Connection probabilities for FK-Ising model and Dyson's circular ensemble

Hao Wu

Tsinghua University, China

2023. 6. 8

Outline

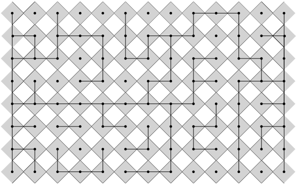
Table of contents

Random-cluster model

Random-cluster model [Fortuin-Kasteleyn, 1970s]

- $G = (V, E)$ is a finite graph.
- $\omega = (\omega_e)_{e\in E} \in \{0,1\}^E$:
  $e \in E$ is open (resp. close) if $\omega_e = 1$ (resp. if $\omega_e = 0$).
- Edge-weight : $p \in (0,1)$. Cluster-weight : $q > 0$.

$$\mathbb{P}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}.$$



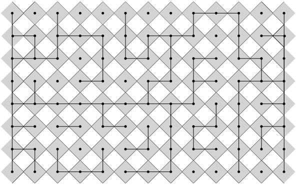
- $q = 1$: Bernoulli bond percolation
- $q = 2$: FK-Ising model

Random-cluster model

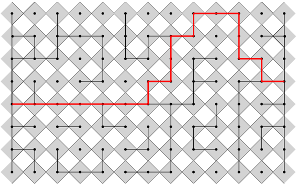
Random-cluster model [Fortuin-Kasteleyn, 1970s]

- $G = (V, E)$ is a finite graph.
- $\omega = (\omega_e)_{e \in E} \in \{0, 1\}^E$:
  $e \in E$ is open (resp. close) if $\omega_e = 1$ (resp. if $\omega_e = 0$).
- Edge-weight : $p \in (0, 1)$. Cluster-weight : $q > 0$.

$$\mathbb{P}[\omega] \propto p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}.$$



- $q = 1$: Bernoulli bond percolation
- $q = 2$: FK-Ising model



Critical edge-weight [Beffara-Duminil-Copin, PTRF 2012]

$$\text{For } q \geq 1, \quad p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

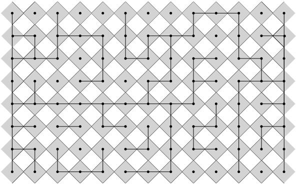
- sub-critical $p < p_c(q)$: connection probability $\to 0$.
- critical $p = p_c(q)$: connection probability is non-trivial.
- super-critical $p > p_c(q)$: connection probability $\to 1$.

Random-cluster model

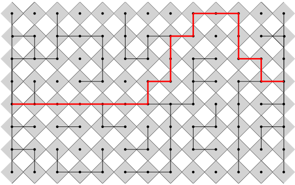
Random-cluster model [Fortuin-Kasteleyn, 1970s]

- $G = (V, E)$ is a finite graph.
- $\omega = (\omega_e)_{e \in E} \in \{0,1\}^E$:
  $e \in E$ is open (resp. close) if $\omega_e = 1$ (resp. if $\omega_e = 0$).
- Edge-weight : $p \in (0,1)$. Cluster-weight : $q > 0$.

$$\mathbb{P}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}.$$



- $q = 1$: Bernoulli bond percolation
- $q = 2$: FK-Ising model



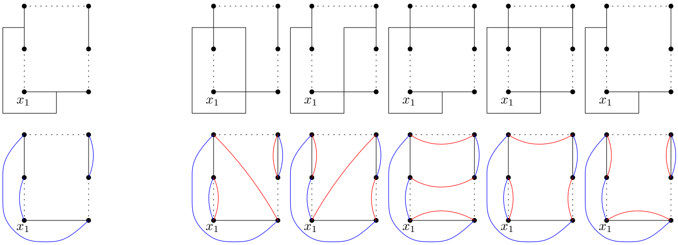
Critical edge-weight [Beffara-Duminil-Copin, PTRF 2012]

$$\text{For } q \geq 1, \quad p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

- sub-critical $p < p_c(q)$: connection probability $\to 0$.
- critical $p = p_c(q)$: connection probability is non-trivial.
- super-critical $p > p_c(q)$: connection probability $\to 1$.

- $q = 1$: Cardy's formula
  Proved by Smirnov for site percolation on $\mathbb{T}$
- $q = 2$: [Chelkak-Smirnov, Invent. 2012]

Connection Probabilities



- A polygon $(\Omega; x_1, \ldots, x_{2N})$
- Alternating boundary conditions : $(x_1 x_2), (x_3 x_4), \ldots, (x_{2N-1} x_{2N})$ are wired.
- N wired arcs are further wired according to a non-crossing partition outside of the polygon.

