Regularity of SLE trace

Yizheng Yuan

University of Cambridge

partly joint work with Nina Holden

06 June 2023

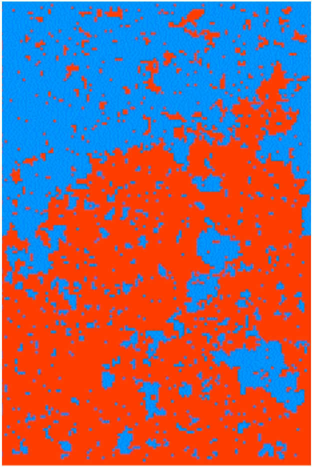


Figure: Ising model at critical temperature (by H. Duminil-Copin and S. Smirnov)

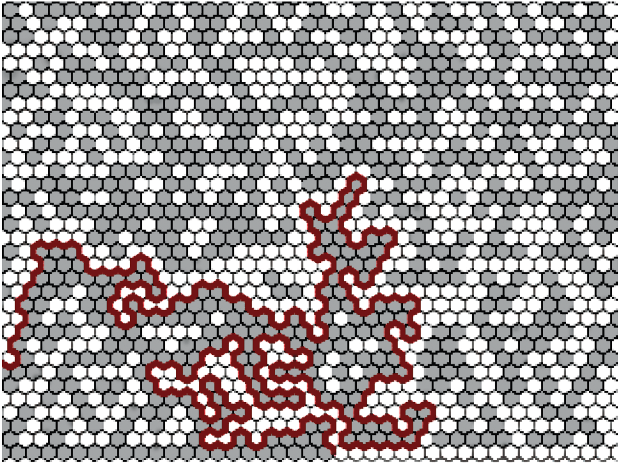


Figure: critical Bernoulli percolation (by C. Garban, G. Pete, O. Schramm)

SLE (Schramm-Löwner evolution)

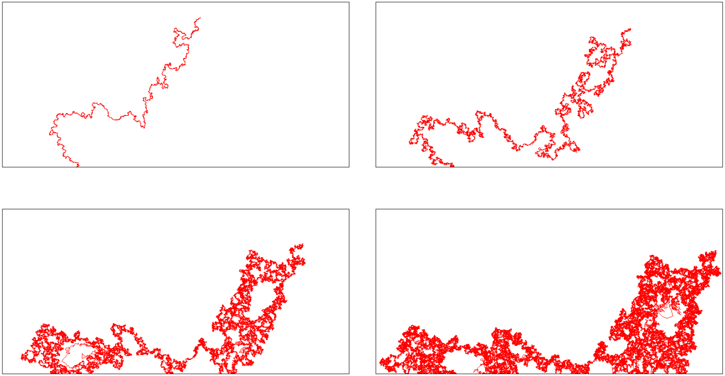


Figure: Simulations by Tom Kennedy

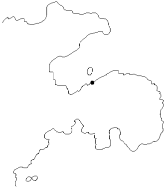
Outline

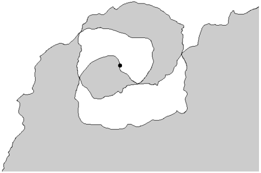
Objects in this talk:

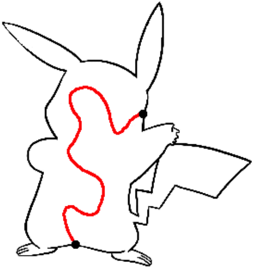
- chordal SLE in a domain
- two-sided whole-plane SLE

$\kappa > 0$ (either space-filling or non-space-filling in case $\kappa \in]4, 8[$),

$d = 1 + \kappa/8$ resp. $d = 2$ dimension of the curve.







