Multiple SLE via conformal welding of Liouville quantum gravity disks

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Random conformal geometry and related fields in Jeju
joint works with Morris Ang and Xin Sun
June 8, 2023

Outline

The Gaussian Free Field

- The GFF on the upper half plane $\mathbb{H}$: The Gaussian random field on $\mathbb{H}$ with mean 0 and covariance

$$\mathrm{Cov}(h(z), h(w)) = G_{\mathbb{H}}(z, w)$$

where $G_{\mathbb{H}}(z, w)$ is the Green's function

$$G_{\mathbb{H}}(z, w) = -\log|z-w| - \log|z-\bar{w}| + 2\log|z|_+ + 2\log|w|_+$$

with $|z|_+ = \max\{|z|, 1\}$.

- h is a well-defined generalized function.

Liouville quantum gravity (LQG)

- Let $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and ϕ be a variant of the GFF, e.g., $\phi = h + f$ where f is a continuous function.
- Area measure: $\mu_\phi(d^2z) = $ " $e^{\gamma\phi(z)}d^2z$".
- Length measure: $\nu_\phi(dx) = $ " $e^{\frac{\gamma}{2}\phi(x)}dx$" .

Liouville conformal field theory on $\mathbb{H}$

- Start with the GFF h on $\mathbb{H}$.
- Sample $(h, \mathbf{c})$ from $P_{\mathbb{H}} \times [e^{-Qc}dc]$, and set $\phi(z) = h(z) - 2Q\log|z|_+ + \mathbf{c}$. Let $\mathrm{LF}_{\mathbb{H}}$ be the law of ϕ [David-Kupiainen-Rhodes-Vargas '14].
- Let $\beta_j \in \mathbb{R}$ and $x_j \in \partial\mathbb{H}$. Liouville field with boundary insertions: $\mathrm{LF}_{\mathbb{H}}^{(\beta_j, x_j)}(d\phi) = \prod_j e^{\frac{\beta_j}{2}\phi(x_j)}\mathrm{LF}_{\mathbb{H}}(d\phi)$.

LQG surfaces

- Let $\gamma \in (0, 2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.
- Say $(D_1, \phi_1) \sim_\gamma (D_2, \phi_2)$, if there exists $f : D_1 \to D_2$ conformal with $\phi_2 = \phi_1 \circ f^{-1} + Q \log |(f^{-1})'|$.
- A quantum surface is an equivalence class over the relation $\sim_\gamma$.

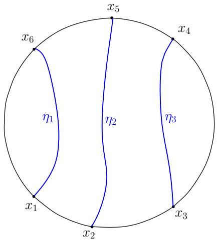
Quantum disks

- Let $W > 0$ be the *weight* parameter. Let $\beta = \gamma + \frac{2-W}{\gamma}$.
- Weight W (thick) quantum disks: $W > \frac{\gamma^2}{2}$, and near each marked point z_0 the field looks like $h - \beta \log|\cdot - z_0|$. Can be viewed as *uniform embedding of* $\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,\infty)}$ (Ang-Holden-Sun'21).
- Special weight $W = 2$: the two marked points can be resampled from the boundary length measure, which defines $\mathrm{QD}_{0,2}$.
- $\mathrm{QD}_{0,n}$: starting from $\mathrm{QD}_{0,2}$ and sample $n - 2$ marked points from the boundary length measure.

Multiple SLE_κ as a probability measure

Let $\kappa \in (0, 4]$, and α be a given link pattern.