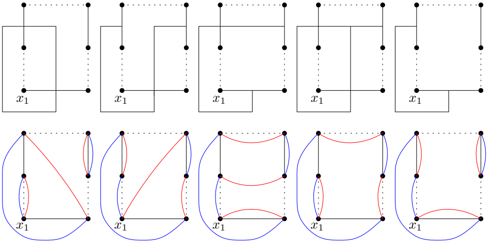
Such boundary condition can be encoded by a planar link pattern with N links of $\{1, 2, \ldots, 2N\}$

$$\text{Boundary condition} : \beta = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_N, b_N\}\}.$$

- N interfaces inside of the polygon : planar link pattern with N links of $\{1, 2, \ldots, 2N\}$

$$\text{Internal link pattern} : \alpha = \{\{c_1, d_1\}, \{c_2, d_2\}, \ldots, \{c_N, d_N\}\}.$$

Connection Probabilities



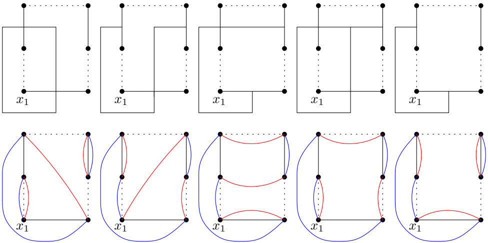
Conjecture on critical random-cluster model [Flores-Kleban-Simmons-Ziff 2011]

Fix parameters $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$. We have the scaling limit of the connection probabilities :

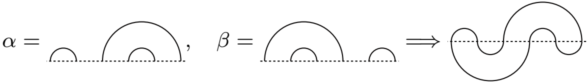
$$\lim_{\delta \to 0} \mathbb{P}^{\delta}_{\beta}[\mathcal{A}^{\delta} = \alpha] = \mathcal{M}_{\alpha,\beta}(q) \frac{\mathcal{Z}_{\alpha}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{G}_{\beta}(\Omega; x_1, \ldots, x_{2N})}.$$

- $\mathcal{M}_{\alpha,\beta}(q)$: Meander matrix
- $\mathcal{G}_{\beta}$: Coulomb gas integrals
- $\mathcal{Z}_{\alpha}$: Pure partition functions

Meander matrix



- Meander formed from α, β :



- Meander matrix

$$\mathcal{M}_{\alpha,\beta}(q) = \sqrt{q}^{\#\text{loops in the meander } (\alpha,\beta)}, \quad \alpha, \beta \in \mathrm{LP}_N.$$

Coulomb gas integrals

- Write the boundary condition as
  $\beta = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_N, b_N\}\}$ with $a_1 < a_2 < \cdots < a_N$ and $a_r < b_r$ for all $1 \le r \le N$.

Coulomb gas integrals with $\kappa \in (4, 8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0, 4)$

$$\mathcal{G}_\beta : \ \{\boldsymbol{x} := (x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N} : x_1 < \cdots < x_{2N}\} \to \mathbb{C},$$

$$\mathcal{G}_\beta(x_1, \ldots, x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2 - 8/\kappa)}{\Gamma(1 - 4/\kappa)^2} \right)^N$$
$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1 \le i < j \le 2N} (x_j - x_i)^{2/\kappa} \prod_{1 \le r < s \le N} (u_s - u_r)^{8/\kappa} \prod_{\substack{1 \le i \le 2N \\ 1 \le r \le N}} (u_r - x_i)^{-4/\kappa}.$$

where the branch of the multivalued integrand is chosen to be real and positive when

$$x_{a_r} < \mathrm{Re}(u_r) < x_{a_r + 1} \qquad \text{for all } 1 \le r \le N.$$

- **BPZ equations** :

$$\left[\frac{\kappa}{2} \partial_i^2 + \sum_{j \ne i} \left(\frac{2}{x_j - x_i} \partial_j - \frac{(6 - \kappa)/\kappa}{(x_j - x_i)^2} \right) \right] F(x_1, \ldots, x_{2N}) = 0.$$

Pure partition functions

Pure partition functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j-x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1,\ldots,x_{2N}) = 0$, where $h=(6-\kappa)/2\kappa$.

COV : $\mathcal{Z}(x_1,\ldots,x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1),\ldots,\varphi(x_{2N}))$.

ASY : $\lim_{x_j,x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\alpha/\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N}), & \text{if } \{j,j+1\}\in\alpha;\\ 0, & \text{else.}\end{cases}$

Pure partition functions

Pure partition functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0$, where $h = (6-\kappa)/2\kappa$.

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N}))$.

ASY : $\lim_{x_j, x_{j+1}\to\xi} \frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\alpha/\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \text{if } \{j, j+1\} \in \alpha; \\ 0, & \text{else.}\end{cases}$

Probability

- PDE : Itô's formula
- ASY : compatible

Pure partition functions

Pure partition functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j-x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1,\ldots,x_{2N}) = 0$, where $h=(6-\kappa)/2\kappa$.

COV : $\mathcal{Z}(x_1,\ldots,x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1),\ldots,\varphi(x_{2N}))$.

ASY : $\lim_{x_j,x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\alpha/\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N}), & \text{if } \{j,j+1\}\in\alpha;\\ 0, & \text{else.}\end{cases}$

Probability

- PDE : Itô's formula
- ASY : compatible

CFT

- PDE : BPZ equations
- ASY : fusion rules

Pure partition functions

Pure partition functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j-x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1,\ldots,x_{2N}) = 0$, where $h=(6-\kappa)/2\kappa$.

COV : $\mathcal{Z}(x_1,\ldots,x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1),\ldots,\varphi(x_{2N}))$.

ASY : $\lim_{x_j,x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\alpha/\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N}), & \text{if } \{j,j+1\}\in\alpha;\\ 0, & \text{else.}\end{cases}$

Probability

- PDE : Itô's formula
- ASY : compatible

CFT

- PDE : BPZ equations
- ASY : fusion rules

PDE

- PDE : 2N variables, 2N PDEs
- ASY : boundary value ?