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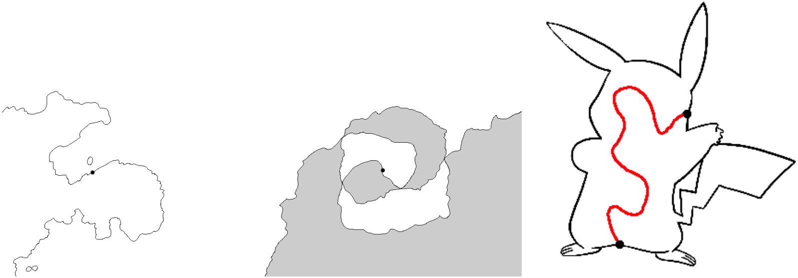

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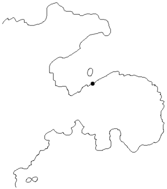

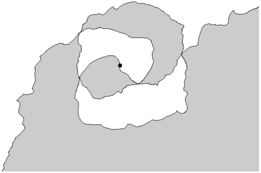

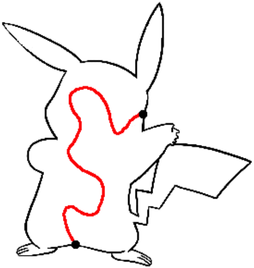

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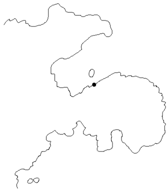

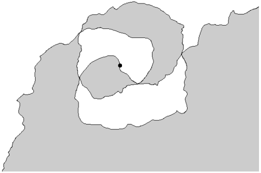

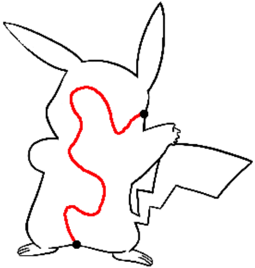

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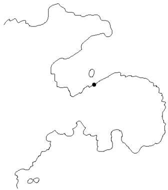

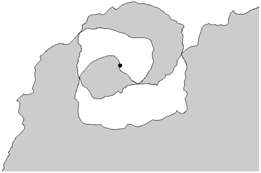

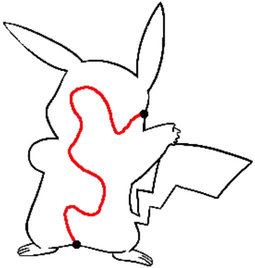

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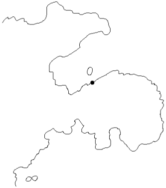
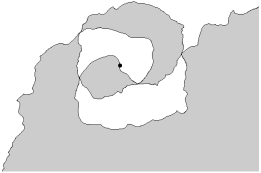
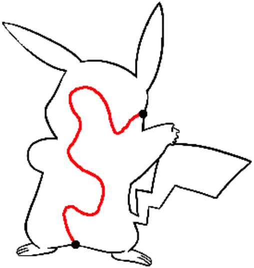
Regularity results:

- ψ-variation
- modulus of continuity
    - natural parametrisation
    - capacity parametrisation
- uniformising maps







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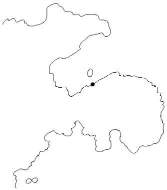

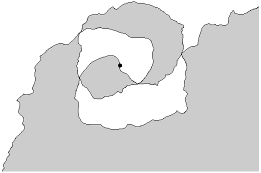

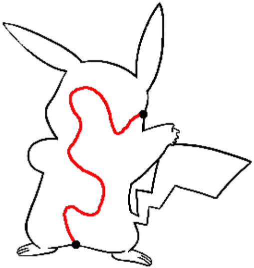

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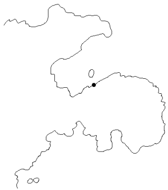

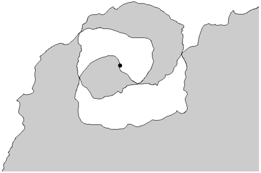

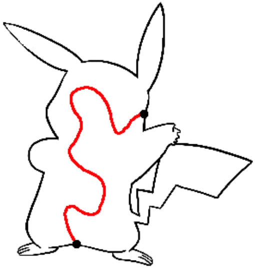

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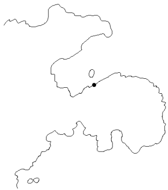