- Local construction via Loewner flow (e.g. Dubédat'07, Graham'07, Kytölä-Peltola'16)
- Global construction by weighting the law of N independent SLE_κ curves (e.g. Kozdron-Lawler'06, Peltola-Wu'19)
- Resampling property: given $N - 1$ curves, the conditional law of the remaining curve is the SLE_κ.
- The resampling property uniquely characterize the multiple SLE probability measure (Miller-Sheffield'12, Beffara-Peltola-Wu'18)



Multiple SLE as non-probability measure

- Let $b = \frac{6-\kappa}{2\kappa}$ be the boundary scaling exponent.
- Multiple SLE pure partition function $\mathcal{Z}_\alpha(D; x_1, ..., x_{2N})$: solution to a 2nd order PDE and is used to construct the local multiple SLE (Peltola-Wu'19).
- Conformal covariance:

$$\mathrm{mSLE}_{\kappa,\alpha}(D, x_1, ..., x_{2N}) = \prod f'(x_i)^b f \circ \mathrm{mSLE}_{\kappa,\alpha}(f(D), f(x_1), ..., f(x_{2N}))$$

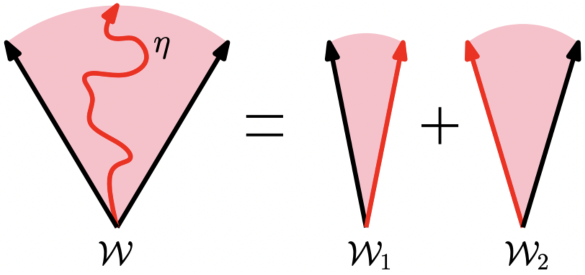
for conformal maps $f : D \to f(D)$.

Conformal welding of quantum wedges

Let $\kappa = \gamma^2 \in (0,4)$.

Theorem (Duplantier-Miller-Sheffield '14)

$$\mathcal{M}^{\text{wedge}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2) = \mathcal{M}^{\text{wedge}}(W^L) \times \mathcal{M}^{\text{wedge}}(W^R). \tag{1}$$

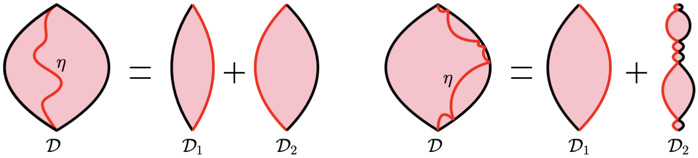


Conformal welding of quantum disks

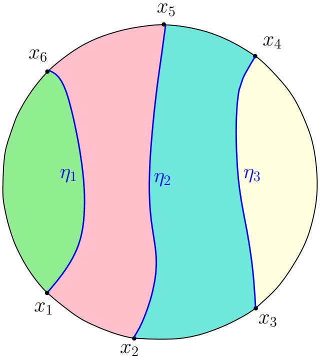
Theorem (Ang-Holden-Sun '20)

Let $\kappa = \gamma^2 \in (0, 4)$.

$$\mathcal{M}_2^{\mathrm{disk}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2) = c \int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{disk}}(W^L; \ell), \mathcal{M}_2^{\mathrm{disk}}(W^R; \ell) d\ell. \quad (2)$$



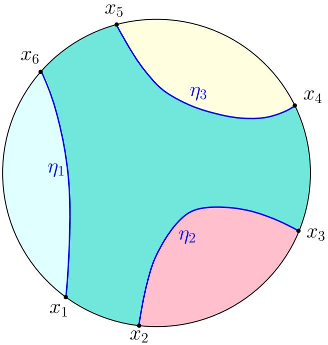
Conformal welding of LQG disks by link pattern



$\alpha = \{\{1,6\},\{2,5\},\{3,4\}\}$, with $\mathrm{Weld}_\alpha(QD)$ written as

$$\int_{\mathbb{R}_+^3} \mathrm{Weld}(\mathrm{QD}_{0,2}(\ell_1), \mathrm{QD}_{0,4}(\ell_1,\ell_2), \mathrm{QD}_{0,4}(\ell_2,\ell_3), \mathrm{QD}_{0,2}(\ell_3))\, d\ell_1\, d\ell_2\, d\ell_3.$$