Pure partition functions

Pure partition functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0$, where $h = (6-\kappa)/2\kappa$.

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N}))$.

ASY : $\lim_{x_j, x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\alpha/\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \text{if } \{j, j+1\}\in\alpha; \\ 0, & \text{else.}\end{cases}$

Probability

- PDE : Itô's formula
- ASY : compatible

CFT

- PDE : BPZ equations
- ASY : fusion rules

PDE

- PDE : 2N variables, 2N PDEs
- ASY : boundary value ?

Questions

Existence and uniqueness ?

Pure partition functions

Uniqueness [Flores-Kleban, CMP 2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

Pure partition functions

Uniqueness [Flores-Kleban, CMP 2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP 2016]
- $\kappa \in (0, 4]$ [Peltola-W. CMP 2019, Beffara-Peltola-W. AOP 2021]
- $\kappa \in (0, 6]$ [W. CMP 2020]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

Pure partition functions

Uniqueness [Flores-Kleban, CMP 2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP 2016]
- $\kappa \in (0, 4]$ [Peltola-W. CMP 2019, Beffara-Peltola-W. AOP 2021]
- $\kappa \in (0, 6]$ [W. CMP 2020]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

Theorem [W. CMP 2020]

Fix $\kappa \in (0, 6]$. The pure partition functions are the recursive collection $\{\mathcal{Z}_\alpha : \alpha \in \cup_N \mathrm{LP}_N\}$ of smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}$ uniquely determined by the following properties :

PDE, COV, ASY as well as **PLB** :

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \leq \prod_{\{a,b\} \in \alpha} |x_b - x_a|^{-2h}, \quad \forall (x_1, \ldots, x_{2N}) \in \mathfrak{X}_{2N}.$$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly independent and forms a basis for the solution space.

Coulomb gas integrals and pure partition functions

Coulomb gas integrals with $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\right)^N$$
$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1\le i<j\le 2N} (x_j - x_i)^{2/\kappa} \prod_{1\le r<s\le N} (u_s - u_r)^{8/\kappa} \prod_{\substack{1\le i\le 2N \\ 1\le r\le N}} (u_r - x_i)^{-4/\kappa}.$$

Pure Partition Functions with $\kappa \in (0,6]$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is the unique collection of smooth functions satisfying PDE, COV, ASY and PLB.

Coulomb gas integrals and pure partition functions

Coulomb gas integrals with $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\right)^N$$
$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1\le i<j\le 2N} (x_j - x_i)^{2/\kappa} \prod_{1\le r<s\le N} (u_s - u_r)^{8/\kappa} \prod_{\substack{1\le i\le 2N \\ 1\le r\le N}} (u_r - x_i)^{-4/\kappa}.$$

Pure Partition Functions with $\kappa \in (0,6]$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is the unique collection of smooth functions satisfying PDE, COV, ASY and PLB.

Theorem [Feng-Peltola-W. 2022]

Fix parameters $\kappa \in (4,6]$ and $q = 4\cos^2(4\pi/\kappa) \in [1,4)$. We have

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) = \sum_\alpha \mathcal{M}_{\alpha,\beta}(q)\mathcal{Z}_\alpha(x_1,\ldots,x_{2N}).$$

Coulomb gas integrals and pure partition functions

Coulomb gas integrals with $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\right)^N$$

$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1\le i<j\le 2N} (x_j-x_i)^{2/\kappa} \prod_{1\le r<s\le N} (u_s-u_r)^{8/\kappa} \prod_{\substack{1\le i\le 2N \\ 1\le r\le N}} (u_r-x_i)^{-4/\kappa}.$$

Pure Partition Functions with $\kappa \in (0,6]$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is the unique collection of smooth functions satisfying PDE, COV, ASY and PLB.

Theorem [Feng-Peltola-W. 2022]

Fix parameters $\kappa \in (4,6]$ and $q = 4\cos^2(4\pi/\kappa) \in [1,4)$. We have

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) = \sum_\alpha \mathcal{M}_{\alpha,\beta}(q)\mathcal{Z}_\alpha(x_1,\ldots,x_{2N}).$$

- As $\mathcal{Z}_\alpha > 0$ for all α, we have $\mathcal{G}_\beta > 0$.

Coulomb gas integrals and pure partition functions

Coulomb gas integrals with $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\right)^N$$

$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1\le i<j\le 2N} (x_j - x_i)^{2/\kappa} \prod_{1\le r<s\le N} (u_s - u_r)^{8/\kappa} \prod_{\substack{1\le i\le 2N \\ 1\le r\le N}} (u_r - x_i)^{-4/\kappa}.$$

Pure Partition Functions with $\kappa \in (0,6]$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is the unique collection of smooth functions satisfying PDE, COV, ASY and PLB.

Theorem [Feng-Peltola-W. 2022]

Fix parameters $\kappa \in (4,6]$ and $q = 4\cos^2(4\pi/\kappa) \in [1,4)$. We have

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) = \sum_\alpha \mathcal{M}_{\alpha,\beta}(q)\mathcal{Z}_\alpha(x_1,\ldots,x_{2N}).$$

- As $\mathcal{Z}_\alpha > 0$ for all α, we have $\mathcal{G}_\beta > 0$.
- $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly indept.