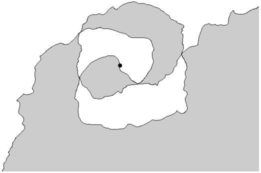

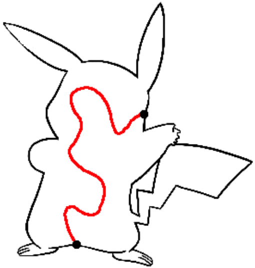

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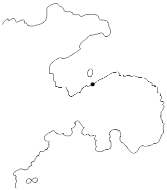

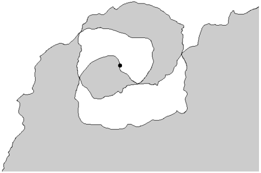

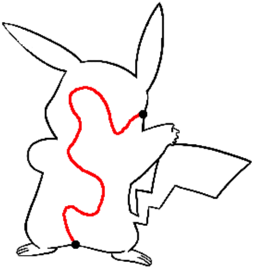

Variation regularity

η has finite ψ-variation if there exists $C < \infty$ such that

$$\sum_i \psi(|\eta(t_{i+1}) - \eta(t_i)|) \le C$$

for any partition $0 = t_0 < t_1 < ... < t_m = 1$,

or equivalently, if it has a parametrisation such that

$$|\tilde{\eta}(t) - \tilde{\eta}(s)| \le \psi^{-1}(|t-s|).$$

- Variation regularity is independent of the choice of parametrisation.
- finite p-variation (i.e. $\psi(x) = x^p$) $\implies \dim_{\mathrm{H}} \le p$.

$$[\eta]_{\psi\text{-var}} := \inf \left\{ M > 0 \;\middle|\; \sup_{t_0 < \dots < t_r} \sum_i \psi\left(\frac{|\eta(t_{i+1}) - \eta(t_i)|}{M}\right) \le 1 \right\}.$$

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For two-sided whole-plane variant (restricted to a compact time interval):

Theorem (Holden, Y.)

SLE_κ has finite ψ-variation for $\psi(x) = x^d (\log\log \frac{1}{x})^{-(d-1)}$, and not for $\tilde\psi$ with $\lim_{x\searrow 0} \frac{\psi(x)}{\tilde\psi(x)} \to 0$.

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Previous results:

- exponent d is optimal:
  Beffara (2008): lower bound.
  Werness (2012), Friz, Tran (2017): upper bound for ordinary SLE.
  Gwynne, Holden, Miller (2020): upper bound for space-filling SLE.

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Remark: The same result should hold for SLE in a (regular) domain.

Modulus of continuity (natural parametrisation)

Minkowski content of $\eta[s,t]$ ("fractal length"): (Lawler, Rezaei (2015))

$$\mathrm{Cont}_d(\eta[s,t]) = \lim_{\varepsilon\searrow 0} \frac{\mathrm{area}\,\{z \mid \mathrm{dist}(z,\eta[s,t]) \le \varepsilon\}}{\varepsilon^{2-d}}$$

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Natural parametrisation: $\mathrm{Cont}_d(\eta[s,t]) = |t-s|$.

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SLE_κ satisfies $|\eta(t)-\eta(s)| \lesssim \varphi(|t-s|)$ for $\varphi(t) = t^{1/d}(\log \frac{1}{t})^{1-1/d}$, and not for $\tilde{\varphi}$ with $\lim_{t\searrow 0} \frac{\tilde{\varphi}(t)}{\varphi(t)} \to 0$.
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Comments on the proofs

General argument: These results follow from probabilistic bounds on the increments of stochastic processes.

Main inputs:

- scaling and translation invariance of two-sided whole-plane SLE
- whole-plane reversibility properties
- domain Markov property

Remark: The same results apply to Brownian motion (with $d = 2$) or uniformly elliptic Markov processes (with $d = d_{\mathrm{w}}$ walk dimension).

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Arguments can also be applied to discrete models converging to SLE.