Conformal welding of LQG disks by link pattern



$\alpha = \{\{1,6\},\{2,3\},\{4,5\}\}$, with $\mathrm{Weld}_\alpha(QD)$ written as

$$\int_{\mathbb{R}_+^3} \mathrm{Weld}(\mathrm{QD}_{0,2}(\ell_1), \mathrm{QD}_{0,2}(\ell_2), \mathrm{QD}_{0,2}(\ell_3), \mathrm{QD}_{0,6}(\ell_1,\ell_2,\ell_3)) d\ell_1\, d\ell_2\, d\ell_3.$$

Conformal welding of LQG disks by link pattern

Theorem (Ang-Sun-Y. '23+)

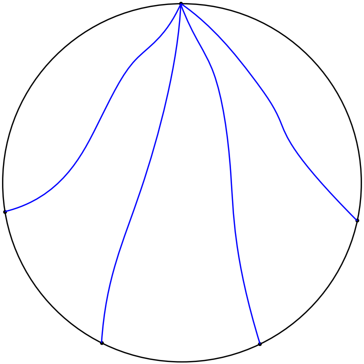
Let $\gamma \in (0,2)$, $\kappa = \gamma^2$ and $\beta = \gamma - \frac{2}{\gamma}$. Let $N \geq 2$ and $\alpha \in \mathrm{LP}_N$ be a link pattern. Then there exists a constant $c \in (0,\infty)$ such that

$$\int_{0<y_1<\ldots<y_{2N-3}<1} \Big[\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,1),(\beta,\infty),(\beta,y_1),\ldots,(\beta,y_{2N-3})} \times \mathrm{mSLE}_{\kappa,\alpha}(\mathbb{H},0,y_1,\ldots,y_{2N-3},1,\infty)\Big] dy_1 \ldots dy_{2N-3} = c\,\mathrm{Weld}_\alpha(\mathrm{QD}) \tag{3}$$

where the left hand side is understood as the law of a curve-decorated quantum surface.

Random modulus = partition function

- The above theorem implies that the random location of the marked points under conformal welding is encoded by multiple SLE pure partition function.
- This implication also works for other settings.



Random modulus = partition function

Theorem (Ang-Sun-Y '23+)

Let $\beta_k = \gamma - \frac{2k}{\gamma}$. The conformal welding of $N-1$ samples from $\mathrm{QD}_{0,3}$ with 2 samples from $\mathrm{QD}_{0,2}$ in the previous picture is given by

$$c \int_{0<x_1<\ldots<x_{N-2}<1} \Big[\mathrm{LF}_{\mathbb{H}}^{(\beta_1,x_0),(\beta_1,x_1),\ldots,(\beta_1,x_{N-1}),(\beta_N,\infty)} \times \prod_{0\le i<j\le N-1} (x_j - x_i)^{\frac{2}{\kappa}} \mathrm{IG}_{x_0,\ldots,x_{N-1}} \Big] dx_1 \ldots dx_{N-2} \tag{4}$$

where $x_0 = 0$, $x_{N-1} = 1$, and $\mathrm{IG}_{x_0,\ldots,x_{N-1}}$ denote the flow lines of the Imaginary Geometry field with marked points $x_0, \ldots, x_{N-1}$ and $\frac{2\pi}{\sqrt{\kappa}}$ change in boundary value near each marked point.

Random modulus = partition function

- The value $\prod_{0\leq i<j\leq N-1}(x_j - x_i)^{\frac{2}{\kappa}}$ can be viewed as the partition function of the Imaginary Geometry field (Dubédat).
- Similar identities holds when the weight 2 is replaced by other weights $W_j > \frac{\gamma^2}{2}$.

Outlook

- (In progress) Joint with Ang, Holden and Sun, we will show that similar result holds also for $\kappa \in (4, 8)$, where the conformal welding of generalized version of quantum disks (or quantum disks decorated with *loop trees*) gives Liouville field decorated with multiple SLE_κ.
- With Ang we will show that the conformal welding of general LQG/LCFT surfaces gives LQG/LCFT surfaces decorated by random curves. This shall give a new aspect to study SLE on more general topological surfaces.

Thanks for listening!