Coulomb gas integrals and pure partition functions

Coulomb gas integrals with $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) := \left(\frac{\sqrt{q}\,\Gamma(2-8/\kappa)}{\Gamma(1-4/\kappa)^2}\right)^N$$

$$\times \int_{x_{a_1}}^{x_{b_1}} \mathrm{d}u_1 \cdots \int_{x_{a_N}}^{x_{b_N}} \mathrm{d}u_N \prod_{1\le i<j\le 2N}(x_j-x_i)^{2/\kappa} \prod_{1\le r<s\le N}(u_s-u_r)^{8/\kappa} \prod_{\substack{1\le i\le 2N\\ 1\le r\le N}}(u_r-x_i)^{-4/\kappa}.$$

Pure Partition Functions with $\kappa \in (0,6]$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is the unique collection of smooth functions satisfying PDE, COV, ASY and PLB.

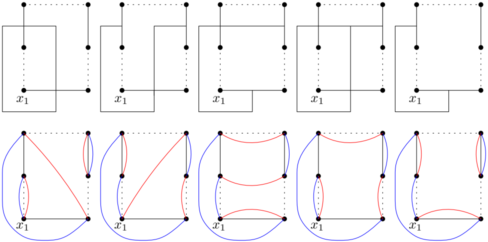
Theorem [Feng-Peltola-W. 2022]

Fix parameters $\kappa \in (4,6]$ and $q = 4\cos^2(4\pi/\kappa) \in [1,4)$. We have

$$\mathcal{G}_\beta(x_1,\ldots,x_{2N}) = \sum_\alpha \mathcal{M}_{\alpha,\beta}(q)\mathcal{Z}_\alpha(x_1,\ldots,x_{2N}).$$

- As $\mathcal{Z}_\alpha > 0$ for all α, we have $\mathcal{G}_\beta > 0$.
- $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly indept.
- $\{\mathcal{M}_{\alpha,\beta}(q) : \alpha,\beta \in \mathrm{LP}_N\}$ may be not invertible.
- When $\kappa = 16/3$ and $q = 2$, it is NOT invertible.

Connection Probabilities



Conjecture on critical random-cluster model [Flores-Kleban-Simmons-Ziff 2011]

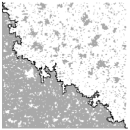
Fix parameters $\kappa \in (4,8)$ and $q = 4\cos^2(4\pi/\kappa) \in (0,4)$. We have the scaling limit of the connection probabilities :

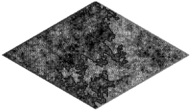
$$\lim_{\delta \to 0} \mathbb{P}^{\delta}_{\beta}[\mathcal{A}^{\delta} = \alpha] = \mathcal{M}_{\alpha,\beta}(q) \frac{\mathcal{Z}_{\alpha}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{G}_{\beta}(\Omega; x_1, \ldots, x_{2N})}.$$

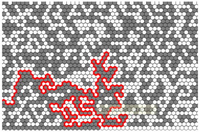
Theorem [Feng-Peltola-W. 2022]

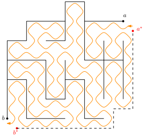
This conjecture holds for FK-Ising model ($q = 2$).

Conformal invariance in 2D critical lattice models



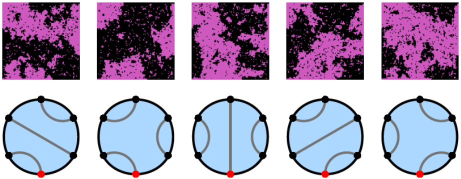






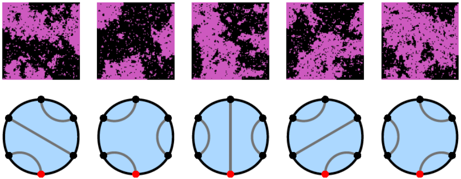
- Loop-erased random walk (LERW) : $\kappa = 2$ [Lawler-Schramm-Werner, AOP 2004]
- Ising model : $\kappa = 3$ [Chelkak-Smirnov et al. 2012]
- Level lines of GFF : $\kappa = 4$ [Schramm-Sheffield, ACTA 2009]
- FK-Ising model : $\kappa = 16/3$ [Chelkak-Smirnov et al. 2012]
- Percolation : $\kappa = 6$ [Smirnov 2001]
- Uniform spanning tree (UST) : $\kappa = 8$ [Lawler-Schramm-Werner, AOP 2004]

Connection probabilities



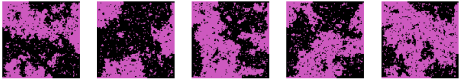
- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]

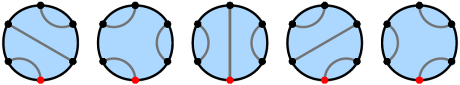
Connection probabilities



- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]

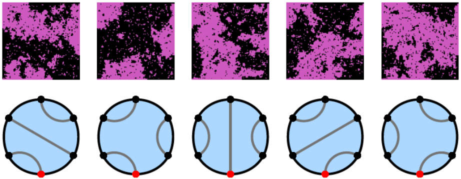
Connection probabilities





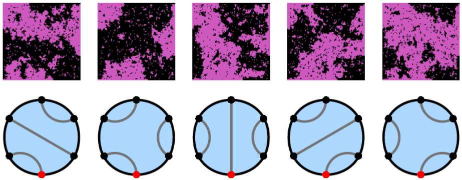
- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]

