_{s\in\mathbb{R}}$ is a **stationary** *$CRBES^{(\delta_+, \delta_-^*)}$, where $\delta_+ = \frac{4}{\kappa}(\rho_+ + 2)$ and $\delta_-^* = \frac{4}{\kappa}(\rho_-^* + 2)$. We may recover W using $\widehat{Z}$.*

This lemma will be applied to the case $\rho_+ = \rho$ and $\rho_-^* = \rho + 2$. In that case, $\delta_+ > 0$ and $\delta_-^* = 4 - \delta_-$ is dual to δ_-.

Radial Bessel Processes

Using the dual relation between $\text{CRBES}^{(\delta_+,\delta_-)}$ and $\text{CRBES}^{(\delta_+,\delta_-^*)}$, we may construct an **infinite** measure $\mu^{\mathbb{R}}_{\delta_+,\delta_-}$ on $\Sigma^{\mathbb{R}} := \bigcup_{T\in\mathbb{R}} C((-\infty,T),\mathbb{R})$ such that a process X with the "law" $\mu^{\mathbb{R}}_{\delta_+,\delta_-}$ satisfies the following Markov property:

- If τ is a finite stopping time, then conditionally on the part of X before τ and the event that X is not complete by τ, the rest part of X is a $\text{CRBES}^{(\delta_+,\delta_-)}$.

Such $\mu^{\mathbb{R}}_{\delta_+,\delta_-}$ is locally absolutely continuous w.r.t. the law $\mu^{\mathbb{R}}_{\delta_+,\delta_-^*}$ of the stationary $\text{CRBES}^{(\delta_+,\delta_-^*)}$ defined on $\mathbb{R}$.

Construction of $\mathrm{SLE}_\kappa(\rho)$ Bubbles

- The last lemma gives an injective map $\Phi : W \mapsto \widehat{Z}$ such that the law $\mu^{0;(0^+,0^-)}_{\kappa;(\rho_+,\rho_-^*)}$ of the driving function of an $\mathrm{SLE}_\kappa(\rho_+,\rho_-)$ curve started from 0 with force points $(0^+,0^-)$ under the pushforward Φ is $\mu^{\mathbb{R}}_{\delta_+,\delta_-^*}$, i.e., $\Phi_*(\mu^{0;(0^+,0^-)}_{\kappa;(\rho_+,\rho_-^*)}) = \mu^{\mathbb{R}}_{\delta_+,\delta_-^*}$.
- Now we define $\mu^{0;(0^+,0^-)}_{\kappa;(\rho_+,\rho_-)} = \Phi_*^{-1}(\mu^{\mathbb{R}}_{\delta_+,\delta_-})$. Then $\mu^{0;(0^+,0^-)}_{\kappa;(\rho_+,\rho_-)}$ is an infinite measure on driving processes. The pushforward measure of $\mu^{0;(0^+,0^-)}_{\kappa;(\rho_+,\rho_-)}$ on curves under the Loewner map satisfies the $\mathrm{SLE}_\kappa(\rho)$-bubble-DMP up to the hitting time on $(-\infty,0]$.
- If $\rho \geq \frac{\kappa}{2}-2$, then η a.s. avoids $(-\infty,0)$ and ends at 0. Then we already get the $\mathrm{SLE}_\kappa(\rho)$ bubble measure.
- If $\rho < \frac{\kappa}{2}-2$, then curve a.s. ends at some point $x \in (-\infty,0)$. To complete the construction, we continue the curve with an $\mathrm{SLE}_\kappa(\rho)$ curve from x to 0 with the force point x^+.

Weak Convergence and Reversibility

- In the case $\rho \leq \frac{\kappa}{2} - 4$, the $\mathrm{SLE}_\kappa(\rho)$ law $\mu^{\mathbb{H}}_{(r;r^+)\to -r}$ converges weakly to the law $\mu^{\mathbb{H}}_{0\circlearrowleft}$ of the ccw $\mathrm{SLE}_\kappa(\rho)$ bubble in $\mathbb{H}$ rooted at 0 as $r \to 0^+$.
- In the case $\rho > \frac{\kappa}{2} - 4$, $(2r)^{-\alpha}\mu^{\mathbb{H}}_{(r;r^+)\to -r}$ converges weakly to $\mu^{\mathbb{H}}_{0\circlearrowleft}$, where
$$\alpha = \frac{(\rho+2)(2\rho+8-\kappa)}{2\kappa}.$$
- The above weak convergence implies that, if $\mathrm{SLE}_\kappa(\rho)$ satisfies reversibility, say $\kappa \in (0,8)$ and $\rho \geq \frac{\kappa}{2} - 4$, then the time-reversal of the ccw bubble measure $\mu^{\mathbb{H}}_{0\circlearrowleft}$ is the cw bubble measure $\mu^{\mathbb{H}}_{0\circlearrowright}$.