Example: Loop-erased random walk (parametrised by length).

$$\eta^\delta \to \mathrm{SLE}_2 \quad \text{with respect to } \|\cdot\|_\infty.$$

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$$\sup_\delta \mathbb{P}([\eta^\delta]_{\varphi\text{-Höl}} > u) \lesssim \exp\left(-cu^{d/(d-1)}\right),$$
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with $\varphi(t) = t^{1/d}(\log \frac{1}{t})^{1-1/d}$ and $\psi(x) = x^d(\log\log \frac{1}{x})^{d-1}$.

In particular, the convergence $\eta^\delta \to \eta$ holds in $\|\cdot\|_{\tilde{\varphi}\text{-Höl}}$ resp. $\|\cdot\|_{\tilde{\psi}\text{-var}}$ for $\lim_{t\searrow 0} \frac{\varphi(t)}{\tilde{\varphi}(t)} = 0$ and $\lim_{x\searrow 0} \frac{\tilde{\psi}(x)}{\psi(x)} = 0$.

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Modulus of continuity (capacity parametrisation)

Half-plane capacity of $\eta[0,t] \subseteq \mathbb{H}$:

$$\mathrm{hcap}(\eta[0,t]) = \lim_{y\to\infty} y\mathbb{E}^{iy}[\operatorname{Im} B_{\tau_{\eta[0,t]\cup\mathbb{R}}}].$$

Capacity parametrisation: $\mathrm{hcap}(\eta[0,t]) = 2t$.

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Half-plane capacity of $\eta[0,t] \subseteq \mathbb{H}$:

$$\operatorname{hcap}(\eta[0,t]) = \lim_{y\to\infty} y\mathbb{E}^{iy}[\operatorname{Im} B_{\tau_{\eta[0,t]\cup\mathbb{R}}}].$$

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For SLE in $\mathbb{H}$, parametrised by half-plane capacity (restricted to a compact time interval):

Theorem

SLE_κ satisfies $|\eta(t)-\eta(s)| \lesssim \varphi(|t-s|)$ for

$$\varphi(t) = \begin{cases} t^\alpha (\log \frac{1}{t})^{\tilde{\beta}} & \text{for } \kappa \neq 8, \\ (\log \frac{1}{t})^{-1/4} (\log\log \frac{1}{t})^{2+\varepsilon} & \text{for } \kappa = 8, \end{cases}$$

where $\alpha = 1 - \frac{\kappa}{24+2\kappa-8\sqrt{8+\kappa}}$, $\tilde{\beta} =$ [explicit formula]$_\kappa < 0.54$. Moreover,

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for some $p > 1$.

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Previous results:

- exponent α is optimal: Viklund, Lawler (2011)
- exponent $1/4$ is optimal: Kavvadias, Miller, Schoug (2021x)

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The optimal m.o.c. should be

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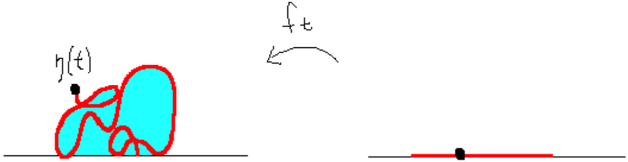
if and only if $\int_1^{\infty} \frac{ds}{s\ell(s)} < \infty.$

Uniformising maps

For SLE in $\mathbb{H}$, consider the conformal maps $f_t \colon \mathbb{H} \to \mathbb{H} \setminus \text{fill}(\eta[0,t])$.



For fixed t,

$$|f_t'(u+iv)| \lesssim \begin{cases} v^{-1+\zeta} & \text{for } \kappa \neq 4, \\ v^{-1}(\log \frac{1}{v})^{-1/3+\varepsilon} & \text{for } \kappa = 4. \end{cases}$$

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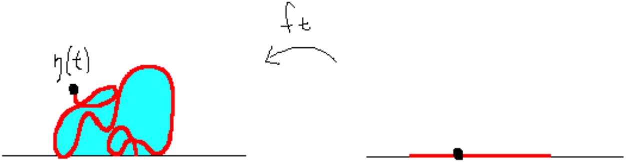

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The proof analyses the one-point martingales (for forward SLE) evaluated at a chosen grid of points.

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