Connection probabilities



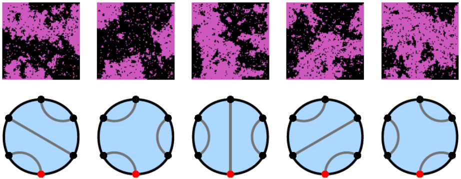
- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]

Connection probabilities



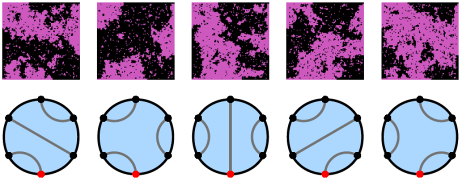
- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]
- Multiple percolation interfaces : $\kappa = 6$. [Liu-Peltola-W. 2021]

Connection probabilities



- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]
- Multiple percolation interfaces : $\kappa = 6$. [Liu-Peltola-W. 2021]
- Multiple Peano curves in UST : $\kappa = 8$.
  [Han-Liu-W. 2020], [Liu-Peltola-W. 2021], [Liu-W. Bernoulli 2023]

Connection probabilities

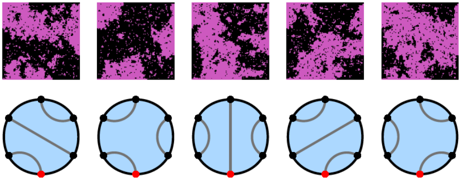


- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]
- Multiple percolation interfaces : $\kappa = 6$. [Liu-Peltola-W. 2021]
- Multiple Peano curves in UST : $\kappa = 8$.
  [Han-Liu-W. 2020], [Liu-Peltola-W. 2021], [Liu-W. Bernoulli 2023]

Strategy :

1. Step 1 : Proper holomorphic observable ϕ_β.

Connection probabilities

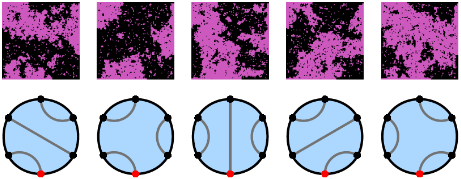


- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]
- Multiple percolation interfaces : $\kappa = 6$. [Liu-Peltola-W. 2021]
- Multiple Peano curves in UST : $\kappa = 8$.
  [Han-Liu-W. 2020], [Liu-Peltola-W. 2021], [Liu-W. Bernoulli 2023]

Strategy :

1. Step 1 : Proper holomorphic observable ϕ_β.
2. Step 2 : A single interfaces $\sim$ Loewner chain associated to $\mathcal{G}_\beta$.

Connection probabilities

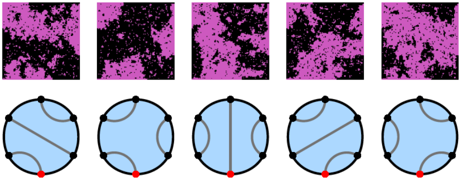


- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]
- Multiple percolation interfaces : $\kappa = 6$. [Liu-Peltola-W. 2021]
- Multiple Peano curves in UST : $\kappa = 8$.
  [Han-Liu-W. 2020], [Liu-Peltola-W. 2021], [Liu-W. Bernoulli 2023]

Strategy :

1. Step 1 : Proper holomorphic observable ϕ_β.
2. Step 2 : A single interfaces $\sim$ Loewner chain associated to $\mathcal{G}_\beta$.
3. Step 3 : Fine analysis on the martingale $\mathcal{Z}_\alpha/\mathcal{G}_\beta$.

Connection probabilities



- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019]
- Multiple Ising interfaces : $\kappa = 3$. [Peltola-W. AAP 2023+]
- Multiple level lines of GFF : $\kappa = 4$.
  [Peltola-W. CMP 2019], [Ding-Wirth-W. AIHP 2022], [Liu-W. EJP 2021]
- Multiple FK-Ising interfaces : $\kappa = 16/3$. [Feng-Peltola-W. 2022]
- Multiple percolation interfaces : $\kappa = 6$. [Liu-Peltola-W. 2021]
- Multiple Peano curves in UST : $\kappa = 8$.
  [Han-Liu-W. 2020], [Liu-Peltola-W. 2021], [Liu-W. Bernoulli 2023]

Strategy :

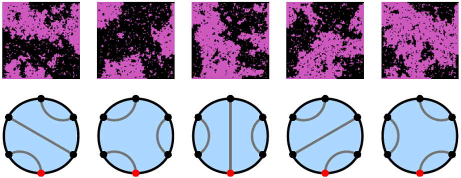
1. Step 1 : Proper holomorphic observable ϕ_β.
2. Step 2 : A single interfaces $\sim$ Loewner chain associated to $\mathcal{G}_\beta$.
3. Step 3 : Fine analysis on the martingale $\mathcal{Z}_\alpha/\mathcal{G}_\beta$.