Minkowski Content

- In the case that $\kappa \in (0,8)$ and ρ is in the boundary touching region, a curve η following the "law" $\mu^{\mathbb{H}}_{0\circlearrowleft}$ intersects $\mathbb{R}$ at a compact fractal set.
- Using the existence of boundary MCM for $\mathrm{SLE}_\kappa(\rho)$ we could prove that $\eta \cap \mathbb{R}$ possesses MCM.
- Moreover, we get a decomposition equality for $\mu^{\mathbb{H}}_{w\circlearrowleft}$, $w \in \mathbb{R}$:

$$\mu^{\mathbb{H}}_{w\circlearrowleft}(d\eta)\overrightarrow{\otimes}\mathcal{M}^{d}_{\eta\cap\mathbb{R}}(du) = \mu^{\mathbb{H}}_{w\leftrightarrows u}(d\eta)\overleftarrow{\otimes}C_{\kappa,\rho}|w-u|^{-2\alpha}\cdot \mathrm{m}(du),$$

where $C_{\kappa,\rho} \in (0,\infty)$ is a constant depending only on κ, ρ, and $\mu^{\mathbb{H}}_{w\leftrightarrows u}$ could be understood as an $\mathrm{SLE}_\kappa(\rho)$ bubble rooted at w conditioned to pass through u.

Unrooted $\mathrm{SLE}_\kappa(\rho)$ Bubbles

- Looking at the marginal measures in the decomposition equality for $\mu^{\mathbb{H}}_{w\circlearrowleft}$, we find that

$$\mathrm{Cont}_d(\eta \cap \mathbb{R}) \cdot \mu^{\mathbb{H}}_{w\circlearrowleft}(d\eta) = C_{\kappa,\rho} \int_{\mathbb{R}\setminus\{w\}} \mu^{\mathbb{H}}_{w\leftrightarrows u} |u|^{-2\alpha}\, \mathrm{m}(du).$$

  Thus, $\mu^{\mathbb{H}}_{w\circlearrowleft}$ may be expressed by

$$\mu^{\mathbb{H}}_{w\circlearrowleft} = \mathrm{Cont}_d(\cdot \cap \mathbb{R})^{-1} \cdot C_{\kappa,\rho} \int_{\mathbb{R}\setminus\{w\}} \mu^{\mathbb{H}}_{w\leftrightarrows u} |w-u|^{-2\alpha}\, \mathrm{m}(du).$$

- This equality motivates us to define the unrooted ccw $\mathrm{SLE}_\kappa(\rho)$ bubble measure $\mu^{\mathbb{H}}_{\circlearrowleft}$ by

$$\mu^{\mathbb{H}}_{\circlearrowleft} = \mathrm{Cont}_d(\cdot \cap \mathbb{R})^{-1} \cdot \int_{\mathbb{R}} \mu^{\mathbb{H}}_{w\circlearrowleft}\, \mathrm{m}(dw)$$

$$= \mathrm{Cont}_d(\cdot \cap \mathbb{R})^{-2} \int_{\mathbb{R}^2} \mu^{\mathbb{H}}_{w\leftrightarrows u} C_{\kappa,\rho} |w-u|^{-2\alpha}\, \mathrm{m}^2(dw \otimes du).$$

Decomposition of Unrooted $\mathrm{SLE}_\kappa(\rho)$ Bubble

We get two decomposition formulas for the unrooted bubble measure:

$$\mu^{\mathbb{H}}_{\circlearrowleft}(d\eta)\overrightarrow{\otimes}\mathcal{M}^d_{\eta\cap\mathbb{R}}(dw) = \mu^{\mathbb{H}}_{w\circlearrowleft}(d\eta)\overleftarrow{\otimes}\,\mathrm{m}(dw).$$

$$\mu^{\mathbb{H}}_{\circlearrowleft}(d\eta)\overrightarrow{\otimes}(\mathcal{M}^d_{\eta\cap\mathbb{R}})^2(dw\otimes du) = \mu^{\mathbb{H}}_{w\leftrightarrows u}(d\eta)\overleftarrow{\otimes}C_{\kappa,\rho}|w-u|^{-2\alpha}\cdot\mathrm{m}^2(dw\otimes du).$$

Conformal Invariance/Covariance

- If $\rho \le \frac{\kappa}{2} - 4$, the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measures satisfy conformal invariance. Suppose f is a Möbius automorphism of $\mathbb{H}$ that fixes $w \in \mathbb{R}$. Then

$$f_*(\mu^{\mathbb{H}}_{w\circlearrowleft}) = \mu^{\mathbb{H}}_{w\circlearrowleft}.$$

- If $\rho > (-2) \vee (\frac{\kappa}{2} - 4)$, then we have conformal covariance:

$$f_*(\mu^{\mathbb{H}}_{w\circlearrowleft}) = f'(w)^\alpha \mu^{\mathbb{H}}_{w\circlearrowleft}.$$

- The unrooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure $\mu^{\mathbb{H}}_{\circlearrowleft}$ satisfies conformal invariance:

$$f_*(\mu^{\mathbb{H}}_{\circlearrowleft}) = \mu^{\mathbb{H}}_{\circlearrowleft}$$

  if f is any Möbius automorphism of $\mathbb{H}$.

Thank you for your attention!