Consequence : given the connectivity α, a single interface $\sim$ Loewner chain associated to $\mathcal{Z}_\alpha$:

$$\mathrm{d}W_t = \mathrm{d}B_t + \partial_j \log \mathcal{Z}_\alpha(V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

Table of contents

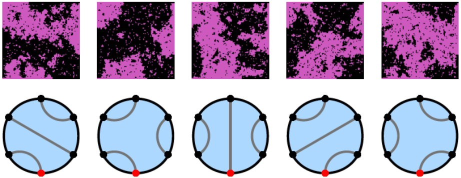
Usual parameterization vs common parameterization



Recall : given the connectivity α, a single interface $\sim$ Loewner chain associated to $\mathcal{Z}_\alpha$:

$$\mathrm{d}W_t = \mathrm{d}B_t + \partial_j \log \mathcal{Z}_\alpha(V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

Usual parameterization vs common parameterization



Recall : given the connectivity α, a single interface $\sim$ Loewner chain associated to $\mathcal{Z}_\alpha$:

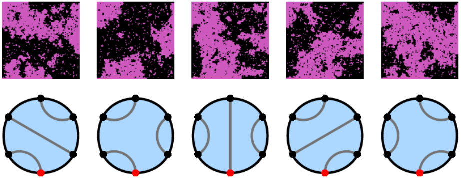
$$\mathrm{d}W_t = \mathrm{d}B_t + \partial_j \log \mathcal{Z}_\alpha(V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

From the upper-half plane to the unit disc :

$$\mathcal{G}_\alpha(\theta^1, \ldots, \theta^{2N}) = \mathcal{Z}_\alpha(\mathbb{D}; \exp(2\mathrm{i}\theta^1), \ldots, \exp(2\mathrm{i}\theta^{2N})),$$

$$\mathrm{d}\xi_t = \mathrm{d}B_t + \partial_j \log \mathcal{G}_\alpha(V_t^1, \ldots, V_t^{j-1}, \xi_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

Usual parameterization vs common parameterization



Recall : given the connectivity α, a single interface $\sim$ Loewner chain associated to $\mathcal{Z}_\alpha$:

$$\mathrm{d}W_t = \mathrm{d}B_t + \partial_j \log \mathcal{Z}_\alpha(V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

From the upper-half plane to the unit disc :

$$\mathcal{G}_\alpha(\theta^1, \ldots, \theta^{2N}) = \mathcal{Z}_\alpha(\mathbb{D}; \exp(2\mathrm{i}\theta^1), \ldots, \exp(2\mathrm{i}\theta^{2N})),$$

$$\mathrm{d}\xi_t = \mathrm{d}B_t + \partial_j \log \mathcal{G}_\alpha(V_t^1, \ldots, V_t^{j-1}, \xi_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

Fix $a = 2/\kappa$. Under a-common parameterization :

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + \partial_j \log \mathcal{G}_\alpha(\theta_t^1, \ldots, \theta_t^{2N})\mathrm{d}t + a\sum_{k\neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, t < T,$$

where $\{B^j\}_{1\leq j\leq 2N}$ are independent Brownian motions and T is the collision time.

Dyson's circular ensemble

Fix $a = 2/\kappa$. Under a-common parameterization :

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + \partial_j \log \mathcal{G}_\alpha(\theta_t^1, \ldots, \theta_t^{2N})\mathrm{d}t + a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, t < T, \tag{1}$$

where $\{B^j\}_{1 \leq j \leq 2N}$ are independent Brownian motions and T is the collision time.

Dyson's circular ensemble

Fix $a = 2/\kappa$. Under a-common parameterization :

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + \partial_j \log \mathcal{G}_\alpha(\theta_t^1, \ldots, \theta_t^{2N})\mathrm{d}t + a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, t < T, \tag{1}$$

where $\{B^j\}_{1 \leq j \leq 2N}$ are independent Brownian motions and T is the collision time.

Proposition [Feng-W.-Yang 2023]

Fix $\kappa \in (0, 4]$ and $a = 2/\kappa$. The solution $\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^{2N})$ to (1) conditioned on $\{T > s\}$ converges in total variation distance as $s \to \infty$ to $2N$ radial Bessel process

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + 2a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, \tag{2}$$

whose invariant density is called Dyson's circular ensemble [Dyson, J. Math. Phys. 1962] :

$$f(\theta^1, \ldots, \theta^{2N}) \propto \prod_{1 \leq j < k \leq 2N} |\sin(\theta^k - \theta^j)|^{4a}. \tag{3}$$

Dyson's circular ensemble

Fix $a = 2/\kappa$. Under a-common parameterization :

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + \partial_j \log \mathcal{G}_\alpha(\theta_t^1, \ldots, \theta_t^{2N})\mathrm{d}t + a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, t < T, \tag{1}$$

where $\{B^j\}_{1 \leq j \leq 2N}$ are independent Brownian motions and T is the collision time.

Proposition [Feng-W.-Yang 2023]

Fix $\kappa \in (0, 4]$ and $a = 2/\kappa$. The solution $\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^{2N})$ to (1) conditioned on $\{T > s\}$ converges in total variation distance as $s \to \infty$ to $2N$ radial Bessel process

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + 2a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, \tag{2}$$

whose invariant density is called Dyson's circular ensemble [Dyson, J. Math. Phys. 1962] :

$$f(\theta^1, \ldots, \theta^{2N}) \propto \prod_{1 \leq j < k \leq 2N} |\sin(\theta^k - \theta^j)|^{4a}. \tag{3}$$

Key ingredients : [Peltola-W. CMP 2019], [Healey-Lawler, PTRF 2021], $2N$-time local martingale :

$$M_{\boldsymbol{t}}^{\alpha} = g_{\boldsymbol{t}}'(0)^{-2N\tilde{b}} \prod_{j=1}^{2N} h_{\boldsymbol{t},j}'(\xi_{t_j}^j)^b g_{\boldsymbol{t},j}'(0)^{\tilde{b}} \times \mathcal{G}_\alpha(\theta_{\boldsymbol{t}}^1, \ldots, \theta_{\boldsymbol{t}}^{2N}) \exp\left(\frac{c}{2} \sum_{j=1}^{2N} \mu_{\boldsymbol{t}}^j\right).$$

Applications : estimates for multiple SLEs

Theorem [Feng-W.-Yang 2023]

Fix $\kappa \in (0,4]$. Fix $\theta^1 < \cdots < \theta^{2N} < \theta^1 + \pi$ and write $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^{2N})$. We denote by $(\gamma_1, \ldots, \gamma_N) \sim \mathbb{P}_\alpha^{(\boldsymbol{\theta})}$ the law of chordal N-SLE$_\kappa$ in polygon $(\mathbb{D}; \exp(2i\theta^1), \ldots, \exp(2i\theta^{2N}))$ associated to link pattern $\alpha \in \mathrm{LP}_N$. We have

$$\mathbb{P}_\alpha^{(\boldsymbol{\theta})}\left[\mathrm{dist}(0,\gamma_j) < r, 1 \le j \le N\right] = CG_\alpha(\boldsymbol{\theta}) r^{A_{2N}}(1 + O(r^u)), \quad \text{as } r \to 0+,$$

where dist is Euclidean distance.

- A_{2N} is $2N$-arm exponent :
$$A_{2N} = \frac{16N^2 - (4-\kappa)^2}{8\kappa};$$
- G_α is Green's function for chordal N-SLE$_\kappa$:
$$G_\alpha(\theta^1, \ldots, \theta^{2N}) := \frac{\mathcal{G}_*(\theta^1, \ldots, \theta^{2N})}{\mathcal{G}_\alpha(\theta^1, \ldots, \theta^{2N})},$$
where $\mathcal{G}_*$ is the partition function for $2N$-sided radial SLE$_\kappa$:
$$\mathcal{G}_*(\theta^1, \ldots, \theta^{2N}) := \prod_{1 \le j < k \le 2N} |\sin(\theta^k - \theta^j)|^{2/\kappa};$$
- $C \in (0,\infty)$ is a constant depending on κ, N, α and $u > 0$ is a constant depending on κ, N.

Applications : estimates for multiple SLEs

Theorem [Feng-W.-Yang 2023]

Fix $\kappa \in (0,4]$. Fix $\theta^1 < \cdots < \theta^{2N} < \theta^1 + \pi$ and write $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^{2N})$. We denote by $(\gamma_1, \ldots, \gamma_N) \sim \mathbb{P}_\alpha^{(\boldsymbol{\theta})}$ the law of chordal N-SLE$_\kappa$ in polygon $(\mathbb{D}; \exp(2i\theta^1), \ldots, \exp(2i\theta^{2N}))$ associated to link pattern $\alpha \in \mathrm{LP}_N$. We have

$$\mathbb{P}_\alpha^{(\boldsymbol{\theta})}\left[\mathrm{dist}(0,\gamma_j) < r, 1 \le j \le N\right] = CG_\alpha(\boldsymbol{\theta}) r^{A_{2N}}(1 + O(r^u)), \quad \text{as } r \to 0+,$$

where dist is Euclidean distance.

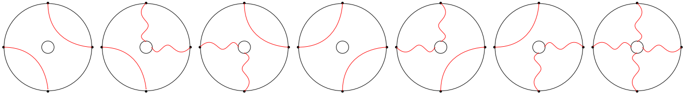
- Proved for $N = 1$ and $\kappa \in (0,8)$: [Lawler-Rezaei, AOP 2015].
- Proved for $N = 2$ and $\kappa \in (0,8)$: [Zhan, CMP 2020].
- We prove it for all $N \ge 1$ and α and $\kappa \in (0,4]$.

Key ingredients : [Peltola-W. CMP 2019], [Healey-Lawler, PTRF 2021].

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

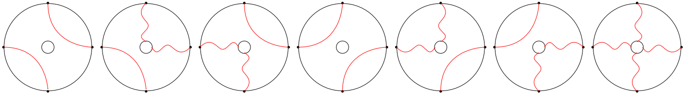
alternating $\oplus / \ominus$ on the outer boundary and free on the inner boundary.



Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

alternating $\oplus/\ominus$ on the outer boundary and free on the inner boundary.



Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \ldots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}[\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

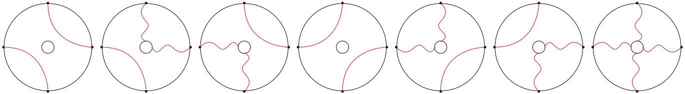
Conditioned on the event $\{\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

$$\mathrm{d}\theta^j_t = \mathrm{d}B^j_t + \frac{4}{3}\sum_{k\neq j}\cot(\theta^j_t - \theta^k_t)\mathrm{d}t, \quad 1 \le j \le 2N. \tag{5}$$

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

alternating $\oplus/\ominus$ on the outer boundary and free on the inner boundary.



Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \ldots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}[\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

Conditioned on the event $\{\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

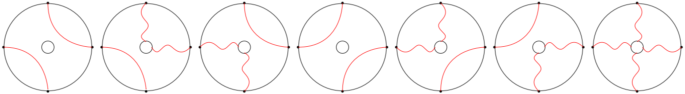
$$\mathrm{d}\theta^j_t = \mathrm{d}B^j_t + \frac{4}{3}\sum_{k\neq j}\cot(\theta^j_t - \theta^k_t)\mathrm{d}t, \quad 1 \leq j \leq 2N. \tag{5}$$

- Estimate (4) recovers [W. AOP 2018].

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

alternating $\oplus/\ominus$ on the outer boundary and free on the inner boundary.



Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \dots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}[\eta_1, \dots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

Conditioned on the event $\{\eta_1, \dots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

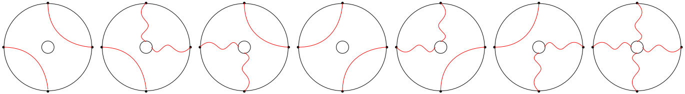
$$\mathrm{d}\theta^j_t = \mathrm{d}B^j_t + \frac{4}{3}\sum_{k\neq j}\cot(\theta^j_t - \theta^k_t)\mathrm{d}t, \quad 1 \le j \le 2N. \tag{5}$$

- Estimate (4) recovers [W. AOP 2018].
- $2N$-sided radial SLE : [Healey-Lawler, PTRF 2021].

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

alternating $\oplus/\ominus$ on the outer boundary and free on the inner boundary.



Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \ldots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}[\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

Conditioned on the event $\{\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\mathrm{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

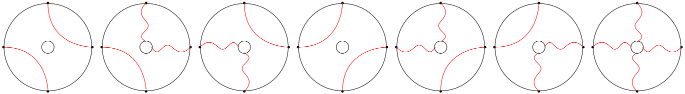
$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + \frac{4}{3}\sum_{k\neq j}\cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \le j \le 2N. \tag{5}$$

- Estimate (4) recovers [W. AOP 2018].
- $2N$-sided radial SLE : [Healey-Lawler, PTRF 2021].
- Eq. (5) : [Cardy, J. Phys. A 2003]

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

alternating $\oplus/\ominus$ on the outer boundary and free on the inner boundary.



Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \ldots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}[\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

Conditioned on the event $\{\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

$$\mathrm{d}\theta^j_t = \mathrm{d}B^j_t + \frac{4}{3}\sum_{k\neq j}\cot(\theta^j_t - \theta^k_t)\mathrm{d}t, \quad 1 \le j \le 2N. \tag{5}$$

- Estimate (4) recovers [W. AOP 2018].
- $2N$-sided radial SLE : [Healey-Lawler, PTRF 2021].
- Eq. (5) : [Cardy, J. Phys. A 2003]
- Similar for level lines of GFF.

Acknowledge to all my coauthors

- Eveliina Peltola (Aalto University)
- Vincent Beffara (CNRS)
- Jian Ding (Peking University)
- Mateo Wirth (UPenn)
- Yong Han (Shenzhen University)
- Mingchang Liu (Tsinghua University $\to$ KTH)
- Yu Feng (Tsinghua University)
- Lu Yang (Tsinghua University)

Thanks !

1. [Peltola-W. CMP 2019] Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. *Comm. Math. Phys.* 366(2) : 469-536, 2019.
2. [W. CMP 2020] Hypergeometric SLE : conformal Markov characterization and applications *Comm. Math. Phys.* 374(2) : 433-484, 2020.
3. [Beffara-Peltola-W. AOP 2021] On the uniqueness of global multiple SLEs *Ann. Probab.* 49(1) : 400-434, 2021.
4. [Liu-W. EJP 2021] Scaling limits of crossing probabilities in metric graph GFF *Electron. J. Probab.* 26 : article no. 37, 1-46, 2021.
5. [Ding-Wirth-W. AIHP 2022] Crossing estimates from metric graph and discrete GFF *Ann. Inst. H. Poincaré Probab. Statist.* 58(3) :1740-1774, 2022.
6. [Liu-W. Bernoulli 2023] Loop-erased random walk branch of uniform spanning tree in topological polygons. *Bernoulli.* 29(2) : 1555-1577, 2023.
7. [Peltola-W. AAP 2023+] Crossing probabilities of multiple Ising interfaces *Ann. Appl. Probab.* to appear. 2023+
8. [Han-Liu-W. 2020] Hypergeometric SLE with $\kappa = 8$: convergence of UST and LERW in topological rectangles. arXiv :2008.00403. 2020.
9. [Liu-Peltola-W. 2021] Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in $c = -2$ logarithm CFT. arXiv :2108.04421. 2021.
10. [Feng-Peltola-W. 2022] Connection probabilities of multiple FK-Ising interfaces. arXiv :2205.08800. 2022.
11. [Feng-W.-Yang 2023] Multiple Ising interfaces in annulus and $2N$-sided radial SLE. arXiv :2302.09124. 2023.